DEHN SURGERY ON SYMMETRIC KNOTS

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For a knot $K$ in $S^3$ and a rational number $r$, let $M(K, r)$ be the closed 3-manifold obtained by Dehn surgery of type $r$ on $K$. $K$ is said to have Property $P_n (n \in \mathbb{Z} \setminus \{0\})$, iff $M(K, 1/n)$ is not simply connected, and $K$ is said to have Property $P$, iff $K$ has Property $P_n$ for any $n \in \mathbb{Z} \setminus \{0\}$. A link $L$ in $S^3$ is said to have Property $P^*$, iff every Dehn surgery of $S^3$ on $L$ does not produce a fake homotopy 3-sphere (cf. [23]).

Montesinos [13] studied the relationship between 2-fold branched coverings and closed 3-manifolds obtained by Dehn surgery on links with symmetry. In particular, he showed that every simply connected 2-fold branched covering of $S^3$ is homeomorphic to $S^3$, iff every strongly invertible link has Property $P^*$. (See p. 227 of [13]. Note that the term “Property P” in [13] means Property $P^*$ in this paper.) Hence, by the positive solution of the homotopy Smith conjecture [20], it follows that every strongly invertible link has Property $P^*$. Nevertheless, it remains open whether every strongly invertible knot has Property $P$.

In this paper, using the techniques of Montesinos [13] and the homotopy Smith conjecture, we prove the Property P conjecture for some classes of knots with symmetry — a class containing all 3-strand pretzel knots of odd type (Theorem 1), and a class containing all 2-bridge knots (Theorem 2). (For 2-bridge knots, the Property P conjecture has been proved by Takahashi [21].)

We apply our method to the knots in the knot table which were not proved by Riley [17] to have Property PP (which is stronger than Property P). Through it, we can conclude that all knots in the table with 9 crossings or less except $8_{17}, 9_{32}$ and $9_{33}$ have Property P (see Section 7). (The exceptional knots $8_{17}, 9_{32}$ and $9_{33}$ are the only knots with 9 crossings or less, which are non-invertible (see [5, 7]).)

1. Dehn surgery on periodic knots

Let $L=O \cup K$ be a 2-component link in $S^3$ with $O$ a trivial knot, and let $n (n>1)$ be a positive integer relatively prime to $\lambda=\text{lk}(O, K)$. The $n$-fold cyclic branched covering $\Sigma_n(O)$ of $S^3$ branched along $O$ is again a 3-sphere, and the lift, $C_n(L)$, of $K$ to $\Sigma_n(O)$ is a periodic knot of period $n$. (By [20], every periodic knot is so obtained.) Let $N(K)$ be a regular neighbourhood of $K$, which is disjoint
from \(O\). For an integer \(k\), the homology sphere \(M(K, 1/k)\) is obtained from \(S^3 - \text{int } N(K)\) by sewing a solid torus. Let \(W_k(L)\) be the knot in \(M(K, 1/k)\) given by \(O \subseteq S^3 - \text{int } N(K) \subseteq M(K, 1/k)\). [Hereafter, for a 2-component ordered link \(L = K_1 \cup K_2\) in \(S^3\), we use the symbol \(C_\mu(L)\) (resp. \(W_\mu(L)\)) to denote the lift of \(K_2\) in \(\Sigma_a(K_1)\) (resp. the knot \(K_1\) in \(M(K_2, 1/k)\)).]

The following is a generalization of Theorem 2 of [13].

**Proposition 1.** For an integer \(q\), \(M(C_\mu(L), 1/q)\) is the \(n\)-fold cyclic branched covering of \((M(K, 1/nq)\) branched along \(W_{\mu q}(L)\).

**Proof.** Let \(p\) be the covering projection \(\Sigma_a(O) \to S^3\), and let \(\tilde{N}(K) = p^{-1}(N(K))\). Then \(\tilde{N}(K)\) is a regular neighbourhood of \(C_\mu(L)\), which is invariant under the \(Z_n\)-action on \(\Sigma_a(O)\). Let \(\ell\) and \(m\) be a preferred longitude (see p. 31 of [19]) and a meridian of \(N(K)\), and let \(\tilde{\ell}\) and \(\tilde{m}\) be a lift of \(\ell\) and \(m\) respectively. Then \(\tilde{\ell}\) and \(\tilde{m}\) are a preferred longitude and a meridian of \(\tilde{N}(K)\) respectively, and their homology classes satisfy the equations \(p_a[\tilde{\ell}] = n[\ell]\) and \(p_a[\tilde{m}] = [m]\) in \(H_1(\partial N(K))\). Now, \(M(C_\mu(L), 1/q)\) is obtained from \(\Sigma_a(O) - \text{int } \tilde{N}(K)\) and a solid torus \(\tilde{T}\) by identifying their boundaries, where a meridian \(\tilde{\mu}\) of \(\tilde{T}\) is identified with a simple loop on \(\partial \tilde{N}(K)\) representing the homology class \(q[\tilde{\ell}] + [\tilde{m}] \in H_1(\partial \tilde{N}(K))\). It can be seen that the \(Z_n\)-action on \(\partial \tilde{N}(K)\) extends to a free \(Z_n\)-action on \(\tilde{T}\), such that \(\tilde{T}/Z_n\) is again a solid torus \(T\). Thus we obtain a \(Z_n\)-action on \(M(C_\mu(L), 1/q)\), such that

1. \(\text{Fix}(Z_n) = p^{-1}(O)\), and
2. \(M(C_\mu(L), 1/q)/Z_n = (\Sigma_a(O) - \text{int } \tilde{N}(K))/Z_n \cup \tilde{T}/Z_n \cong (S^3 - \text{int } N(K)) \cup T\).

Here a meridian \(\mu\) of \(T\) is identified with a simple loop on \(\partial N(K)\) representing the homology class \(nq[\ell] + [m] \in H_1(\partial N(K))\), since \(p_a([\tilde{\mu}]) = p_a(q[\tilde{\ell}] + [\tilde{m}]) = nq[\ell] + [m]\) in \(H_1(\partial N(K))\). This completes the proof.

From the above, we have the following proposition.

**Proposition 2.** \(C_\mu(L)\) has Property \(P_q\) \((q \in \mathbb{Z}\setminus\{0\})\), if \(K\) has Property \(P_{nq}\) or the knot \(W_{\mu q}(L)\) in \(M(K, 1/nq)\) is non-trivial.

**Proof.** This follows from the positive solution of the homotopy Smith Conjecture [20] and the fact that the homomorphism \(\pi_1(M(C_\mu(L), 1/q)) \to \pi_1(M(K, 1/nq))\) induced by the covering projection is an epimorphism.

**Example 1.** The knot \(8_{18}\) is a periodic knot of period 4. Let \(h\) be a periodc map on \(S^3\) realizing the 4-fold symmetry. Then the knot \(8_{18}/h^2 \subset S^3/h^2 \cong S^3\) is a figure-eight knot, which has Property \(P\) by [1, 3]. Hence \(8_{18}\) has Property \(P\).
EXAMPLE 2. Let $L = O \cup K$ be a link as illustrated in Fig. 1. Then $C_2(L)$ is a figure-eight knot (see Fig. 2).

![Fig. 1](image1)

![Fig. 2](image2)

Since $K$ is a trivial knot, $M(C_2(L), 1/q)$ is the 2-fold branched covering of $S^3$ branched along the knot $W_{2q}(L)$, which is obtained from $O$ by $(-2q)$ right-hand full twists along $K$ (see Fig. 3).

![Fig. 3](image3)

The Alexander polynomial of $W_{2q}(L)$, which is calculated in Example 4 of Section 3, is nontrivial. So, we can conclude that the figure-eight knot has Property P.

2. Dehn surgery on strongly invertible knots

Let $O$ be a trivial knot in $S^3$, and let $J$ be an arc in $S^3$ such that $J \cap O = \partial J$. The 2-fold branched covering $\Sigma_2(O)$ of $S^3$ branched along $O$ is a 3-sphere, and the inverse image, $I(J)$, of the arc $J$ in $\Sigma_2(O)$ is a strongly invertible knot. (By [24], every strongly invertible knot is so obtained.)

Montesinos proved that $M(I(J), r) \ (r \in Q \cup \{\infty\})$ is a 2-fold branched covering of $S^3$ (see Theorem 1 of [13]). Let $F_q(J) \ (q \in Z)$ be the branch line of the branched covering $M(I(J), 1/q) \to S^3$. Then, by the homotopy Smith conjecture [20], we have the following.
PROPOSITION 3. \( I(J) \) has Property \( P_q \) \((q \in \mathbb{Z} \setminus \{0\})\), iff the knot \( F_q(J) \) is nontrivial.

Here, we describe the branch line \( F_q(J) \) according to [14] (cf. [12]). Let \( B \) be a regular neighbourhood of \( J \), such that \( O \) intersects \( B \) in two disjoint proper arcs (see Fig. 4). Then the inverse image \( \beta \) of \( B \) is a regular neighbourhood of \( I(J) \), which is invariant under the nontrivial covering transformation \( t \) (see Fig. 5).

Let \( \ell \) be a preferred longitude of \( \beta \), such that \( \pi(\ell) \cap \ell = \emptyset \), and let \( \gamma = p(\ell) \), where \( p \) is the projection \( \Sigma_2(O) \to S^3 \) (see Fig.'s 4, 5 and 6). Let \( D_\gamma \) be a homeomorphism on \( B \) as illustrated in Fig. 6.

Then the branch line \( F_q(J) \) is given by

\[
F_q(J) = (O - (O \cap \text{int} B)) \cup D_\gamma^{-q}(O \cap B) \subset (S^3 - B) \cup B = S^3.
\]

Let \( L_0(J) = O \cup \gamma \) and \( L_1(J) = F_1(J) \cup \gamma \). Then, we have

\[
F_q(J) \cong \begin{cases} 
W_q(L_0(J)) & (q = 2k) \\
W_q(L_1(J)) & (q = 2k + 1).
\end{cases}
\]

(Recall the definition of \( W_q(\cdot) \) given in Section 1.)
Example 3. Let $O$ and $J$ be a trivial knot and an arc in $S^3$ as illustrated in Fig. 7. Then $I(J)$ is a figure-eight knot (see Fig. 8).

![Fig. 7](image1.png)  
![Fig. 8](image2.png)

The knot $F_q(J) = (O - (O \cap \text{int } B)) \cup D^{-q}_\gamma(O \cap B)$ is illustrated in Fig. 9.

![Fig. 9](image3.png)

The Alexander polynomial of $F_q(J)$, which is calculated in Example 5 of Section 3, is nontrivial. So, we can again conclude that the figure-eight knot has Property P.

3. The effect of the transformation $W_k(\ )$ on the Alexander polynomials

As discussed in the previous sections, the Property P conjecture for periodic
knots and strongly invertible knots is reduced to proving the nontriviality of knots obtained from certain 2-component links through the operation $W_k(\ )$.

In this section, we give formulas of the Alexander polynomials of such knots, one of which was formulated by Kidwell [8].

Let $L = K_1 \cup K_2$ be an oriented ordered link in an oriented $S^3$. Recall that $W_k(L)$ denotes the knot $K_1$ in $M(K_2, 1/k)$. If $K_2$ is a trivial knot, $W_k(L)$ is a knot in $S^3$ obtained from $K_1$ by $(-k)$ right-hand full twists along $K_2$. Let $\Delta(x, y)$ be the Alexander polynomial of the link $L$, $\lambda = lk(K_1, K_2)$, and $\Delta_\lambda(t)$ be the Alexander polynomial of the knot $W_k(L)$. In case $\lambda = 0$, define a polynomial $A(t)$ as follows. Let $V = M(K_2, 0) - \text{int } N(K_1)$, where $N(K_1)$ is a regular neighbourhood of $K_1$, and let $\bar{V}$ be the infinite cyclic cover of $V$ corresponding to the composite homomorphism $\pi_1(V) \to Z$ of the abelianization and $\tau: H_1(V) \cong Z \oplus Z \to Z$ where $\tau$ carries the meridians of $K_1$ and $K_2$ to a generator and zero respectively. Define $A(t)$ to be the determinant of a square presentation matrix of the $\mathbb{Z}\langle t \rangle$-module $H_1(\bar{V})$ (see Section 2 of [9]). We call $A(t)$ the $A\tau$-polynomial of $L$.

**Proposition 4.** (1) (Corollary 3.2 of Kidwell [8]) In case $\lambda \neq 0$, we have $\Delta_\lambda(t) = \Delta(t, t^{-\lambda})/\rho_\lambda(t)$, where $\rho_\lambda(t) = (t^\lambda - 1)(t - 1)$.

(2) In case $\lambda = 0$, we have $\Delta_0(t) = \Delta_0(t) + kA(t)$.

**Remark.** Since the polynomials $\Delta_\lambda(t)$ and $A(t)$ are well-defined only up to units of the group ring $\mathbb{Z}\langle t \rangle$, there remains some ambiguity in the formula (2). The precise meaning of it is as follows: For a suitable representation of the polynomials

$$A(t) = a_0 + \sum_{i \geq 0} a_i(t^i + t^{-i}) \quad \text{and} \quad \Delta_\lambda(t) = b_0 + \sum_{i \geq 0} b_i(t^i + t^{-i}),$$

we have $\Delta_\lambda(t) = (b_0 + ka_0) + \sum_{i \geq 0} (b_i + ka_i)(t^i + t^{-i})$.

**Proof.** (1) This is a generalization of the Torres' formula, and, in fact, is proved by Kidwell [8] by using it.

(2) To prove this formula, we use arguments of Kojima-Yamasaki [9] and Rolfsen [18]. By [18], there are disjoint solid tori $T_1, \ldots, T_n$ in $S^3$ and a self-homeomorphism $h$ on $S^3 - \text{int } (T_1 \cup \cdots \cup T_n)$, such that

1. $h(K_1)$ is unknotted in $S^3$,
2. $lk(T_r, K_1) = lk(T_r, h(K_1)) = 0$ for all $r$,
3. $h(\partial T_r) = \partial T_r$ and $lk(\mu_r, T_r) = \pm 1$, where $\mu_r$ is a meridian of $T_r$ and $\mu_r' = h(\mu_r)$.

Since $h(K_1)$ is unknotted, the infinite cyclic cover of $S^3 - \text{int } h(N(K_1))$ is

$$p: R^1 \times D^2 \longrightarrow S^1 \times D^2 \cong S^3 - \text{int } h(N(K_1)).$$
Since $\text{lk}(h(K), K) = 0$ (resp. $\text{lk}(T, h(K)) = 0$), a lift $\hat{N}(K)$ (resp. $\hat{T}$) of $N(K)$, a regular neighbourhood of $K$ (resp. $T$), is homeomorphic to $D^2 \times S^1$. Let $t$ be a generator of the covering transformation group, $\delta$ and $m$ be a preferred longitude and a meridian of $K$ respectively, and $\delta$ (resp. $m$, $\mu$) be the lift of $\delta$ (resp. $m$, $\mu$) to $\partial \hat{N}(K)$ (resp. $\partial \hat{K}$). Let $E = M(K, 1/k) - \text{int} N(W(L))$, where $N(W(L))$ is a regular neighbourhood of the knot $W(L)$, and let $E_k$ be the infinite cyclic cover of $E$. Then $\hat{V}$ (resp. $\hat{E}_k$) is obtained from $R^1 \times D^2$ by removing each $\text{int} t^i(\hat{N}(K))$, $\text{int} t^i(\hat{T})$ ($i, j \in \mathbb{Z}$), and sewing back a solid torus so that its meridian coincides with $t^i \delta$ (resp. $t^i (k \delta + m)$) or $t^i \mu$ (resp. $t^i \mu$). Then, by Proposition 4 of [9], $H_1(\hat{V})$ has a presentation matrix

$$\begin{pmatrix} a & b \\ c^r & D \end{pmatrix}$$

where $b = (b_1, \ldots, b_n)$, $c = (c_1, \ldots, c_n)$, $D = (d_{ij})_{1 \leq i, j \leq n}$

with $a = \sum \text{lk}(\delta, t^i \hat{K}), b_i = \sum \text{lk}(\delta, t^i \hat{T}), c_i = \sum \text{lk}(\mu, t^i \hat{K}), d_{ij} = \sum \text{lk}(\mu, t^i \hat{T})$.

Here $\hat{K}$ is the lift of $K$ to $\hat{N}(K)$, and $\text{lk}(\ , \ )$ is the linking number in $R^1 \times D^2$.

By [18], $D$ is a presentation matrix of $H_1(\hat{E}_k)$. Recall that, in constructing $\hat{E}_k$, the meridian of the solid torus attached to $\partial \hat{N}(K)$ is identified with $k \delta + m$.

From this fact, we can see that $H_1(\hat{E}_k)$ has a presentation matrix

$$\begin{pmatrix} 1 + ka & kb \\ c^r & D \end{pmatrix}$$

Hence $A_k(t) = \det\begin{pmatrix} 1 + ka & kb \\ c^r & D \end{pmatrix}$

$$= \det\begin{pmatrix} 1 & kb \\ 0 & D \end{pmatrix} + \det\begin{pmatrix} ka & kb \\ c^r & D \end{pmatrix}$$

$$= \det D + k \det\begin{pmatrix} a & b \\ c^r & D \end{pmatrix}$$

$$= A_0(t) + kA(t).$$

This completes the proof. (The statement in the remark follows from the fact that $A_k(t)$ and $A(t)$ are symmetric (see [6]).)

**Example 4.** Consider the same setting as that of Example 2. Then $\lambda = \text{lk}(O, K) = 3$, and the Alexander polynomial $A(x, y)$ of $L = O \cup K$ is $x + (1 - x +$
$x^2)y + xy^2$. Hence, by Proposition 4, the Alexander polynomial $\Delta^{(q)}(t)$ of the knot $W_{2q}(L)$ is given by

$$\Delta^{(q)}(t) = \Delta(t, t^{-6q})/(1 + t + t^2) = (t + t^{-6q} - t^{-6q+1} + t^{-6q+2} + t^{-12q+1})/(1 + t + t^2).$$

In particular, $\text{deg } \Delta^{(q)}(t) = 12|q| - 2$ ($q \neq 0$), and therefore $\Delta^{(q)}(t) \neq 1$.

**Example 5.** Consider the same setting as that of Example 3. Recall that the knot $F_{q}(J)$ is equivalent to $W_{k}(L_{0}(J))$ or $W_{k}(L_{1}(J))$ according to whether $q = 2k$ or $q = 2k + 1$, where $L_{0}(J) = O \cup \gamma$ and $L_{1}(J) = F_{1}(J) \cup \gamma$. Here $lk(O, \gamma) = lk(F_{1}(J), \gamma) = 0$. Hence, by Proposition 3, the Alexander polynomial $\Delta^{(q)}(t)$ of the knot $F_{q}(J)$ is given as follows.

$$\Delta^{(q)}(t) = \begin{cases} 1 + kA_{0}(t) & (q = 2k) \\ \Delta^{(1)}(t) + kA_{1}(t) & (q = 2k + 1), \end{cases}$$

where $A_{0}(t)$ and $A_{1}(t)$ are the $\Delta \tau$-polynomials of the links $L_{0}(J)$ and $L_{1}(J)$ respectively. By direct calculation, we have

$$A_{0}(t) = [-2, 1, 1, -1]$$

$$A_{1}(t) = [0, 1, -2, 1, 1, -2, 1, -1]$$

where $[a_{0}, a_{1}, \ldots, a_{n}]$ means $a_{0} + \sum_{i=1}^{n} a_{i}(t^{i} + t^{-i})$. In particular, $\text{deg } \Delta^{(q)}(t)$ ($q \neq 0$) is equal to 6, 10 or 12, according to whether $q$ is even, $-1$ or one of the rest, and therefore $\Delta^{(q)}(t) \neq 1$.

**Remark.** Since the figure-eight knot is amphicheiral, we have $M_{q} \cong M_{-q}$, where $M_{q} = M$ (figure-eight, $1/q$). So the knots $W_{\pm 2q}(L)$ in Example 2 and the knots $F_{\pm q}(J)$ in Example 3 have the same homology 3-sphere $M_{q}$ as 2-fold branched coverings. On the other hand, by Examples 4 and 5, we have

1. $W_{2q}(L) \cong W_{-2q}(L)$ for any $q$,
2. $F_{q}(J) \cong F_{-q}(J)$ for any $q$,
3. $F_{q}(J) \not\cong W_{\pm 2q}(L) \not\cong F_{-q}(J)$, if $|q| > 1$,
4. $F_{1}(J) \not\cong W_{\pm 2}(L) \cong F_{-1}(J)$.

Hence, if $|q| > 1$ (resp. $|q| = 1$), there are three (resp. two) inequivalent knots in $S^{3}$ whose 2-fold branched coverings are homeomorphic to the same homology 3-sphere $M_{q}$. Takahashi [22] constructed such knots from a different point of view. In fact, it can be seen that they are $F_{q}(J)$ and $F_{-q}(J)$.
At the end of this section, we explain a convenient method for calculating Alexander polynomials of 2-component links given by Cooper [2], which we use in Sections 4 and 5.

Let $D$ and $D'$ be P. L. embedded bicollared disks in $S^3$, such that $D \cap D'$ has only clasp singularities. Let $\{\gamma_1, \ldots, \gamma_n\}$ be a basis of the free abelian group $H_1(D \cup D')$. We define two matrices $A$ and $B$ as follows. Let $u_i$ be a 1-cycle representing $\gamma_i$, such that $u_i \cap (D \cap D')$ has a neighbourhood in $S^3$ of the form as shown in Fig. 10.

![Fig. 10](image)

Then, define $A=(lk(u_i^-, u_j))$ and $B=(lk(u_i^+, u_j))$, where $u_i^-$ (resp. $u_i^+$) is the 1-cycle in $S^3$ obtained by lifting $u_i$ off $D \cup D'$ in the negative normal direction off $D$ and in the negative (resp. positive) normal direction off $D'$. The following is a special case of Theorem 2.1 of Cooper [2].

**Proposition 5.** The Alexander polynomial of the link $L=\partial D \cup \partial D'$ is $\det(xyA + A^T - xB - yB^T)$.

4. A class of knots containing 3-strand pretzel knots of odd type

For a 3-tuple of integers $(r_1, r_2, r_3)$, let us consider an oriented link $L(r_1, r_2, r_3) = K_2 \cup K_2$ as illustrated in Fig. 11.

![Fig. 11](image)
Here, we assume that \((r_i, r_j) \neq (0, -1)\) for any \(i, j\) \((1 \leq i, j \leq 3)\). (If \((r_i, r_j) = (0, -1)\) for some \(i, j\), then \(L(r_1, r_2, r_3)\) is the Hopf link.) For a positive integer \(n\) \((n \geq 2)\) relatively prime to the linking number \(\lambda = lk(K_1, K_2)\), let \(K_n(r_1, r_2, r_3)\) be the periodic knot \(C_n(L(r_1, r_2, r_3))\) generated by the link \(L(r_1, r_2, r_3)\). (Recall the definition of \(C_n(\ )\) given in Section 1.) In particular, \(K_2(r_1, r_2, r_3)\) is the pretzel knot of type \((2r_1 + 1, 2r_2 + 1, 2r_3 + 1)\).

**Theorem 1.** \(K_n(r_1, r_2, r_3)\) has property \(P\).

**Proof.** Since \(L(r_1, r_2, r_3)\) is equivalent to \(L(-r_1 - 1, -r_2 - 1, -r_3 - 1)\) and \(L(r^{\sigma(1)}, r^{\sigma(2)}, r^{\sigma(3)})\) for any permutation \(\sigma\) on \(\{1, 2, 3\}\), we may assume that \(L(r_1, r_2, r_3)\) is of one of the following two types.

**Type 1.** \(L(2l_1, 2l_2, 2l_3)\)

**Type 2.** \(L(2l_1, 2l_2, 2l_3 + 1)\) \((l_i, l_3) \neq (0, -1)\) for each \(i = 1, 2\)

If \(L(r_1, r_2, r_3)\) is of **Type 1** (resp. **Type 2**), then the linking number \(\lambda\) is 3 (resp. 1). Let \(A(x, y)\) be the Alexander polynomial of \(L(r_1, r_2, r_3)\). Then, by Propositions 2 and 4, we have only to prove that \(\deg A(t, t^k) > \lambda - 1\) for each integer \(k\) \((|k| \geq 2)\).

To calculate \(A(x, y)\), let us consider bicollared disks \(D\) and \(D'\) in \(S^3\) with \(\partial D = K_1\) and \(\partial D' = K_2\) as illustrated in Fig. 12. Choose 1-cycles \(u_1\) and \(u_2\), which form a basis of \(H_1(D \cup D')\), as illustrated in Fig. 12.

![Fig. 12](image)

Shaded sides of \(D\) and \(D'\) are negative sides.

Then, by Proposition 5, \(A(x, y) = \det(xyA + AT - xB - yB^T)\), where the matrices \(A\) and \(B\) are given as follows:

**Type 1:** \[
A = \begin{pmatrix}
l_1 + l_2 & -l_2 \\
-l_2 & l_2 + l_3
\end{pmatrix}, \quad
B = \begin{pmatrix}
l_1 + l_2 & -l_2 - 1 \\
-l_2 & l_2 + l_3 + 1
\end{pmatrix},
\]

**Type 2:** \[
A = \begin{pmatrix}
l_1 + l_2 & -l_2 \\
-l_2 & l_2 + l_3 + 1
\end{pmatrix}, \quad
B = \begin{pmatrix}
l_1 + l_2 + 1 & -l_2 \\
-l_2 - 1 & l_2 + l_3 + 1
\end{pmatrix}.
\]
Hence, we have $\Delta(x, y) = \sum_{1 \leq i, j \leq 3} w_{ij} x^{i-1} y^{j-1}$, where the coefficient matrix $(w_{ij})_{1 \leq i, j \leq 3}$ is given as follows.

$$
\begin{array}{ccc}
1 & x & x^2 \\
1 & \beta & -\alpha - 2\beta & \alpha + \beta + 1 \\
\text{Type 1;} & y & (-\alpha - 2\beta) & 2\alpha + 4\beta + 1 & -\alpha - 2\beta \\
y^2 & (\alpha + \beta + 1) & -\alpha - 2\beta & \beta \\
1 & x & x^2 \\
1 & \beta + l_1 + l_2 & -\alpha - 2\beta - 1 - l_1 - l_2 & \alpha + \beta + 1 \\
\text{Type 2;} & y & (-\alpha - 2\beta - 1 - l_1 - l_2) & 2\alpha + 4\beta + 1 + 2(l_1 + l_2) & -\alpha - 2\beta - 1 - l_1 - l_2 \\
y^2 & (\alpha + \beta + 1) & -\alpha - 2\beta - 1 - l_1 - l_2 & \beta + l_1 + l_2,
\end{array}
$$

where $\alpha = l_1 + l_2 + l_3, \beta = l_1 l_2 + l_1 l_3 + l_2 l_3$.

Now, we prove that $\deg \Delta(t, t^{k}) > \lambda - 1$ for any integer $k (|k| \geq 2)$.

Type 1: It is clear that $(\alpha + \beta + 1, -\alpha - 2\beta, \beta) \neq (0, 0, 0)$. Hence, we have

$$
\deg \Delta(t, t^{k}) = \deg \Delta(t, t^{k}) \\
\geq \begin{cases} 
6k - 2 & (k \geq 2) \\
0 - (6k + 2) & (k \leq -2)
\end{cases}
$$

$$
> 2 = \lambda - 1
$$

Type 2: It can be seen that $(\alpha + \beta + 1, -\alpha - 2\beta - 1 - l_1 - l_2) = (0, 0)$ (resp. $(-\alpha - 2\beta - 1 - l_1 - l_2, \beta + l_1 + l_2) = (0, 0)$), if $l_3 = -1$ and $l_1 l_2 = 0$. But this does not occur by the assumption. Hence, for any $k (|k| \geq 2)$, we have

$$
\deg \Delta(t, t^{k}) = \deg \Delta(t, t^{k}) \geq |(2k + 1) - 1| > 0 = \lambda - 1.
$$

This completes the proof.

5. A class of knots containing 2-bridge knots

Let $L(2p, q) = K_1 \cup K_2$ be an oriented 2-bridge link of type $(2p, q)$, where $1 \leq q < 2p$ and g.c.d. $(2p, q) = 1$ (see Fig. 13). Here, we assume that $p \neq 1$. $(L(2, 1)$ is the Hopf link.) For a positive integer $n (n \geq 2)$ relatively prime to the linking number $\lambda = lk(K_1, K_2)$, let $K_n(p, q)$ be the periodic knot $C_n(L(2p, q))$ generated by the link $L(2p, q)$. In particular, $K_n(p, q)$ is a 2-bridge knot of type $(p, q)$.

**Theorem 2.** $K_n(p, q)$ has property $P$. 

PROOF. \( \frac{2p}{q} \) has the following continued fraction:

\[
\frac{2p}{q} = 2b_1 + \frac{1}{-2b_2 + \frac{1}{2b_3 + \cdots + \frac{1}{-2b_{m-1} + \frac{1}{2b_m}}}},
\]

where \( b_i \) is a non-zero integer for each \( 1 \leq i \leq m \), and \( m \) is an odd integer. Then \( L(2p, q) \) is equivalent to the link as illustrated in Fig. 13.

![Diagram of continued fraction and link](image)

Fig. 13

We may assume that the linking number \( \lambda \) is positive. Let \( \Delta(x, y) \) be the Alexander polynomial of \( L(2p, q) \). Then, by Propositions 2 and 4, we have only to prove that \( \deg \Delta(t, t^{k2}) > \lambda - 1 \) for each \( k \mid k \mid \geq 2 \). To calculate the polynomial, let us consider bicollared disks \( D \) and \( D' \) with \( \partial D = K_1 \) and \( \partial D' = K_2 \) as illustrated in Fig. 14. \( D \cap D' \) consists of \( |b_1| + |b_3| + \cdots + |b_m| \) clasp singularities. Choose 1-cycles \( u_1, u_2, \ldots, u_s \) which form a basis of \( H_1(D \cup D') \) as illustrated in Fig. 14, where \( s = |b_1| + |b_3| + \cdots + |b_m| - 1 \). Note that \( s \geq 1 \), since \( p \neq 1 \).

![Diagram of bicollared disks and 1-cycles](image)

Fig. 14

Let \( V = t^{k2+1}A + A^T - tB - t^{k2}B^T \), where \( A \) and \( B \) are matrices defined in Section 3. Then, by Proposition 5, \( \Delta(t, t^{k2}) = \det V \).

The matrices \( A \) and \( B \) are given as follows;
where $B_i$ and $v_j$ are given as follows ($i = 1, 3, 5, ..., m$, $j = 0, 2, 4, ..., m + 1$).
$$B_i = \begin{cases} v_{i-1} & (b_i > 0) \\
-1 & -1 \\
-1 & 1 \\
-1 & 1 \\
1 & v_{i+1} \end{cases}$$

$$v_0 = \begin{cases} -1 & (b_1 > 0) \\
1 & (b_1 < 0) \end{cases}$$

$$v_j = b_j + \epsilon_j/2 \quad \text{with} \quad \epsilon_j = -\{b_{j-1}/|b_{j-1}| + b_{j+1}/|b_{j+1}|\} \quad (j = 2, 4, ..., m - 1)$$

$$v_{m+1} = \begin{cases} -1 & (b_m > 0) \\
1 & (b_m < 0) \end{cases}$$

Therefore the matrix $V = t^{k+1}A + A^T - tB - t^{k+1}B^T$ is of the following form.

$$V = \begin{bmatrix}
T_0 \\
V_1 \\
T_2 \\
V_3 \\
T_4 \\
V_{m-1} \\
V_m \\
T_{m+1}
\end{bmatrix}$$

where $V_i$ and $T_j$ are given as follows ($i = 1, 3, 5, ..., m$, $j = 0, 2, 4, ..., m + 1$).
\[
V_i = \begin{cases}
T_{i-1} & -t \\
-t^{k\lambda} & t^{k\lambda} + t & -t \\
& -t^{k\lambda} & t^{k\lambda} + t & -t \\
& & -t^{k\lambda} & t^{k\lambda} & T_{i+1} \\
& & & & (b_1 > 0)
\end{cases}
\]

\[
T_0 = \begin{cases}
t^{k\lambda} + t & (b_1 > 0) \\
-t^{k\lambda} - t & (b_1 < 0)
\end{cases}
\]

\[
T_j = b_j(t^{k\lambda+1}) - (b_j + \varepsilon/2)(t^{k\lambda} + t) \quad (j = 2, 4, \ldots, m - 1)
\]

\[
T_{m+1} = \begin{cases}
t^{k\lambda} + t & (b_m > 0) \\
-t^{k\lambda} - t & (b_m < 0)
\end{cases}
\]

Let \(W_i(1 \leq i \leq s)\) be the submatrix of \(V\) consisting of \((v, \mu)\) entries of \(V\) with \(v, \mu > s - i\). Define \(d_i(t) = \text{det} \ W_i\ (1 \leq i \leq s)\), and \(d_0(t) = 1\). Especially, \(d_m(t) = \text{det} \ V = \Delta(t, t^{k\lambda})\).

**Lemma 1.** For each integer \(i\ (1 \leq i \leq s - 1)\), the following equation holds.

\[
d_{i+1}(t) = F_{s-i}(t)d_i(t) - t^{k\lambda+1}d_{i-1}(t).
\]

Here, \(F_j(t)\) is the \((j, j)\) entry of \(V\).

**Proof.** By expanding the first column of \(W_i\), we obtain the equation immediately.

For a Laurent polynomial \(f(t)\), let \(\text{Max}(f(t))\) (resp. \(\text{Min}(f(t))\)) be the maximal (resp. minimal) \(t\)-power of any term of \(f(t)\).

**Lemma 2.** For each integer \(i\ (1 \leq i \leq s)\), we have the followings.

1. If \(k \geq 2\), \(\text{Max}(d_i(t)) \geq \text{Max}(d_{i-1}(t)) + k\lambda\) \((a_i)\), \(\text{Min}(d_i(t)) \leq \text{Min}(d_{i-1}(t)) + 1\) \((b_i)\).
2. If \(k \leq -2\), \(\text{Max}(d_i(t)) \geq \text{Max}(d_{i-1}(t))\) \((c_i)\), \(\text{Min}(d_i(t)) \leq \text{Min}(d_{i-1}(t)) + k\lambda + 1\) \((d_i)\).
PROOF. (1) $k \geq 2$: Note that $\text{Max}(F_0(t)) \geq k\lambda$ and $\text{Min}(F_0(t)) \leq 1$, for each $i$ ($1 \leq i \leq s$). We prove the inequality $(\alpha_i)$ inductively. Since $\text{Max}(d_1(t)) = \text{Max}(F_0(t)) \geq k\lambda$, $(\alpha_1)$ holds. Suppose that $(\alpha_j)$ holds for some $j$ ($1 \leq j \leq s-1$).

Then

$$\text{Max}(F_{s-j}(t)d_j(t)) = \text{Max}(F_{s-j}(t)) + \text{Max}(d_j(t))$$

$$\geq k\lambda + (\text{Max}(d_{j-1}(t)) + k\lambda)$$

$$(k\lambda + 1) + \text{Max}(d_{j-1}(t))$$

$$= \text{Max}(t^{k\lambda+1}d_{j-1}(t)).$$

Hence, by Lemma 1, $\text{Max}(d_{j+1}(t)) = \text{Max}(F_{s-j}(t)d_j(t)) \geq \text{Max}(d_j(t)) + k\lambda$, and $(\alpha_{j+1})$ holds. Therefore $(\alpha_i)$ holds for any $i$ ($1 \leq i \leq s$). Next, we prove $(\beta_i)$ inductively. Since $\text{Min}(d_1(t)) = \text{Min}(F_0(t)) \leq 1$, $(\beta_1)$ holds. Suppose that $(\beta_j)$ holds for some $j$ ($1 \leq j \leq s-1$). Then

$$\text{Min}(F_{s-j}(t)d_j(t)) = \text{Min}(F_{s-j}(t)) + \text{Min}(d_j(t))$$

$$\leq 1 + (\text{Min}(d_{j-1}(t)) + 1)$$

$$< (k\lambda + 1) + \text{Min}(d_{j-1}(t))$$

$$\leq \text{Min}(t^{k\lambda+1}d_{j-1}(t)).$$

Hence, by Lemma 1, $\text{Min}(d_{j+1}(t)) = \text{Min}(F_{s-j}(t)d_j(t)) \leq \text{Min}(d_j(t)) + 1$, and $(\beta_{j+1})$ holds. Therefore $(\beta_i)$ holds for any $i$ ($1 \leq i \leq s$).

(2) $k \leq -2$: Note that $\text{Max}(F_0(t)) \geq 0$ and $\text{Min}(F_0(t)) \leq k\lambda + 1$, for each $i$ ($1 \leq i \leq s$). Then, by a similar argument as the above, we can prove the inequalities $(\gamma_i)$ and $(\delta_i)$ ($1 \leq i \leq s$).

From the above lemma, we have $\text{deg}(d_i(t)) > \text{deg}(d_{i-1}(t))$, for any $i$ ($2 \leq i \leq s$), and $\text{deg}(d_1(t)) > \lambda - 1$. Therefore,

$$\text{deg} d(t, t^{k\lambda}) = \text{deg}(d_1(t)) > \text{deg}(d_{s-1}(t)) > \cdots > \text{deg}(d_1(t)) > \lambda - 1.$$

This completes the proof of Theorem 2.

6. Even pretzel knots

Let $K(p, q, 2r)$ be an even pretzel knot. ($p$ and $q$ are odd integers.) Riley [17] proved that, if $p+q \neq 0$, then $K(p, q, 2r)$ has Property PP. So, we consider $K(p, -p, 2r)$. Note that $K(p, -p, 2r)$ is nontrivial, iff $|p| \neq 1$.

**Theorem 3.** $K(p, -p, 2r)$ ($p$: odd, $|p| \neq 1$) has Property $P_{2k+1}$ for any integer $k$.

**Proof.** Since $K(p, -p, 2r) = K(p, p, 2r)$, we may assume that $p = 4p' + 1$. Let $O$ and $J$ be a trivial knot and an arc in $S^3$ as illustrated in Fig. 15. Then $K(p, -p, 2r) = I(J)$. 

(Satoshi Furusawa and Makoto Sakuma)
By Proposition 3, we have only to prove that the knot $F_{2k+1}(J)$ is nontrivial. Recall that $F_{2k+1}(J) = W_4(L_1(J))$. So, by Proposition 4, the Alexander polynomial $A^{(2k+1)}(t)$ of $F_{2k+1}(t)$ is given by

$$A^{(2k+1)}(t) = A^{(1)}(t) + kA_1(t),$$

where $A_1(t)$ is the $A_1$-polynomial of the link $L_1(J)$. By direct calculation, we have

$$A^{(1)}(t) = \begin{cases} 
4p^2 + 2p' + 1, & - p'^2 - p', - 2p'^2 - p', p'^2 + p' \\
4p^2 + 2p' + 1, & - p'^2, - 2p'^2 - p', p'^2 
\end{cases} \quad (r: \text{odd})$$

and $A_1(t) = 0$.

Since $|p| = |4p' + 1| \neq 1$ by the assumption, we have $p' \neq 0$. Therefore $A^{(2k+1)}(t) \neq 1$. This completes the proof.

**Remark.** Since the link $L_0(J)$ is slice in the strong sense, the $A_1$-polynomial of $L_0(J)$ is zero (see [9]). So, the Alexander polynomial of the knot $F_{2k}(J)$ is 1.

7. **Knots with 9 crossings or less**

Riley [17] proved that all knots with 9 crossings or less have Property PP except $8_{10}, 8_{17},$ and $9_n$ for $n = 24, 29, 32, 33, 34, 38, 39, 41, 46, 47,$ and $49$. In this section, we apply our method to them, and prove that all of them except $8_{17}, 9_{32},$ and $9_{33}$ have Property P.

First, we study $8_{10}$ and $9_{24}$ from a different point of view. $8_{10}$ and $9_{24}$ are "ribbon concordant" to $3_1$ and $4_1$, respectively, and therefore, there are epimorphisms from the knot groups of $8_{10}$ and $9_{24}$ to those of $3_1$ and $4_1$, respectively, which carry meridians to meridians and longitudes to longitudes. Therefore, it follows that $8_{10}$ and $9_{24}$ have Property P, since $3_1$ and $4_1$ do. (Recently, Osborn [16] proved that $8_{10}$ has Property P by a different method.)
Next, we use the method of Section 1. Among the knots in consideration, only $9_{41}$, $9_{46}$, $9_{47}$, and $9_{49}$ are periodic (see [15]). $9_{46}$ is a pretzel knot of type $(3, 3, -3)$; so, by Theorem 1, $9_{46}$ has Property P. $9_{41}$, $9_{47}$, and $9_{49}$ belong to the class of knots considered in Section 5. In fact, $9_{41} \cong K_3(9, 5)$, $9_{47} \cong K_3(8, 3)$, and $9_{49} \cong K_3(7, 3)$ (see [4]). Thus, by Theorem 2, they have Property P.

For the remaining knots $9_{29}$, $9_{34}$, $9_{38}$, and $9_{39}$, which are strongly invertible, we use the method of Section 2. The following is a list of the corresponding 0-curves $O \cup J$, the $A_1$-polynomials $A_0(t)$ and $A_1(t)$ of the links $L_0(J)$ and $L_1(J)$, and the Alexander polynomials $A^{(1)}(t)$ of the knots $F_1(J)$ (cf. Example 5).

$$\begin{align*}
9_{29} & \\
A_0(t) &= [2, -1, -1, 1] \\
A^{(1)}(t) &= [3, 0, -3, 0, 2, 0, -1] \\
A_1(t) &= \pm [6, -2, -4, 3, 2, -2, -1, 1] \\

9_{34} & \\
A_0(t) &= [0, 1, -1, -1, 1] \\
A^{(1)}(t) &= [1, 0, 1, -1, -2, 1, 1] \\
A_1(t) &= \pm [0, -1, 2, 1, -3, 0, 1] \\

9_{38} & \\
A_0(t) &= [4, 0, -2] \\
A^{(1)}(t) &= [-1, 1, -1, 1, -1, 1, 0, -3, 2, 2, -1, -2, 1] \\
A_1(t) &= \pm [-2, 2, -2, 2, -2, 1, 3, -5, 0, 3, 1, -3, 1] \\

9_{39} & \\
A_0(t) &= [-4, 0, 2] \\
A^{(1)}(t) &= [1, -1, 1, -1, 0, 1, -2, 1, 1, 0, -1] \\
A_1(t) &= \pm [2, -2, 2, -2, 1, -3, 3, 0, -1, -1, 1] \\
\end{align*}$$

From the above list and Propositions 3 and 4, it follows that $9_{29}$, $9_{34}$, $9_{38}$, and $9_{39}$ have Property P.

8. Final Remark

Litherland [10, 11] proved that, for a 2-component link $L = O \cup K$ in $S^3$ with $O$ a trivial knot, if one of the following conditions holds, then the exterior of the knot $W_0(L)$ in $M(K, 1/k)$ is not a homotopy solid torus.
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(1) \(|lk(O, K)| \geq 3\) and \(k \neq 0\).
(2) \(|lk(O, K)| = 2\) and \(|k| \geq 2\).
(3) \(|lk(O, K)| = 1\), \(wr(O, K) \geq 2\), and \(|k| \geq 6\). (Here \(wr(O, K)\) is the minimum number of intersections of \(K\) with a disk bounded by \(O\).)

Hence, the following holds by Proposition 2.

**Theorem 4.** \(C_n(L)\) has Property \(P\), if one of the following conditions holds.

(1) \(|lk(O, K)| \geq 2\).
(2) \(|lk(O, K)| = 1\), \(n \geq 6\), and \(L\) is not a Hopf link.

References


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