Non-free-periodicity of Amphicheiral Hyperbolic Knots

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A knot $K$ in the 3-sphere $S^3$ is said to have free period $n$ if there is an orientation-preserving homeomorphism $f$ on $S^3$ such that

1. $f(K)=K$,
2. $f$ is a periodic map of period $n$,
3. $\text{Fix}(f^i)=\emptyset$ ($1 \leq i \leq n-1$).

Hartley [3] has given very effective methods for determining the free periods of a knot, and has identified the free periods of all prime knots with 10 crossings or less with eight exceptions. Since then, Boileau [1] has calculated the symmetry groups of the "large" Montesinos knots, and has shown that four of the rest have no free periods. The remaining knots are $8_{10}$, $8_{23}$, $10_{98}$ and $10_{123}$ (cf. [5]). By Hartley-Kawauchi [4], $10_{98}$ and $10_{123}$ are the only prime knots with 10 crossings or less which are strongly positive amphicheiral. Moreover, it follows from the Theorem of [4] that the polynomial condition given by [3] (Theorem 1.2) does not work for determining whether a strongly positive amphicheiral knot has free period 2 or not.

The purpose of this paper is to prove the following theorem:

**Theorem.** Any amphicheiral hyperbolic knot has no free periods.

In particular, $10_{98}$ and $10_{123}$ have no free periods. A circumstantial evidence for this theorem is given by the non-trivial torus knots, which have infinitely many free periods and are not amphicheiral.

§ 1. Some lemmas

Let $K$ be a knot in $S^3$ which has free period $n$, and $f$ be a periodic map on $S^3$ realizing the free period $n$. Let $N$ be an equivariant tubular neighbourhood of $K$ and put $E=S^3-N$.

**Lemma 1.** $K$ does not have an $f$-invariant longitude curve. That is, $f(l) \neq l$, for any simple loop $l$ in $\partial N$ such that $l \sim K$ in $H_1(N)$ and $l \sim 0$ in $H_1(E)$.

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Proof. See [2] p. 180, where this lemma is proved for the case \( n = 2 \). The same argument works even if \( n \geq 3 \).

Lemma 2. Suppose that \( K \) is a hyperbolic knot. Then the restriction of \( f \) to \( \hat{E} \) is equivalent to an isometry.

Proof. Put \( E' = E/f \). Then \( E' \) is a compact manifold with \( \partial E' \cong T^2 \), and \( E' \) is irreducible since \( E \) is so. We show that \( E' \) is homotopically atroidal (cf. [6, 14]). Suppose that \( E' \) is not homotopically atroidal. Then, by the torus theorem (see [6] p. 156), either \( E' \) is a special Seifert fibered space or there is an essential embedding of \( T^2 \) in \( E' \). Since \( E \) is hyperbolic, \( E' \) cannot be a Seifert fibered space. So there is an essential torus \( T \) in \( E' \). Then the lift \( \hat{T} \) of \( T \) in \( E \) is an incompressible torus in \( E \). Since \( E \) is hyperbolic, \( \hat{T} \) is boundary parallel, that is, there is a submanifold \( Q \) of \( E \), such that \( Q \cong T^2 \times I \) and \( \partial Q = \partial E' \cup \hat{T} \). \( Q \) is \( f \)-invariant, and \( Q/f \) forms a submanifold of \( E' \) which is homeomorphic to \( T^2 \times I \) with \( \partial(Q/f) = \partial E' \cup T \); this is a contradiction. Hence \( E' \) is homotopically atroidal. Thus, by Thurston [14], \( \hat{E}' \) admits a hyperbolic structure, and therefore, \( \hat{E} \) admits a hyperbolic structure with respect to which \( f \) is an isometry.

Lemma 3. Suppose that \( (S^3, K) \) admits an action of \( Z_2 + Z_2 \cong \langle f \mid f^2 = 1 \rangle + \langle \tau \mid \tau^2 = 1 \rangle \), such that

1. \( f \) is an orientation-preserving free involution,
2. \( \tau \) reverses the orientation of \( S^3 \).

Then \( K \) is a trivial knot or a composite knot.

Proof. By Livesay [8], \( S^3/f \) is homeomorphic to the 3-dimensional projective space \( P^1 \). Since \( f \) and \( \tau \) are commutative, \( \tau \) induces an orientation reversing involution \( \hat{\tau} \) on \( P^3 \). Then, by Kwon [7], \( \text{Fix}(\hat{\tau}) \) is a disjoint union of \( P^2 \) and \( P^0 \). Let \( x \) be a point of \( P^2 \subset \text{Fix}(\hat{\tau}) \) and let \( \hat{x} \) be a lift of \( x \) in \( S^3 \). Let \( \hat{\tau}' : S^3 \to S^3 \) be the lift of \( \hat{\tau} \) such that \( \hat{\tau}'(\hat{x}) = \hat{x} \). Then \( \text{Fix}(\hat{\tau}') \) contains the inverse image of \( P^3 \), which is homeomorphic to a 2-sphere. Thus \( \hat{\tau}' \) is a reflection along a 2-sphere. Since \( \hat{\tau}' \) is equal to \( \hat{\tau} \) or \( f\hat{\tau}, \hat{\tau}' \) preserves the knot \( K \). Hence \( K \) must be a trivial knot or a composite knot.

§ 2. Proof of Theorem

Let \( K \) be a hyperbolic knot. Then the knot group \( G = \pi_1(E) \) is identified with a discrete subgroup of Isom \( H^3 \), the isometry group of the 3-dimensional hyperbolic space \( H^3 \), and \( \hat{E} \) is identified with \( H^3/G \). We use the upper-half space model \( H^3 = \mathbb{C} \times (0, +\infty) \), and identify Isom \( H^3 \)
with \( P\Gamma L(C) \), the group of all conformal or anti-conformal mappings of the Riemann sphere \( C \cup \{\infty\} \), which is identified with the sphere at infinity of \( H^3 \). Then the orientation-preserving isometry group \( \text{Isom}^+ H^3 \) is identified with \( PSL(C) \), the group of all Möbius transformations. Let \( A \) be the normalizer of \( G \) in \( P\Gamma L(C) \). Then, by Mostow's rigidity theorem (cf. [14]), the automorphism group \( \text{Aut}(G) \) of \( G \) is identified with \( A \), and \( \text{Isom} \hat{E} \cong \text{Out}(G) \) is identified with \( A/G \). Here, an element \( \alpha \in A \) represents the element of \( \text{Aut}(G) \) which sends \( x (\in G) \) to \( \alpha x \alpha^{-1} \). Let \( P \) be the peripheral subgroup of \( G \) generated by a longitude \( l \) and a meridian \( m \). Since \( P \cong \mathbb{Z} + \mathbb{Z} \), we may assume that \( l \) and \( m \) are identified with the Möbius transformations \( l(z) = z + \lambda \) and \( m(z) = z + 1 \) respectively, where \( \lambda \) is a complex number with \( \text{Im}(\lambda) \neq 0 \). Then, as isometries of \( H^3 \), we have \( l(z, t) = (z + \lambda, t) \) and \( m(z, t) = (z + 1, t) \), and an end of \( \hat{E} \) is obtained from \( C \times [t_0, +\infty) \) by identifying each set \( (z + Z\lambda + Z1, t) \) with a point, where \( t_0 \) is a sufficiently large number. Let \( A_\omega \) be the subgroup of \( A (= \text{Aut}(G)) \) consisting of those elements which preserve \( P \). Noting that any automorphism of \( G \) preserves the subgroups \( P \) and \( \langle l \rangle \) up to a conjugation, Riley observed the following (see Section 1 of [11]).

**Lemma 4.**
1. \( \text{Isom} \hat{E} \cong A_\omega/P \).
2. Any element \( \psi \) of \( A_\omega \) is of one of the following types.
   1. \( \psi(z) = z + c \ (c \in C) \),
   2. \( \psi(z) = -z + c \ (c \in C) \),
   3. \( \psi(z) = \epsilon z + c \ (|\epsilon| = 1, c \in C) \).
3. \( K \) is amphicheiral, iff there is an element of \( A_\omega \) which is of type (ii) with \( \epsilon = \pm 1 \), and \( \lambda \) is a purely imaginary number.

**Remark 5.** Let \( A_\omega^* \) be the subgroup of \( A_\omega \) which consists of type (i) elements. Then \( A_\omega^* \) is a normal subgroup of \( A_\omega \); in particular, if \( \psi(z) = z + c \) and \( \xi(z) = \epsilon z + c' \ (\epsilon = \pm 1) \), then \( \xi \psi \xi^{-1}(z) = z + \epsilon c \).

Put \( \text{Isom}^* \hat{E} = A_\omega^*/P \). Then, by Smith conjecture [9], we have the following (cf. [10] p. 124, [12] Lemma 3.3).

**Lemma 6.** \( \text{Isom}^* \hat{E} \) is a normal subgroup of \( \text{Isom} \hat{E} \) (of index at most 4), and is isomorphic to a finite cyclic group.

The proof of the Theorem is divided into two assertions.

**Assertion I.** The Theorem is true for free period \( n \geq 3 \).

**Proof.** Suppose that \( K \) is hyperbolic, amphicheiral, and has free period \( n \geq 3 \). By Lemma 2, there is an isometry \( f \) of \( \hat{E} \) which realizes the free period \( n \). Let \( \psi \) be an element of \( A_\omega \) representing \( f \) (cf. Lemma 4).
Since \( f \) preserves a longitude and a meridian homologically, \( \psi \) is of type (i); so \( \psi(z) = z + c \) for some \( c \in \mathbb{C} \). Since \( f \) has period \( n \), \( c = (p\lambda + qI)/n \) for some integers \( p \) and \( q \).

**Lemma 7.** The greatest common divisors \((p, n)\) and \((q, n)\) are equal to 1.

**Proof.** Put \( r = n/(p, n) \). Then

\[
\psi^r(z) = z + (p\lambda + qI)/(p, n)
= l^r(p, n)(z) + qI/(p, n).
\]

Thus the isometry \( f^r \) has an invariant meridian curve. (Recall the structure of an end of \( E \).) By Smith conjecture \([9]\), we have \( f^r = \text{id} \) and therefore \((p, n) = 1\). Put \( s = n/(q, n) \). Then

\[
\psi^s(z) = z + (p\lambda + qI)/(q, n)
= m^s/(q, n)(z) + p\lambda/(q, n).
\]

Thus the isometry \( f^s \) has an invariant longitude curve. So, by Lemma 1, we have \( f^s = \text{id} \), and therefore \((q, n) = 1\).

Since \( K \) is amphicheiral, \( \lambda \) is a purely imaginary number, and \( \hat{E} \) admits an orientation-reversing isometry \( \tau \), which is represented by an element \( \xi \) of \( A_\infty \) such that \( \xi(z) = \varepsilon z + b \) (\( \varepsilon = \pm 1 \), \( b \in \mathbb{C} \)) (see Lemma 4). By remark 5,

\[
\xi \psi \xi^{-1}(z) = z + \varepsilon(p\lambda + qI)/n
= z + \varepsilon(-p\lambda + qI)/n.
\]

By Lemma 6, there is an integer \( r \) (\( 0 \leq r \leq n - 1 \)) such that \( \tau f \tau^{-1} = f^r \), that is, \( \xi \psi \xi^{-1} \equiv \psi^r \text{ mod } P \). Hence we have

\[
e(-p\lambda + qI)/n \equiv r(p\lambda + qI)/n \text{ mod } \{\lambda, 1\}.
\]

This is equivalent to

\[
\begin{cases}
-ep \equiv rp \mod n \\
eq \equiv rq \mod n.
\end{cases}
\]

Since \((p, n) = (q, n) = 1\) by Lemma 7, we have

\[-\varepsilon \equiv r \equiv \varepsilon \mod n.
\]

This is a contradiction, since \( n \geq 3 \). Thus Assertion I is proved.
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Assertion II. *The Theorem is true for free period 2.*

Proof. Assume that $K$ is hyperbolic, amphicheiral, and has free period 2. Then $\text{Isom}^*\hat{E}$ is a cyclic group of order $2n$ ($n \in \mathbb{N}$), and the free period 2 is realized by the isometry $f = f_0^\varepsilon$, where $f_0$ is a generator of $\text{Isom}^*\hat{E}$. Let $\psi_0$ be an element of $A_\infty$ representing $f_0$. Then by an argument similar to the proof of Lemma 7, we can see that $\psi_0(z) = z + (p\lambda + q1)/2n$, where $p$ is an integer such that $(p, 2n) = 1$ and $q$ is an odd integer. Let $\xi$ be an element of $A_\infty$ representing an orientation-reversing isometry $\tau$ of $\hat{E}$. Then $\xi(z) = \varepsilon z + b$ ($\varepsilon = \pm 1$, $b \in \mathbb{C}$). Note that $\xi^2(z) = z + (\varepsilon b + b)$.

Case 1. $\varepsilon = +1$. Then $\xi^2(z) = z + 2\text{Re}(b)$. Thus $\tau^2$ has an invariant meridian curve, and therefore $\tau^2 = \text{id}$ by Smith conjecture. Since $f$ is the order 2 element of the cyclic normal subgroup $\text{Isom}^*\hat{E} \cong \mathbb{Z}_{2n}$, we have $\tau^2 f \tau^{-1} = f$. So $f$ and $\tau$ generate a $\mathbb{Z}_2 + \mathbb{Z}_2$ action on $(S^3, K)$ which satisfies the condition of Lemma 3. This is a contradiction, since a hyperbolic knot is non-trivial and prime.

Case 2. $\varepsilon = -1$. Then $\tau^2(z) = z + 2\text{Im}(b)i$. By an argument similar to the final step of the proof of Assertion I, we have $\tau^2 f \tau^{-1} = f_0$. Let $u$ be an integer such that $\tau^2 = f_0^u \in \text{Isom}^*\hat{E}$.

Subcase 1. $u$ is even. Put $\tau' = \tau f_0^{v}$, where $v = u/2$. Then $(\tau')^2 = \text{id}$. So $\tau'$ and $f$ generate a $\mathbb{Z}_2 + \mathbb{Z}_2$ action on $(S^3, K)$ satisfying the condition of Lemma 3; a contradiction.

Subcase 2. $u$ is odd. Note that $\xi^2(z) = \psi_0^2(z) = z + (up\lambda + uq1)/2n \text{ mod } \{\lambda, 1\}$. Since $q$ is odd, $uq/2n \equiv 0 \text{ mod } 1$, and therefore $(up\lambda + uq1)/2n \equiv a$ purely imaginary number mod $\{\lambda, 1\}$.

This contradicts the fact that $\xi^2(z) = z + 2\text{Im}(b)i$. This completes the proof of the Theorem.

§ 3. Further discussion

The Theorem does not hold for composite knots. In fact, the connected some of $n$-copies of an amphicheiral knot is amphicheiral, but has free period $n$. However, as shown in [13], the free periods of a composite knot are completely determined by the free periods of its prime factors,
and the Theorem holds for prime knots except free period 2; that is, any amphicheiral prime knot does not have free periods greater than 2. It remains open whether there is an amphicheiral prime knot which has free period 2.

I also calculated the symmetry groups of the "small" Montesinos knots by using the results of Thurston [15]. In particular, it follows that $8_{6}$ and $8_{10}$ have no free periods.* This completes the enumeration of the free periods of the prime knots with 10 crossings or less.

* Boileau informed me that he proved the non-free-periodicity of the small Montesinos knots without using [15].

References


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