COMPUTATIONAL DATA OF THE PAPER
“THE AUTOMORPHISM GROUP OF A SUPERSINGULAR K3
SURFACE WITH ARTIN VARIANT 1
IN CHARACTERISTIC 3”

ICHIRO SHIMADA

We present the computational data that are used for the study in the paper
[KS] Shigeyuki Kondo and Ichiro Shimada. The automorphism group of a su-
persingular K3 surface with Artin variant 1 in characteristic 3.

This paper [KS] is available from
http://www.math.sci.hiroshima-u.ac.jp/~shimada/preprints.html

The folder containing this document includes the following files:
• compdataAutFQChar3.txt (file size: 1.1 MB),
• FQprojaut.txt (file size: 636 MB),
• FQprojautS.txt (file size: 870 MB).

The two big files FQprojaut.txt and FQprojautS.txt are compressed to
• FQprojautfolder.tar.gz (file size: 274 MB).

1. Summary of the paper [KS]

Let $X \subset \mathbb{P}^3$ be the Fermat quartic surface in characteristic 3, that is

$$X = \{w^4 + x^4 + y^4 + 1 = 0\},$$

where $w, x, y$ are the affine coordinates of $\mathbb{P}^3$. We denote by

$$X \xrightarrow{\psi_i} Y_i \xrightarrow{\pi_i} \mathbb{P}^2$$

the Stein factorization of the morphism $\phi_i : X \rightarrow \mathbb{P}^2$ defined in Proposition 1.1 of
the paper [KS]. The involution $g_i : X \rightarrow X$ is obtained as the deck-transformation
of $\pi_i : Y_i \rightarrow \mathbb{P}^2$. The birational morphism $\psi_i : X \rightarrow Y_i$ is given by the rational map

$$(w, x, y) \mapsto [u : x_0 : x_1 : x_2] = [G_i : F_{i0} : F_{i1} : F_{i2}]$$

to the weighted projective space $\mathbb{P}(3, 1, 1, 1)$, where $G$ and $H_{ij}$ are polynomials in
$w, x, y$ with coefficients in $F_9$. The normal K3 surface $Y_i$ is defined in $\mathbb{P}(3, 1, 1, 1)$
by

$$u^2 + f_i(x_0, x_1, x_2) = 0,$$

2010 Mathematics Subject Classification. 14J28, 14G17, 11H06.
where \( f_i \) is the homogeneous polynomial of degree 6 given in Proposition 1.2. The involution \( g_i : X \to X \) is given by the rational map

\[
(w, x, y) \mapsto [H_{i0} : H_{i1} : H_{i2} : H_{i3}]
\]
to \( P^3 \), where \( H_{ij} \) are polynomials in \( w, x, y \) with coefficients in \( \mathbb{F}_9 \).

The Fermat quartic surface \( X \) contains 112 lines \( \ell_v \). The Néron-Severi lattice \( S \) of \( X \) is spanned by the classes of these lines. In particular, the classes of the lines in (3.1) of the paper [KS] form a basis of \( S \). We use this basis or its dual to indicate elements of \( S \otimes \mathbb{Q} \). Let \( h_0 \in S \) denote the class of the hyperplane section of \( X \). Let \( T \) be a negative-definite root lattice of type \( 2A_2 \). The positive cone \( \mathcal{P}_S \) of \( S \otimes \mathbb{R} \) containing \( h_0 \) is decomposed into \( \mathcal{R}_S \)-chambers by the embedding \( S \hookrightarrow L \) and the Weyl vector \( w_0 \in L \), where \( L \) is an even unimodular overlattice of \( S \otimes T \) obtained by adding vectors \( a_1, a_2 \in S^\vee \otimes T^\vee \) to \( S \otimes T \). The \( \mathcal{R}_S \)-chamber \( D_{S0} \subset \mathcal{P}_S \) containing \( h_0 \) is bounded by \( 112 + 648 + 5184 \) walls. Each wall is defined as the hyperplane \((r_S)^\perp \subset S \otimes \mathbb{R}\) perpendicular to \( r_S \), where \( r \in L \) is a Leech root with respect to \( w_0 \) such that its projection \( r_S \) to \( S^\vee \) satisfies \((r_S, r_S)_S < 0\). The projective automorphism group \( \text{Aut}(X, h_0) = \text{PGU}(4, \mathbb{F}_9) \) of order 13,063,680 acts on \( D_{S0} \) and hence on the set of the vectors

\[
\mathcal{W}(D_{S0}) := \{ r_S \mid (r_S)^\perp \text{ bounds } D_{S0} \}.
\]

This action decomposes \( \mathcal{W}(D_{S0}) \) into three orbits \( \mathcal{W}_{112}, \mathcal{W}_{648}, \mathcal{W}_{5184} \). The set \( \mathcal{W}_{112} \) coincides with the set of the classes of the lines \( \ell_v \) on \( X \). The vector \( b_1 \) is an element of \( \mathcal{W}_{648} \), and the vector \( b_2 \) is an element of \( \mathcal{W}_{5184} \).

For \( i = 1 \) and 2, the morphism \( \phi_i : X \to \mathbb{P}^2 \) is given by the polarization \( m_i \in S \) of degree 2, which is located on the wall \((b_i)^\perp_S \subset D_{S0}\). Using the expression (5.2), (5.3) of \( m_i \), we can regard \( F_{i0}, F_{i1}, F_{i2} \) as global sections of the line bundle corresponding to \( m_i \).

The morphism \( g_i : X \to X \) is given by the polarization \( h_i \in S \) of degree 4. Using the expression (7.1), (7.2) of \( h_i \), we can regard \( H_{i0}, H_{i1}, H_{i2}, H_{i3} \) as global sections of the line bundle corresponding to \( h_i \).

The action \( g_{i*} \) of \( g_i \) on \( S \) is given by \( v \mapsto vA_i \), so that we have \( h_i = h_0A_i \). The key point of the proof of the main result

\[
\text{Aut}(X) = \langle \text{Aut}(X, h_0), g_1, g_2 \rangle
\]
is that

\[
h_0 A_1 = h_0 + 3b, \quad \text{and} \quad h_0 A_2 = h_0 + 9b_2
\]
holds. The matrix \( A_i \) is calculated by the following two methods, which are independent of each other.
(1) The eigenspace of $A_i$ in $S \otimes \mathbb{Q}$ with eigenvalue 1 is spanned by $m_i$ and the
vectors $\gamma + \gamma'$, where $\{\gamma, \gamma'\}$ is a pair of the classes of $(-2)$-curves contracted
by $\phi_i$ that are interchanged by $g_i$.

(2) The image of a line $\ell_\nu$ by $g_i$ is calculated by the parametric representation
$\rho_\nu$ of $\ell_\nu$ and the polynomials $H_{i0}, H_{i1}, H_{i2}, H_{i3}$, provided that $\ell_\nu$ is not
contained in the common zero of $H_{i0}, H_{i1}, H_{i2}, H_{i3}$. Hence the images $[\ell_\nu]^g_i$
of classes $[\ell_\nu]$ that span $S \otimes \mathbb{Q}$ are calculated.

2. File compdataAutFQChar3.txt

In the file “compdataAutFQChar3.txt”, the following data are given.

- The elements of the finite field $\mathbb{F}_9$ are written as
  
  $\mathbb{F}_9 := [0, \sqrt{2}, 2 \cdot \sqrt{2}, 1, 1 + \sqrt{2}, 1 + 2 \cdot \sqrt{2}, 2, 2 + \sqrt{2}, 2 + 2 \cdot \sqrt{2}]$.

- The data $FQpsi[i]$ is the list of 4 polynomials $G_i, F_{i0}, F_{i1}, F_{i2}$ such that the rational map
  
  $(w, x, y) \mapsto [u : x_0 : x_1 : x_2] = [G_i : F_{i0} : F_{i1} : F_{i2}]$

  induces $\psi_i$. Thus the polynomials in Table 1.1 are the 2nd, 3rd and the
  4th polynomials of $FQpsi[i]$, while the first polynomial of $FQpsi[i]$ is the
  polynomial $G_i$ in the proof of Proposition 1.2 in Section 5.

- The polynomial $FQdefeqB[i]$ is the defining equation $f_i$ (with homogeneous coordinates $[X : Y : Z]$) of the branch curve $B_i \subset \mathbb{P}^2$ of $\pi_i : Y_i \to \mathbb{P}^2$.

- The data $FQmorphg[i]$ is the list of 4 polynomials $H_{i0}, H_{i1}, H_{i2}, H_{i3}$ such that the rational map
  
  $(w, x, y) \mapsto [H_{i0} : H_{i1} : H_{i2} : H_{i3}]$

  from $X$ to $\mathbb{P}^3$ induces the involution $g_i$ of $X$.

- The data $FQlines$ is the list of affine defining ideals of the lines on $X$. The $\nu$th entry of $FQlines$ is the defining ideal of $\ell_\nu$. Each member of $FQlines$ is a set of two linear (inhomogeneous) polynomials of $w, x, y$.

- The data $FQparalines$ is the list of parametric representations
  
  $\rho_\nu : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$

  of lines $\ell_\nu$. The $\nu$th entry of $FQparalines$ is a list of 4 linear homogeneous
  polynomials of $U$ and $V$ that gives the parametric representation $\rho_\nu$, where
  $[U : V]$ is homogeneous coordinates of $\mathbb{P}^1$.

- In the list $FQF9pts$, the 280 rational points of $X$ over $\mathbb{F}_9$ are given. All
  points are written in terms of the homogeneous coordinates of $\mathbb{P}^3$ such that
  the first non-zero entry is 1.
• The list $FQ\text{incidenceptline}$ is the list of $[a, b]$ such that the $a$th rational point in $FQ\text{9pts}$ is contained in the $b$th line in $FQ\text{lines}$.
• The data $FQ\text{NSbasis}$ is the list of indices of the lines in (3.1), whose classes form a basis of the Néron-Severi lattice $S$ of $X$.

By the (non-dual) basis, we mean the basis of $S$ given by $FQ\text{NSbasis}$. By the dual basis, we mean the basis of $S^\vee$ dual to the (non-dual) basis.

• The matrix $FQ\text{GramS}$ is the Gram matrix of $S$ with respect to the (non-dual) basis.
• The data $FQ\text{lineclasses}$ is the list of classes $[\ell_\nu]$ of the lines on $X$. The $\nu$th member of $FQ\text{lineclasses}$ is $[\ell_\nu]$ written in terms of the (non-dual) basis.
• The vector $FQ\text{h0}$ is $h_0$ written in terms of the (non-dual) basis.
• The matrix $FQ\text{GramT}$ is the Gram matrix of $T$.
• The vectors $FQ\text{a1}$ and $FQ\text{a2}$ are the vectors $a_1$ and $a_2$ that are added to $S \oplus T$ so that
  $$L = (S \oplus T) + \langle a_1 \rangle + \langle a_2 \rangle$$
is even and unimodular. They are written in terms of the (non-dual) basis of $S \oplus T$.
• The vector $FQ\text{w0}$ is $w_0$ in terms of the (non-dual) basis of $S \oplus T$, while $FQ\text{w0dual}$ is the Weyl vector $w_0$ in terms of the dual basis of $S^\vee \oplus T^\vee$.
• The vector $FQ\text{w1dual}$ is the vector $w'_0$ in the proof of Proposition 4.2 written in terms of the dual basis.
• The data $FQ\text{lambdasdual}$ is a basis $\lambda_1, \ldots, \lambda_{24}$ of $U^\perp = \langle w_0, w'_0 \rangle^\perp$ in the proof of Proposition 4.2 written in terms of the dual basis.
• The vector $FQ\text{LRw0}$ is the list $LR(w_0, S)$. Each vector is written in terms of the dual basis of $S^\vee \oplus T^\vee$.
• The data $FQ\text{WW112}$, $FQ\text{WW648}$ and $FQ\text{WW5184}$ are $\tilde{W}_{112}$, $\tilde{W}_{648}$ and $\tilde{W}_{5184}$, respectively. Each vector is written in terms of the dual basis of $S^\vee$.
• The vector $FQ\text{b[i]}$ is the vector $b_i$ in Proposition 4.5 written in terms of the dual basis of $S^\vee$.
• The vector $FQ\text{m[i]}$ is the polarization $m_i$ of degree 2 in Proposition 5.1 written in terms of (non-dual) basis of $S$.
• The data $FQ\text{mm[i]}$ is the expressions (5.2) and (5.3) of $m_i$. The first entry of $FQ\text{mm[i]}$ is the coefficient of $h_0$ and the members $[k, a]$ of the second entry indicate the term $-a[\ell_k]$.
• The matrix $FQ\text{A[i]}$ is the matrix $A_i$ that represents the action of the involution $g_i$ on $S$ with respect to the (non-dual) basis of $S$.
• The data $FQ\text{hh[i]}$ is the expressions (7.1) and (7.2) of $h_i = h_0 A_i$. 
The data FQcontract[i] is the list of the lines contracted by $\phi_i : X \to \mathbb{P}^2$. A member

$$[[k_1, \ldots, k_r], [a_0, a_1, a_2]]$$

of FQcontract[i] indicates that the set of lines contracted to the singular point $[a_0 : a_1 : a_2]$ of $B_i$ is $\{\ell_{k_1}, \ldots, \ell_{k_r}\}$, and that $\ell_{k_1}, \ldots, \ell_{k_r}$ form an $A_r$-chain of $(-2)$-curves in this order.

The data FQlines[i] indicates the parametric representations $g_i \circ \rho_\nu : \mathbb{P}^1 \to \mathbb{P}^3$ of the image $\ell_{g_i \nu}$ of the line $\ell_\nu$ by the involution $g_i$ of $X$. If the line $\ell_\nu$ appears in the right hand side of (7.1) when $i = 1$ or (7.2) when $i = 2$, then the $\nu$th entry of FQlines[i] is not calculated.

The data FQlinepsis[i] indicates the parametric representations $\psi_i \circ \rho_\nu : \mathbb{P}^1 \to \mathbb{P}(3, 1, 1, 1)$ of the image $\ell_{\psi_i \nu}$ of the line $\ell_\nu$ by the morphism $\psi_i : X \to Y_i$. If the line $\ell_\nu$ appears in the right hand side of (5.2) when $i = 1$ or (5.3) when $i = 2$, then the $\nu$th entry of FQlinepsis[i] is not calculated.

The matrix FQAF is the matrix $A_F$ that represents the Frobenius action of $\mathbb{F}_9$ on $S$.

The list FQperm280[i] indicates the permutation on $X(\mathbb{F}_9)$ induced by the involution $g_i$. The $\nu$th point in FQF9pts is mapped by $g_i$ to the $\nu'$th point in FQF9pts, where $\nu'$ is the $\nu$th entry of the list FQperm280[i].

The list FQF9ptpsis[i] indicates the images of the $\mathbb{F}_9$-rational points of $X$ by $\psi_i : X \to Y_i$. The $\nu$th point in FQF9pts is mapped by $\psi_i$ to $\nu$th point in FQF9ptpsis[i], which is written in terms of the weighted homogeneous coordinates of $\mathbb{P}(3, 1, 1, 1)$.

### 3. File FQprojaut.txt

In the file “FQprojaut.txt”, the list FQprojaut of the elements of the projective automorphism group $\text{Aut}(X, h_0) = \text{PGU}_4(\mathbb{F}_9)$ of order 13,063,680 is presented in the following way. The list FQprojaut consists of 13,063,680 integer vectors of length 16. An element

$$a := a + b \sqrt{2} \in \mathbb{F}_9$$

with $0 \leq a < 3$ and $0 \leq b < 3$ is expressed by a single integer

$$\tilde{a} := a + 3b.$$
If an element $\tau \in \text{PGU}_4(\mathbb{F}_9)$ is represented by a matrix
\[
\begin{bmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\
\alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\
\alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44}
\end{bmatrix}
\in \text{GU}_4(\mathbb{F}_9),
\]
then $\tau$ is expressed by
\[
[\tilde{\alpha}_{11}, \tilde{\alpha}_{12}, \tilde{\alpha}_{13}, \tilde{\alpha}_{14}, \tilde{\alpha}_{21}, \tilde{\alpha}_{22}, \tilde{\alpha}_{23}, \tilde{\alpha}_{24}, \tilde{\alpha}_{31}, \tilde{\alpha}_{32}, \tilde{\alpha}_{33}, \tilde{\alpha}_{34}, \tilde{\alpha}_{41}, \tilde{\alpha}_{42}, \tilde{\alpha}_{43}, \tilde{\alpha}_{44}].
\]

4. File FQprojautS.txt

In the file “FQprojautS.txt”, the representation
\[
\tau \mapsto T_\tau
\]
of Aut$(X, h_0) = \text{PGU}_4(\mathbb{F}_9)$ on the lattice $S$ is given in the list FQprojautS of 13,063,680 vectors of length 22. Suppose that $\tau \in \text{PGU}_4(\mathbb{F}_9)$ is given as the $i$th element of FQprojaut. Then the $i$th element of FQprojautS is the list of 22 indices $[k_1, \ldots, k_{22}]$ such that
\[
\ell^\tau_{[\nu]} = \ell_{k_\nu},
\]
where $\ell_{[\nu]}$ is the $\nu$th line in (3.1). Therefore $T_\tau \in O^+(S)$ is the matrix whose $\nu$th row vector is the $k_\nu$th vector in the list FQlineclasses.