COMPUTATIONAL DATA OF THE PAPER "THE AUTOMORPHISM GROUP OF A SUPERSINGULAR $K3$ SURFACE WITH ARTIN VARIANT 1 IN CHARACTERISTIC 3"

ICHIRO SHIMADA

We present the computational data that are used for the study in the paper

[KS] Shigeyuki Kondo and Ichiro Shimada. The automorphism group of a supersingular $K3$ surface with Artin variant 1 in characteristic 3. preprint, 2012.

This paper [KS] is available from

`http://www.math.sci.hiroshima-u.ac.jp/~shimada/preprints.html`

The folder containing this document includes the following files:

- `compdataAutFQChar3.txt` (file size: 1.1 MB),
- `FQprojaut.txt` (file size: 636 MB),
- `FQprojautS.txt` (file size: 870 MB).

The two big files `FQprojaut.txt` and `FQprojautS.txt` are compressed to

- `FQprojautfolder.tar.gz` (file size: 274 MB).

1. Summary of the paper [KS]

Let $X \subset \mathbb{P}^3$ be the Fermat quartic surface in characteristic 3, that is

$$X = \{w^4 + x^4 + y^4 + 1 = 0\},$$

where w, x, y are the affine coordinates of $\mathbb{P}^3$. We denote by

$$X \xrightarrow{\psi_i} Y_i \xrightarrow{\pi_i} \mathbb{P}^2$$

the Stein factorization of the morphism $\phi_i : X \to \mathbb{P}^2$ defined in Proposition 1.1 of the paper [KS]. The involution $g_i : X \to X$ is obtained as the deck-transformation of $\pi_i : Y_i \to \mathbb{P}^2$. The birational morphism $\psi_i : X \to Y_i$ is given by the rational map

$$(w, x, y) \mapsto [u : x_0 : x_1 : x_2] = [G_i : F_{i0} : F_{i1} : F_{i2}]$$

to the weighted projective space $\mathbb{P}(3, 1, 1, 1)$, where G and H_{ij} are polynomials in w, x, y with coefficients in $\mathbb{F}_9$. The normal $K3$ surface Y_i is defined in $\mathbb{P}(3, 1, 1, 1)$ by

$$u^2 + f_i(x_0, x_1, x_2) = 0,$$

2010 *Mathematics Subject Classification.* 14J28, 14G17, 11H06.

where f_i is the homogeneous polynomial of degree 6 given in Proposition 1.2. The involution $g_i : X \to X$ is given by the rational map

$$(w, x, y) \mapsto [H_{i0} : H_{i1} : H_{i2} : H_{i3}]$$

to P^3, where H_{ij} are polynomials in w, x, y with coefficients in $\mathbb{F}_9$.

The Fermat quartic surface X contains 112 lines ℓ_ν. The Néron-Severi lattice S of X is spanned by the classes of these lines. In particular, the classes of the lines in (3.1) of the paper [KS] form a basis of S. We use this basis or its dual to indicate elements of $S \otimes \mathbb{Q}$. Let $h_0 \in S$ denote the class of the hyperplane section of X. Let T be a negative-definite root lattice of type $2A_2$. The positive cone $\mathcal{P}_S$ of $S \otimes \mathbb{R}$ containing h_0 is decomposed into $\mathcal{R}_S$-chambers by the embedding $S \hookrightarrow L$ and the Weyl vector $w_0 \in L$, where L is an even unimodular overlattice of $S \oplus T$ obtained by adding vectors $a_1, a_2 \in S^\vee \oplus T^\vee$ to $S \oplus T$. The $\mathcal{R}_S$-chamber $D_{S0} \subset \mathcal{P}_S$ containing h_0 is bounded by $112 + 648 + 5184$ walls. Each wall is defined as the hyperplane $(r_S)^\perp \subset S \otimes \mathbb{R}$ perpendicular to r_S, where $r \in L$ is a Leech root with respect to w_0 such that its projection r_S to $S^\vee$ satisfies $(r_S, r_S)_S < 0$. The projective automorphism group $\mathrm{Aut}(X, h_0) = \mathrm{PGU}(4, \mathbb{F}_9)$ of order $13,063,680$ acts on D_{S0} and hence on the set of the vectors

$$\widetilde{\mathcal{W}}(D_{S0}) := \{ \ r_S \ \mid \ (r_S)^\perp \text{ bounds } D_{S0} \ \}.$$

This action decomposes $\widetilde{\mathcal{W}}(D_{S0})$ into three orbits $\widetilde{\mathcal{W}}_{112}$, $\widetilde{\mathcal{W}}_{648}$, $\widetilde{\mathcal{W}}_{5184}$. The set $\widetilde{\mathcal{W}}_{112}$ coincides with the set of the classes of the lines ℓ_ν on X. The vector b_1 is an element of $\widetilde{\mathcal{W}}_{648}$, and the vector b_2 is an element of $\widetilde{\mathcal{W}}_{5184}$.

For $i = 1$ and 2, the morphism $\phi_i : X \to \mathbb{P}^2$ is given by the polarization $m_i \in S$ of degree 2, which is located on the wall $(b_i)^\perp_S$ of D_{S0}. Using the expression (5.2), (5.3) of m_i, we can regard F_{i0}, F_{i1}, F_{i2} as global sections of the line bundle corresponding to m_i.

The morphism $g_i : X \to X$ is given by the polarization $h_i \in S$ of degree 4. Using the expression (7.1), (7.2) of h_i, we can regard $H_{i0}, H_{i1}, H_{i2}, H_{i3}$ as global sections of the line bundle corresponding to h_i.

The action g_{i*} of g_i on S is given by $v \mapsto vA_i$, so that we have $h_i = h_0 A_i$. The key point of the proof of the main result

$$\mathrm{Aut}(X) = \langle \mathrm{Aut}(X, h_0), g_1, g_2 \rangle$$

is that

$$h_0\, A_1 = h_0 + 3\, b_, \quad \text{and} \quad h_0 A_2 = h_0 + 9\, b_2$$

holds. The matrix A_i is calculated by the following two methods, which are independent of each other:

(1) The eigenspace of A_i in $S \otimes \mathbb{Q}$ with eigenvalue 1 is spanned by m_i and the vectors $\gamma+\gamma'$, where $\{\gamma, \gamma'\}$ is a pair of the classes of (-2)-curves contracted by ϕ_i that are interchanged by g_i.
(2) The image of a line ℓ_ν by g_i is calculated by the parametric representation ρ_ν of ℓ_ν and the polynomials $H_{i0}, H_{i1}, H_{i2}, H_{i3}$, provided that ℓ_ν is not contained in the common zero of $H_{i0}, H_{i1}, H_{i2}, H_{i3}$. Hence the images $[\ell_\nu]^{g_i}$ of classes $[\ell_\nu]$ that span $S \otimes \mathbb{Q}$ are calculated.

2. File `compdataAutFQChar3.txt`

In the file "`compdataAutFQChar3.txt`", the following data are given.

- The elements of the finite field $\mathbb{F}_9$ are written as

$$\texttt{F9} := [0, \texttt{sqrt}(2), 2 * \texttt{sqrt}(2), 1, 1 + \texttt{sqrt}(2),$$
$$1 + 2 * \texttt{sqrt}(2), 2, 2 + \texttt{sqrt}(2), 2 + 2 * \texttt{sqrt}(2)].$$

- The data `FQpsi[i]` is the list of 4 polynomials $G_i, F_{i0}, F_{i1}, F_{i2}$ such that the rational map

$$(w, x, y) \mapsto [u : x_0 : x_1 : x_2] = [G_i : F_{i0} : F_{i1} F_{i2}]$$

induces ψ_i. Thus the polynomials in Table 1.1 are the 2nd, 3rd and the 4th polynomials of `FQpsi[i]`, while the first polynomial of `FQpsi[i]` is the polynomial G_i in the proof of Proposition 1.2 in Section 5.
- The polynomial `FQdefeqB[i]` is the defining equation f_i (with homogeneous coordinates $[\texttt{X} : \texttt{Y} : \texttt{Z}]$) of the branch curve $B_i \subset \mathbb{P}^2$ of $\pi_i : Y_i \to \mathbb{P}^2$.
- The data `FQmorphg[i]` is the list of 4 polynomials $H_{i0}, H_{i1}, H_{i2}, H_{i3}$ such that the rational map

$$(w, x, y) \mapsto [H_{i0} : H_{i1} : H_{i2} : H_{i3}]$$

from X to $\mathbb{P}^3$ induces the involution g_i of X.
- The data `FQlines` is the list of affine defining ideals of the lines on X. The νth entry of `FQlines` is the defining ideal of ℓ_ν. Each member of `FQlines` is a set of two linear (inhomogeneous) polynomials of w, x, y.
- The data `FQparalines` is the list of parametric representations

$$\rho_\nu : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$

of lines ℓ_ν. The νth entry of `FQparalines` is a list of 4 linear homogeneous polynomials of `U` and `V` that gives the parametric representation ρ_ν, where $[\texttt{U} : \texttt{V}]$ is homogeneous coordinates of $\mathbb{P}^1$.
- In the list `FQF9pts`, the 280 rational points of X over $\mathbb{F}_9$ are given. All points are written in terms of the homogeneous coordinates of $\mathbb{P}^3$ such that the first non-zero entry is 1.

- The list `FQincidenceptline` is the list of $[a, b]$ such that the ath rational point in `FQF9pts` is contained in the bth line in `FQlines`.
- The data `FQNSbasis` is the list of indices of the lines in (3.1), whose classes form a basis of the Néron-Severi lattice S of X.

By the (non-dual) basis, we mean the basis of S given by `FQNSbasis`. By the *dual* basis, we mean the basis of $S^\vee$ dual to the (non-dual) basis.

- The matrix `FQGramS` is the Gram matrix of S with respect to the (non-dual) basis.
- The data `FQlineclasses` is the list of classes $[\ell_\nu]$ of the lines on X. The νth member of `FQlineclasses` is $[\ell_\nu]$ written in terms of the (non-dual) basis.
- The vector `FQh0` is h_0 written in terms of the (non-dual) basis.
- The matrix `FQGramT` is the Gram matrix of T.
- The vectors `FQa1` and `FQa2` are the vectors a_1 and a_2 that are added to $S \oplus T$ so that

$$L = (S \oplus T) + \langle a_1 \rangle + \langle a_2 \rangle$$

  is even and unimodular. They are written in terms of the (non-dual) basis of $S \oplus T$.
- The vector `FQw0` is the Weyl vector w_0 in terms of the (non-dual) basis of $S \oplus T$, while `FQw0dual` is the Weyl vector w_0 in terms of the *dual* basis of $S^\vee \oplus T^\vee$.
- The vector `FQw1dual` is the vector w_0' in the proof of Proposition 4.2 written in terms of the *dual* basis.
- The data `FQlambdasdual` is a basis $\lambda_1, \ldots, \lambda_{24}$ of $U^\perp = \langle w_0, w_0' \rangle^\perp$ in the proof of Proposition 4.2 written in terms of the *dual* basis.
- The data `FQLRw0` is the list $LR(w_0, S)$. Each vector is written in terms of the *dual* basis of $S^\vee \oplus T^\vee$.
- The data `FQWW112`, `FQWW648` and `FQWW5184` are $\widetilde{W}_{112}$, $\widetilde{W}_{648}$ and $\widetilde{W}_{5184}$, respectively. Each vector is written in terms of the *dual* basis of $S^\vee$.
- The vector `FQb[i]` is the vector b_i in Proposition 4.5 written in terms of the *dual* basis of $S^\vee$.
- The vector `FQm[i]` is the polarization m_i of degree 2 in Proposition 5.1 written in terms of (non-dual) basis of S.
- The data `FQmm[i]` is the expressions (5.2) and (5.3) of m_i. The first entry of `FQmm[i]` is the coefficient of h_0 and the members $[k, a]$ of the second entry indicate the term $-a\,[\ell_k]$.
- The matrix `FQA[i]` is the matrix A_i that represents the action of the involution g_i on S with respect to the (non-dual) basis of S.
- The data `FQhh[i]` is the expressions (7.1) and (7.2) of $h_i = h_0 A_i$.

- The data FQcontract[i] is the list of the lines contracted by $\phi_i : X \to \mathbb{P}^2$. A member

$$[[k_1, \dots, k_r], [a_0, a_1, a_2]]$$

of FQcontrct[i] indicates that the set of lines contracted to the singular point $[a_0 : a_1 : a_2]$ of B_i is $\{\ell_{k_1}, \dots, \ell_{k_r}\}$, and that $\ell_{k_1}, \dots, \ell_{k_r}$ form an A_r-chain of (-2)-curves in this order.
- The data FQlinegs[i] indicates the parametric representations

$$g_i \circ \rho_\nu : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$

of the image $\ell_\nu^{g_i}$ of the line ℓ_ν by the involution g_i of X. If the line ℓ_ν appears in the right hand side of (7.1) when $i = 1$ or (7.2) when $i = 2$, then the νth entry of FQlinegs[i] is not_calculated.
- The data FQlinepsis[i] indicates the parametric representations

$$\psi_i \circ \rho_\nu : \mathbb{P}^1 \hookrightarrow \mathbb{P}(3, 1, 1, 1)$$

of the image $\ell_\nu^{\psi_i}$ of the line ℓ_ν by the morphism $\psi_i : X \to Y_i$. If the line ℓ_ν appears in the right hand side of (5.2) when $i = 1$ or (5.3) when $i = 2$, then the νth entry of FQlinepsis[i] is not_calculated.
- The matrix FQAF is the matrix A_F that represents the Frobenius action of $\mathbb{F}_9/\mathbb{F}_3$ on S.
- The list FQperm280[i] indicates the permutation on $X(\mathbb{F}_9)$ induced by the involution g_i. The νth point in FQF9pts is mapped by g_i to the ν'th point in FQF9pts, where ν' is the νth entry of the list FQperm280[i].
- The list FQF9ptpsis[i] indicates the images of the $\mathbb{F}_9$-rational points of X by $\psi_i : X \to Y_i$. The νth point in FQF9pts is mapped by ψ_i to νth point in FQF9ptpsis[i], which is written in terms of the weighted homogeneous coordinates of $\mathbb{P}(3, 1, 1, 1)$.

3. File FQprojaut.txt

In the file "FQprojaut.txt", the list FQprojaut of the elements of the projective automorphism group $\mathrm{Aut}(X, h_0) = \mathrm{PGU}_4(\mathbb{F}_9)$ of order $13,063,680$ is presented in the following way. The list FQprojaut consists of $13,063,680$ integer vectors of length 16. An element

$$\alpha := a + b\sqrt{2} \in \mathbb{F}_9$$

with $0 \leq a < 3$ and $0 \leq b < 3$ is expressed by a single integer

$$\tilde{\alpha} := a + 3\,b.$$

If an element $\tau \in \mathrm{PGU}_4(\mathbb{F}_9)$ is represented by a matrix

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix} \in \mathrm{GU}_4(\mathbb{F}_9),$$

then τ is expressed by

$$[\tilde{\alpha}_{11}, \tilde{\alpha}_{12}, \tilde{\alpha}_{13}, \tilde{\alpha}_{14}, \tilde{\alpha}_{21}, \tilde{\alpha}_{22}, \tilde{\alpha}_{23}, \tilde{\alpha}_{24}, \tilde{\alpha}_{31}, \tilde{\alpha}_{32}, \tilde{\alpha}_{33}, \tilde{\alpha}_{34}, \tilde{\alpha}_{41}, \tilde{\alpha}_{42}, \tilde{\alpha}_{43}, \tilde{\alpha}_{44}].$$

4. File `FQprojautS.txt`

In the file "`FQprojautS.txt`", the representation

$$\tau \mapsto T_\tau$$

of $\mathrm{Aut}(X, h_0) = \mathrm{PGU}_4(\mathbb{F}_9)$ on the lattice S is given in the list `FQprojautS` of $13,063,680$ vectors of length 22. Suppose that $\tau \in \mathrm{PGU}_4(\mathbb{F}_9)$ is given as the ith element of `FQprojaut`. Then the ith element of `FQprojautS` is the list of 22 indices $[k_1, \dots, k_{22}]$ such that

$$\ell^\tau_{[\nu]} = \ell_{k_\nu},$$

where $\ell_{[\nu]}$ is the νth line in (3.1). Therefore $T_\tau \in \mathrm{O}^+(S)$ is the matrix whose νth row vector is the k_νth vector in the list `FQlineclasses`.

Department of Mathematics, Graduate School of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526 JAPAN

E-mail address: `shimada@math.sci.hiroshima-u.ac.jp`