

# Construction of non-Galois triple coverings over $\mathbf{P}^2$ branched along quintic curves by pull-back construction

Tadasuke YASUMURA

Joint work with Hiro-o Tokunaga

Department of Mathematics and Information Sciences, Tokyo Metropolitan University

## Definition

Let  $X$  and  $Y$  be normal projective varieties. We denote the function fields of  $X$  and  $Y$  by  $\mathbf{C}(X)$  and  $\mathbf{C}(Y)$ , respectively. Assume that there exists a finite surjective morphism  $\pi : X \rightarrow Y$ . Note that  $\pi$  induces the field extension  $\mathbf{C}(X)/\mathbf{C}(Y)$ . Then

- $\pi : X \rightarrow Y$ : non-Galois triple covering  $\stackrel{\text{def}}{\iff} \mathbf{C}(X)/\mathbf{C}(Y)$ : non-Galois cubic extension.
- $\Delta_\pi := \{y \in Y \mid \sharp(\pi^{-1}(y)) < 3\}$  is called the branch locus of  $\pi$ .
- $p \in \Delta_\pi$ : totally ramified point  $\stackrel{\text{def}}{\iff} \sharp\pi^{-1}(p) = 1$ .
- $p \in \Delta_\pi$ : simply ramified point  $\stackrel{\text{def}}{\iff} \sharp\pi^{-1}(p) = 2$ .

## Fact

1. For all non-Galois triple covering  $\pi : X \rightarrow Y$ , there exists an algebraic element  $\xi$  over  $\mathbf{C}(Y)$  with minimal equation  $\xi^3 + 3\alpha\xi + 2\beta = 0$  ( $\alpha, \beta \in \mathbf{C}(Y)$ ) such that  $\mathbf{C}(X) = \mathbf{C}(Y)(\xi)$
2. (H. Tokunaga) Let  $\pi : \Sigma \rightarrow \mathbf{P}^2$  be a non-Galois triple covering with  $\deg(\Delta_\pi) = 5$ . Then  $\Delta_\pi$  has one of the following two forms:

I.  $\Delta_\pi = C_2 + C_3$  ( $C_2$ : a conic,  $C_3$ : a cubic).

$\pi$  is  $\begin{cases} \text{simply} \\ \text{totally} \end{cases}$  ramified along  $\begin{cases} C_2 \\ C_3 \end{cases}$ .

II.  $\Delta_\pi = Q + L$  ( $Q$ : a quartic,  $L$ : a line).

$\pi$  is  $\begin{cases} \text{simply} \\ \text{totally} \end{cases}$  ramified along  $\begin{cases} Q \\ L \end{cases}$ .

In terms of the singularities of  $Q$  and relative position between  $Q$  and  $L$ , there exist 18 types for Type II.

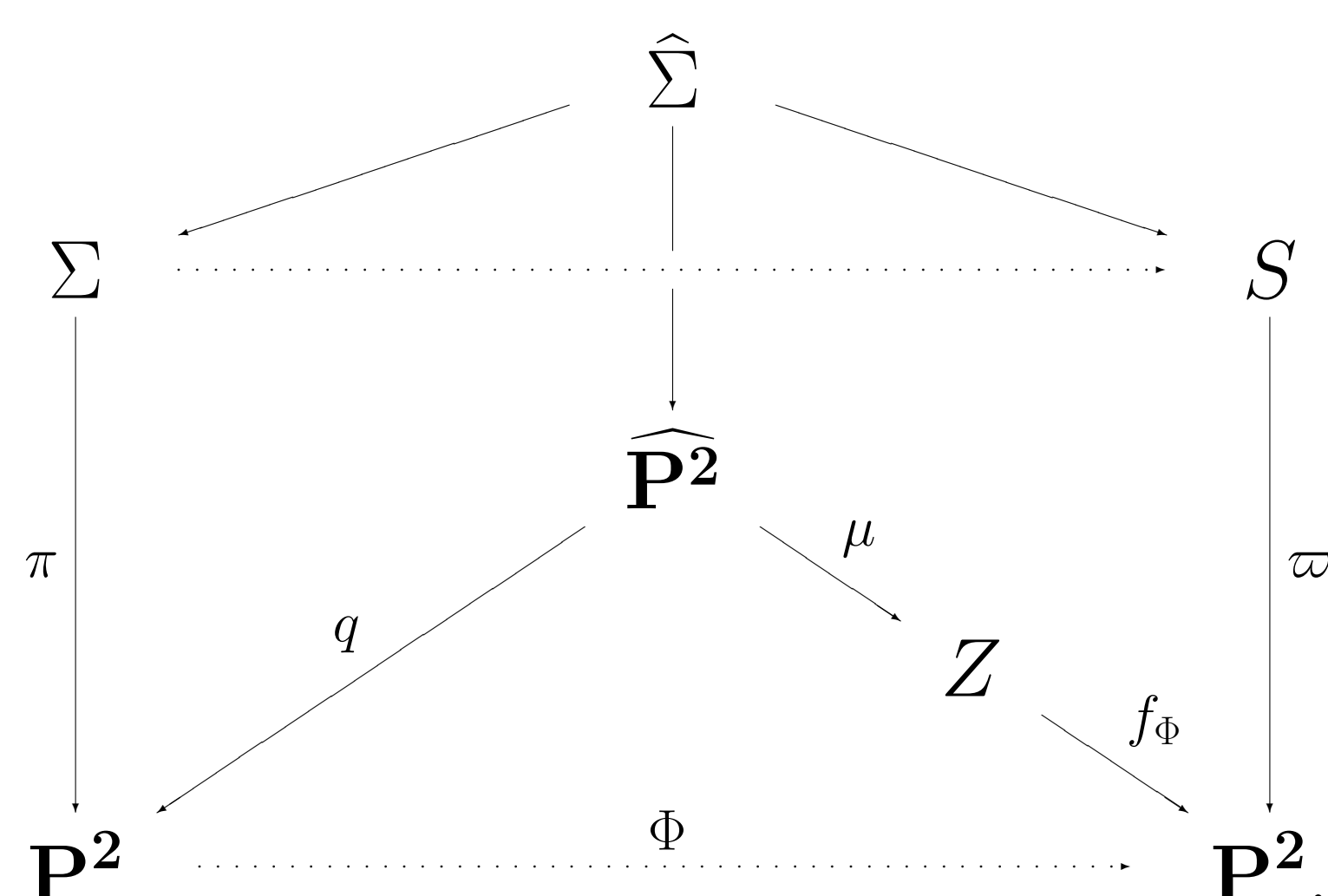
## Pull-back construction

Let  $\varpi : S \rightarrow \mathbf{P}^2$  be a non-Galois triple covering given by the cubic equation  $\zeta^3 + 3u\zeta + 2v$ , where  $(u, v)$  is inhomogeneous coordinate of  $\mathbf{P}^2$ . We consider the rational map

$$\Phi : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2 \\ (x, y) \mapsto (u, v) = (\alpha, \beta),$$

where  $(x, y)$  is inhomogeneous coordinate of  $\mathbf{P}^2$ , and  $\alpha, \beta \in \mathbf{C}(\mathbf{P}^2)$ .

Then, from Fact 1, we obtain a commutative diagram as follows:



Here,  $q$  is a resolution of indeterminacy of  $\Phi$ .  $\nu : Z \rightarrow \mathbf{P}^2$  is a stein factorization of the induced morphism by  $q$ . Then,  $\hat{\Sigma}$  is birationally equivalent to

$S \times_{\mathbf{P}^2} \widehat{\mathbf{P}^2}$  over  $\mathbf{P}^2$ . In other words,  $\Sigma$  is obtained as a ‘‘rational’’ pull-back of  $\varpi : S \rightarrow \mathbf{P}^2$ .

## Result

We apply ‘‘rational’’ pull-back construction to non-Galois triple coverings of Type II in Fact 2 and obtain the rational maps explicitly as follows:

$\Phi_i$	$Q$ 's singularities	$Q \cap L$	$\Phi_\pi^* u$	$\Phi_\pi^* v$	$f_\Phi$
$\Phi_1$	$2a_2$	(i)	$x$	$(y-1)(x-y)$	a double covering branched along an irreducible conic
$\Phi_2$	$2a_2 + a_1$		$4x$	$3x^2 - 6x - 1 - y^2$	
$\Phi_3$	$3a_2$		$-2x$	$6x^2 + 2x + 1 - y^2$	
$\Phi_4$	$a_5$		$x$	$(x-y)(y-2) - 1$	
$\Phi_5$	$e_6$		$x$	$y(x-y)$	
$\Phi_6$	$a_5 + a_1$		$x$	$x^2 - y^2 - x/4$	
$\Phi_7$	$2a_2$	(ii)	$x$	$y^2 - y$	a double covering branched along distinct two lines
$\Phi_8$	$2a_2 + a_1$		$x$	$y^2 - 3x - 4$	
$\Phi_9$	$a_5$		$x$	$y^2 - x$	
$\Phi_{10}$	$e_6$	(iii), $a_3$	$x$	$y - xy$	an isomorphism
$\Phi_{11}$	$a_3 + a_2$		$x - 1$	$1 - xy$	
$\Phi_{12}$	$a_3 + a_2 + a_1$	(iii), $a_6$	$x$	$1 - xy$	the image of $f_\Phi$ is a curve
$\Phi_{13}$	$a_6$	(v), $a_4$	$x$	$y - x^2$	
$\Phi_{14}$	$a_4 + a_2$	(iv), $2a_3$	$-1$	$xy$	the image of $f_\Phi$ is a curve
$\Phi_{15}$	$2a_3$		$-1$	$1 - xy$	
$\Phi_{16}$	$2a_3 + a_1$	(v), $a_7$	$-1$	$x - y^2$	the image of $f_\Phi$ is a curve
$\Phi_{17}$	$a_7$		$-1$	$x - y^2$	
$\Phi_{18}$	ordinary 4-ple point	(v), ordinary 4-ple point	$-1$	$y(y-3)$	

(i)  $L$  is a bitangent line to  $Q$  at two distinct smooth points.

(ii)  $L$  is a tangent line to  $Q$  at a smooth point with multiplicity 4.

(iii)  $L$  is tangent to  $Q$  at one smooth point

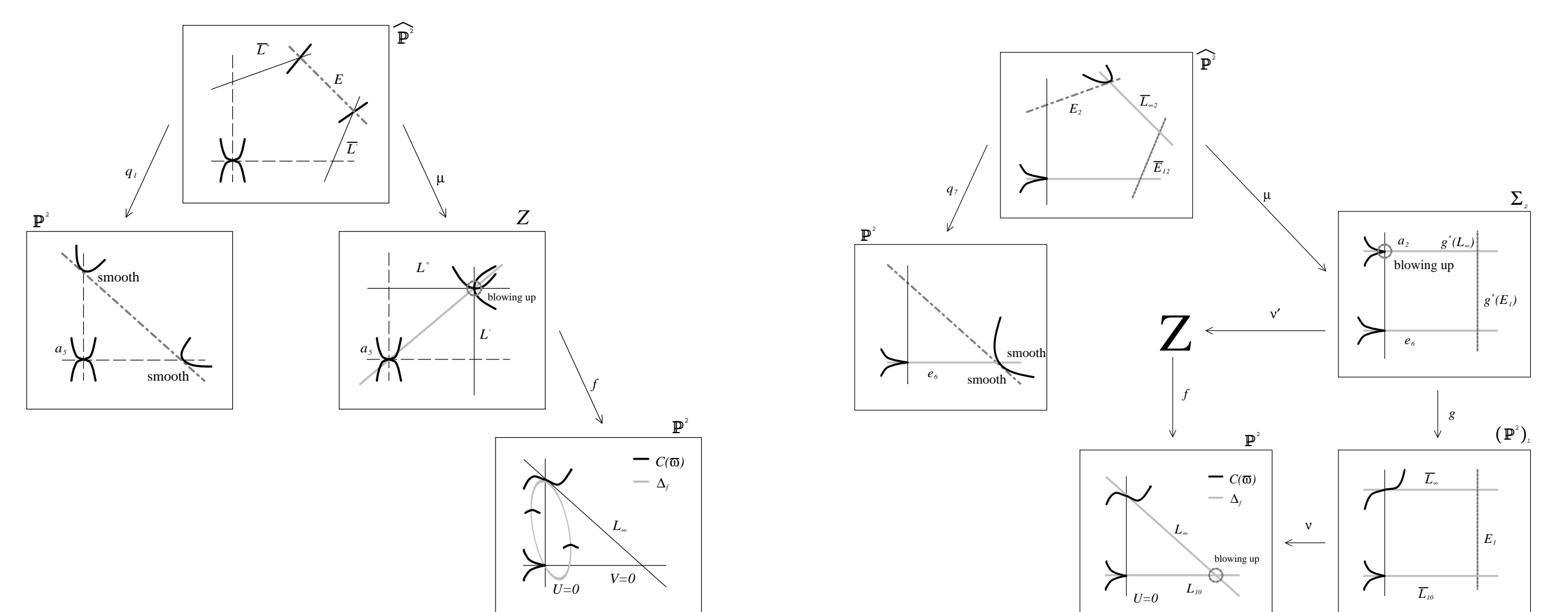
and passes through one singular point of  $Q$ .

(iv)  $L$  passes through two distinct singular points of  $Q$ .

(v)  $L$  intersects  $Q$  at just one singular point.

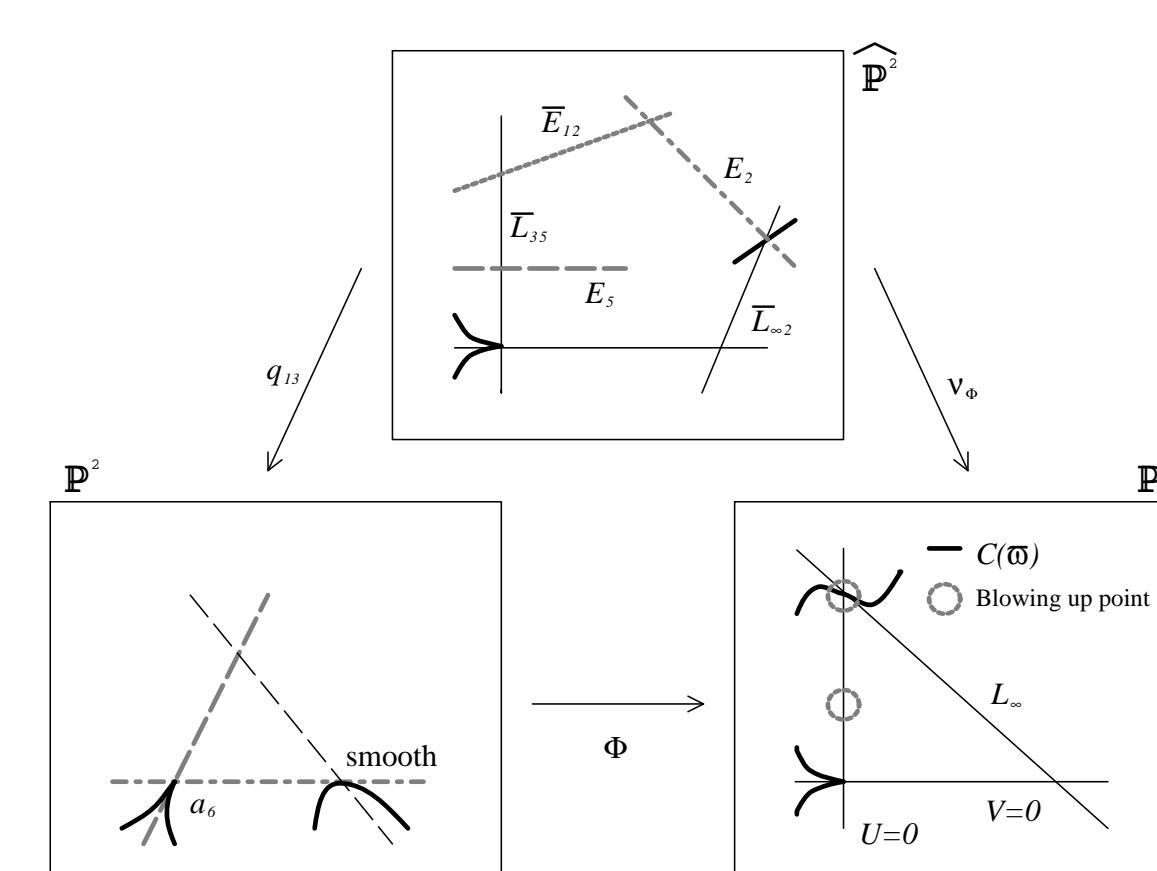
## The rational maps

Following four pictures are the rational maps  $\Phi_4$ ,  $\Phi_{10}$ ,  $\Phi_{13}$  and  $\Phi_{16}$  in above table.

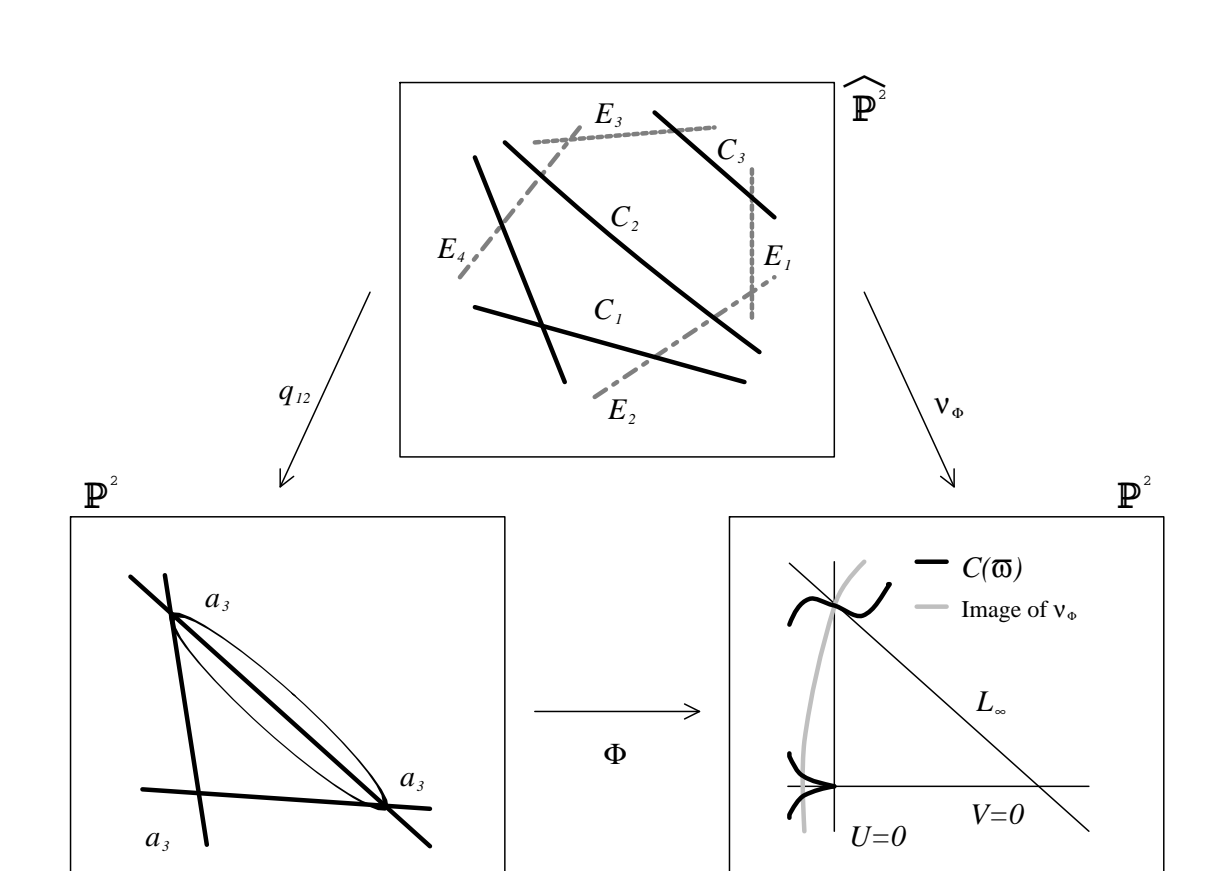


$\Phi_4$

$\Phi_{10}$



$\Phi_{13}$



$\Phi_{16}$