SUPERSINGULAR $K3$ SURFACE IN CHARACTERISTIC 5: COMPUTATIONAL DATA

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1. Introduction

In this note, we explain the computational data that appear in the paper

[KKS] T. Katsura, S. Kondo, I. Shimada: On the supersingular K3 surface in characteristic 5 with Artin invariant 1,

and are available from the author's web page

`http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3.html`.

These computational data are divided in two parts and written in three files: the first part is the data of the generalized Borcherds' method, and the second part is the geometric data of curves on the superspecial abelian surface A in characteristic 5.

2. The data of the generalized Borcherds' method

2.1. The Néron-Severi lattice S_X and its embedding into L. The following data are given in the file

`compdataB.txt`.

We work over $\mathbb{F}_{25} = \mathbb{F}_5(\sqrt{2})$:

$$\begin{aligned}\texttt{F25} \;:=\; & [0,1,2,3,4,\texttt{sqrt}(2),1+\texttt{sqrt}(2),2+\texttt{sqrt}(2),3+\texttt{sqrt}(2),4+\texttt{sqrt}(2),\\ & 2*\texttt{sqrt}(2),1+2*\texttt{sqrt}(2),2+2*\texttt{sqrt}(2),3+2*\texttt{sqrt}(2),4+2*\texttt{sqrt}(2),\\ & 3*\texttt{sqrt}(2),1+3*\texttt{sqrt}(2),2+3*\texttt{sqrt}(2),3+3*\texttt{sqrt}(2),4+3*\texttt{sqrt}(2),\\ & 4*\texttt{sqrt}(2),1+4*\texttt{sqrt}(2),2+4*\texttt{sqrt}(2),3+4*\texttt{sqrt}(2),4+4*\texttt{sqrt}(2)].\end{aligned}$$

The list

`FSF25`

of size 126 is the list of the $\mathbb{F}_{25}$-rational points on the Fermat sextic curve

$$C_F : x^6 + y^6 + z^6 = 0$$

in characteristic 5, sorted as in Table 4.1 of [KKS]. The 252×252 matrix

`M252`

is the intersection matrix of the h_F-lines

$$l_1^+, l_1^-, l_2^+, l_2^-, l_3^+, l_3^-, \;\ldots,\; l_{125}^+, l_{125}^-, l_{126}^+, l_{126}^-. \tag{2.1}$$

The 22×22 matrix

$$\texttt{GramSX}$$

is the Gram matrix of the Néron-Severi lattice S_X of the Fermat double sextic plane

$$X : w^2 = x^6 + y^6 + z^6$$

of characteristic 5, with respect to the basis

$$\ell_1 := l_1^+,\ \ \ell_2 := l_1^-,\ \ \ell_3 := l_2^+,\ \ \ell_4 := l_3^+,\ \ \ell_5 := l_4^+,\ \ \ell_6 := l_5^+,\ \ \ell_7 := l_7^+,\ \ \ell_8 := l_8^+,$$
$$\ell_9 := l_9^+,\ \ \ell_{10} := l_{10}^+,\ \ \ell_{11} := l_{13}^+,\ \ \ell_{12} := l_{14}^+,\ \ \ell_{13} := l_{15}^+,\ \ \ell_{14} := l_{16}^+,\ \ \ell_{15} := l_{17}^+,$$
$$\ell_{16} := l_{19}^+,\ \ \ell_{17} := l_{21}^+,\ \ \ell_{18} := l_{22}^+,\ \ \ell_{19} := l_{24}^+,\ \ \ell_{20} := l_{25}^+,\ \ \ell_{21} := l_{27}^+,\ \ \ell_{22} := l_{34}^+.$$

The vector

$$\texttt{LineClass[i]} \quad (i = 1, \dots, 252)$$

is the class of the ith h_F-line in (2.1) represented with respect to this basis. The 22×22 matrix

$$\texttt{Frob}$$

is the isometry of S_X induced by

$$[l_i^{\pm}] \mapsto [\text{the } \mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)\text{-conjugate of } l_i^{\pm}].$$

(Note that we let $\mathrm{O}(S_X)$ act on S_X *from the right*, so that we have

$$\texttt{Frob} \cdot \texttt{GramSX} \cdot {}^t\texttt{Frob} = \texttt{GramSX},$$

where ${}^t\texttt{Frob}$ is the transpose of $\texttt{Frob}$.) The 22×22 matrix

$$\texttt{Flip}$$

is the action of the deck-transformation of $X \to \mathbb{P}^2$ on S_X:

$$[l_i^{\pm}] \mapsto [l_i^{\mp}].$$

The matrix

$$\texttt{discSX} := [[2/5, 0], [0, 4/5]]$$

is the Gram matrix of the discriminant form

$$q_S : S_X^{\vee}/S_X \to \mathbb{Q}/2\mathbb{Z}$$

of S_X with respect to the basis

$$\alpha_1 := [\ell_3]^{\vee} \bmod S_X \quad \text{and} \quad \alpha_2 := [\ell_4]^{\vee} \bmod S_X.$$

Using this basis of $S_X^{\vee}/S_X \cong \mathbb{F}_5^2$, we present the group

$$\texttt{OqS} := \mathrm{O}(q_S) = \{\ \bar{g} \in \mathrm{GL}_2(\mathbb{F}_5)\ \mid\ \bar{g} \text{ preserves } q_S\ \}$$

of order 12 as a list of 2×2 matrices with entries in $\mathbb{F}_5$. By means of the matrices

$$\texttt{TransAS} \quad \text{and} \quad \texttt{TransBS}$$

of size 2×22 and 22×2, respectively, we can calculate the action $\bar{g} \in \mathrm{O}(q_S)$ on $S_X^\vee/S_X = \langle \alpha_1, \alpha_2 \rangle$ induced by a given isometry $g \in \mathrm{O}(S_X)$:

$$\bar{g} = \texttt{TransAS} \cdot \texttt{GramSX}^{-1} \cdot g \cdot \texttt{GramSX} \cdot \texttt{TransBS} \mod 5.$$

Then g preserves the period $\mathcal{K}_X$ of X if and only if $\bar{g} \in \mathrm{O}(q_S)$ is one of the following six matrices:

$$\begin{aligned} \texttt{AutPeriod} \;:= [\;\; & [[1,0],[0,1]],\;\; [[2,1],[3,2]],\;\; [[2,4],[2,2]],\\ & [[3,1],[3,3]],\;\; [[3,4],[2,3]],\;\; [[4,0],[0,4]]\;]. \end{aligned}$$

The 4×4 matrix

$$\texttt{GramR}$$

is the Gram matrix of the lattice R with respect to the basis $u_1, \dots, u_4$. We present the group

$$\texttt{OR} := \mathrm{O}(R) = \{\; g \in \mathrm{GL}_4(\mathbb{Z}) \;\mid\; g \cdot \texttt{GramR} \cdot {}^t g = \texttt{GramR} \;\}$$

of order 72. (Recall again that we let $\mathrm{O}(R)$ act on R *from the right.*) The matrix

$$\texttt{discR} := [[8/5, 0], [0, 6/5]]$$

is the Gram matrix of the discriminant form

$$q_R : R^\vee/R \to \mathbb{Q}/2\mathbb{Z}$$

of R with respect to the basis

$$\beta_1 := [u_4]^\vee \bmod R \quad \text{and} \quad \beta_2 := [u_2]^\vee \bmod R.$$

Using this basis, we present

$$\texttt{OqR} := \mathrm{O}(q_R) = \{\; \bar{g} \in \mathrm{GL}_2(\mathbb{F}_5) \;\mid\; \bar{g} \text{ preserves } q_R \;\}.$$

(Since $q_S \cong -q_R$, we have $\mathrm{O}(q_S) \cong \mathrm{O}(q_R)$. We have chosen the bases α_1, α_2 and β_1, β_2 in such a way that `OqS` and `OqR` are equal sets of matrices.) By means of the matrices

$$\texttt{TransAR} \quad \text{and} \quad \texttt{TransBR}$$

of size 2×4 and 4×2, respectively, we can calculate the induced action $\bar{g} \in \mathrm{O}(q_R)$ on $R^\vee/R = \langle \beta_1, \beta_2 \rangle \cong \mathbb{F}_5^2$ of a given isometry $g \in \mathrm{O}(R)$:

$$\bar{g} = \texttt{TransAR} \cdot \texttt{GramR}^{-1} \cdot g \cdot \texttt{GramR} \cdot \texttt{TransBR} \mod 5.$$

The fact that $g \mapsto \bar{g}$ is a surjective homomorphism from $\mathrm{O}(R)$ to $\mathrm{O}(q_R)$ is now readily verified.

The 26×26 matrix

$$\texttt{GramL}$$

is the Gram matrix of the even unimodular hyperbolic lattice L of rank 26 with respect to a certain basis $v_1, \dots, v_{26}$. (This matrix `GramL` and the basis $v_1, \dots, v_{26}$

do not appear in the paper [KKS]. They play, however, a crucial role in the actual execution of the generalized Borcherds' method.) The 26×26 matrix

$$\texttt{Emb}$$

gives the embedding $\iota : S_X \oplus R \hookrightarrow L$ with respect to the basis $[\ell_1], \dots, [\ell_{22}], u_1, \dots, u_4$ and $v_1, \dots, v_{26}$. Vectors v in $S_X \oplus R$ are *row* vectors, and ι is given by

$$v \mapsto v \cdot \texttt{Emb}.$$

By definition, `Emb` is an invertible matrix with integer entries such that

$$\texttt{Emb} \cdot \texttt{GramL} \cdot {}^t\texttt{Emb} = \left[\begin{array}{cc} \texttt{GramSX} & O \\ O & \texttt{GramR} \end{array} \right].$$

The projections $L \to S_X^\vee$ and $L \to R^\vee$ are easily calculated by `Emb`.

2.2. **The data of the induced chambers.** The data of the three induced chambers D_i $(i = 0, 1, 2)$ are given in the file

`compdataChams.txt`.

The Weyl vector w_i of D_i is given in terms of the *dual* basis

$$[\ell_1]^\vee, \dots, [\ell_{22}]^\vee, u_1^\vee, \dots, u_4^\vee$$

of $S_X^\vee \oplus R^\vee$:

$$\begin{aligned} \texttt{w[0]} &:= [1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,-2,-1,0,1]^\vee, \\ \texttt{w[1]} &:= [1,2,2,1,1,2,1,2,1,2,2,1,2,1,1,2,2,2,2,2,2,2,2,1,1,0]^\vee, \\ \texttt{w[2]} &:= [4,4,7,4,1,4,4,4,4,4,7,1,4,4,4,7,7,4,4,4,7,7,2,1,-1,0]^\vee. \end{aligned}$$

Hence its projection $w_{i,S}$ to $S_X^\vee$ is obtained from `w[i]` by deleting the last 4 coordinates. The polarizations are given by

$$h_1 := w_{1,S}, \quad h_2 := 5\, w_{2,S}, \quad h_3 := 5\, w_{3,S}.$$

Their representations by the *non-dual* basis $[\ell_1], \dots, [\ell_{22}]$ of S_X are

$$\begin{aligned} \texttt{h[0]} &:= [1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0], \\ \texttt{h[1]} &:= [14,16,-4,-6,-5,-11,12,-8,-5,0,10,8,-13,3,-3,5,-8,10,7,-2,5,-10], \\ \texttt{h[2]} &:= [14,11,3,6,21,15,-3,18,6,-6,-27,0,9,-12,3,-15,-3,-9,-18,12,0,15]. \end{aligned}$$

A *primitive defining vector of a wall* of an induced chamber D_i is a vector $v \in S_X^\vee$ primitive in $S_X^\vee$ such that $(v)^\perp$ is a wall of D_i and $\langle v, x\rangle > 0$ holds for a (and hence any) vector x in the interior of D_i. For each wall of D_i, there exists a unique primitive defining vector. We express each primitive defining vector in terms of the *dual* basis $[\ell_1]^\vee, \dots, [\ell_{22}]^\vee$ of $S_X^\vee$. The group $Aut_X(D_i) \cong \mathrm{Aut}(X, h_i)$ acts on the set of primitive defining vectors of walls. The list

$$\texttt{walls[i]}$$

is the list of orbits under the action of $Aut_X(D_i)$ on the set of primitive defining vectors of walls of D_i.

The list `walls`[0] consists of 3 lists, the first of which consists of 252 primitive vectors defining the (-2)-walls of the chamber D_0. If they are converted to the representations in terms of the *non-dual* basis $[\ell_1], \dots, [\ell_{22}]$ of S_X, they coincide with `LineClass[i]` ($\texttt{i} = 1, \dots, 252$).

The list `walls`[1] consists of 18 lists, the first of which consists of 168 primitive vectors defining the (-2)-walls of the chamber D_1.

The list `walls`[2] consists of 27 lists. The first member of `walls`[2] consists of 48 primitive defining vectors, and the second member also consists of 48 vectors. These 96 vectors define the (-2)-walls of the chamber D_2.

The group $Aut_X(D_i)$ is given by the following method. For $i = 0, 1, 2$, we fix a reference vector `refv`[`i`] $\in S_X$:

$$\begin{aligned}
\texttt{refv}[0] &:= [1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-1,0],\\
\texttt{refv}[1] &:= [1,1,-1,0,0,0,1,0,0,0,0,0,-1,-1,0,0,0,0,0,0,1,0],\\
\texttt{refv}[2] &:= [1,0,1,1,2,2,0,1,0,-1,-2,1,0,-1,0,-1,-1,0,-1,1,0,1],
\end{aligned}$$

represented in terms of the *non-dual* basis. (This vector `refv`[`i`] is a defining vector of a (-2)-wall of D_i, and hence it corresponds to a (-2)-curve on X.) Then the list

$$\texttt{SAutXD}[\texttt{i}]$$

is the stabilizer subgroup of `refv`[`i`] in $Aut_X(D_i)$, and the list

$$\texttt{TAutXD}[\texttt{i}]$$

is the list of representatives of the right coset of `SAutX`[`i`] in $Aut_X(D_i)$. Hence each element of $Aut_X(D_i)$ is uniquely written as a product

$$\sigma \cdot \tau \qquad (\sigma \in \texttt{SAutXD}[\texttt{i}], \;\; \tau \in \texttt{TAutXD}[\texttt{i}]).$$

The lists

$$\texttt{SS}[0,0], \;\; \texttt{SS}[0,1], \;\; \texttt{SS}[0,2], \;\; \texttt{SS}[1,0], \;\; \texttt{SS}[1,1], \;\; \texttt{SS}[1,2]$$

are the classes of smooth rational curves in $\mathcal{S}_{00}, \mathcal{S}_{01}, \mathcal{S}_{02}, \mathcal{S}_{10}, \mathcal{S}_{11}, \mathcal{S}_{12}$ (the decomposition of the 96 smooth rational curves corresponding to the (-2)-walls in the chamber D_2 into the sets of disjoint 16 smooth rational curves) written in terms of the *non-dual* basis of S_X. If they are converted to the representation in terms of the *dual* basis of $S_X^{\vee}$, their union coincide with the union of the first and the second lists of `walls`[2].

The vector

$$\texttt{sv}[1] := [0,1,1,0,0,1,0,1,0,1,1,0,1,0,0,1,1,1,1,1,1,1]^\vee \in S_X^\vee$$

is a member of the second list of $\texttt{walls}[0]$, and it defines the wall separating the chambers D_0 and D_1. The vector $(-1)*\texttt{sv}[1]$ is a member of the second list of $\texttt{walls}[1]$. The vector

$$\texttt{sv}[2] := [1,1,2,1,0,1,1,1,1,1,2,0,1,1,1,2,2,1,1,1,2,2]^\vee \in S_X^\vee$$

is a member of the third list of $\texttt{walls}[0]$, and it defines the wall separating the chambers D_0 and D_2. The vector $(-1)*\texttt{sv}[2]$ is a member of the eleventh list of $\texttt{walls}[2]$.

3. The data of curves on the abelian surface A

The following data are given in the list

`compdataKm.txt`.

The list `F25` of elements of $\mathbb{F}_{25} = \mathbb{F}_5(\sqrt{2})$ is included in this file. We put

$$\texttt{omega} := 2 + 3 * \texttt{sqrt}(2),$$

which is a cubic root of unity in $\mathbb{F}_{25}$. We exhibit $16 \times 6 = 96$ smooth rational curves on the Kummer surface $\mathrm{Km}(A)$, where $A = E \times E$ is the product of the elliptic curve defined by $\texttt{DefE} = 0$, where

$$\texttt{DefE} := \texttt{y}^2 + 4 * \texttt{x}^3 + 1.$$

The addition $m : E \times E \to E$ of the elliptic curve E with the origin at $x = \infty$

$$((x_1, y_1), (x_2, y_2)) \mapsto (x_3, y_3) = (\alpha(x_1, x_2), \tilde{\alpha}(x_1, y_1, x_2, y_2))$$

is given by the pair of rational functions

$$\texttt{addE} := [\alpha, \tilde{\alpha}].$$

The automorphism $\gamma : E \to E$ of E is given by

$$\texttt{gammaE} := [(2 + 3 * \texttt{sqrt}(2)) * \texttt{x}, 4 * \texttt{y}],$$

and the endomorphism $\phi_{E,2} : E \to E$ of degree 2 is given by

$$\texttt{phiE2} := [(2 * \texttt{x}^2 + 3 * \texttt{x} + 1)/(\texttt{x} + 4), 2 * \texttt{sqrt}(2) * \texttt{y} * (\texttt{x}^2 + 3 * \texttt{x} + 3)/(\texttt{x} + 4)^2].$$

The composite $\gamma \circ \phi_{E,2} : E \to E$ is

$$\begin{aligned} \texttt{gammaEphiE2} \quad := \quad & [(\texttt{x} + 3) * (\texttt{x} + 1) * (4 + \texttt{sqrt}(2))/(\texttt{x} + 4), \\ & (3 * (\texttt{x} + 3 * \texttt{sqrt}(2) + 4)) * (\texttt{x} + 2 * \texttt{sqrt}(2) + 4) * \texttt{y} * \texttt{sqrt}(2)/(\texttt{x} + 4)^2]. \end{aligned}$$

By these data, the curves $B_1, \ldots, B_6$ in Proposition 9.1 of [KKS] are obtained.

The Gram matrix of the Néron-Severi lattice S_A of A with respect to the basis $[B_1], \dots, [B_6]$ is given by

$$\texttt{GramSA}.$$

Let A_2 denote the kernel of the homomorphism $[2]_A : A \to A$. A point $(p_1, p_2) \in E \times E$ of A_2 is given by the x-coordinates of $p_1 \in E$ and $p_2 \in E$. They are sorted as follows:

$$
\begin{aligned}
\texttt{A2Pts} \quad := [\quad & [\texttt{infinity}, \texttt{infinity}], [\texttt{infinity}, 1], \\
& [\texttt{infinity}, 2 + 3 * \texttt{sqrt}(2)], [\texttt{infinity}, 2 + 2 * \texttt{sqrt}(2)], \\
& [1, \texttt{infinity}], [1, 1], [1, 2 + 3 * \texttt{sqrt}(2)], [1, 2 + 2 * \texttt{sqrt}(2)], \\
& [2 + 3 * \texttt{sqrt}(2), \texttt{infinity}], [2 + 3 * \texttt{sqrt}(2), 1], \\
& [2 + 3 * \texttt{sqrt}(2), 2 + 3 * \texttt{sqrt}(2)], [2 + 3 * \texttt{sqrt}(2), 2 + 2 * \texttt{sqrt}(2)], \\
& [2 + 2 * \texttt{sqrt}(2), \texttt{infinity}], [2 + 2 * \texttt{sqrt}(2), 1], \\
& [2 + 2 * \texttt{sqrt}(2), 2 + 3 * \texttt{sqrt}(2)], [2 + 2 * \texttt{sqrt}(2), 2 + 2 * \texttt{sqrt}(2)] \quad].
\end{aligned}
$$

By the blow-up $b : \tilde{A} \to A$ at the points of A_2, the lattice S_A is embedded into the Néron-Severi lattice $S_{\tilde{A}}$ of $\tilde{A}$. Let E_k denote the exceptional curve over the kth point of A2Pts. Let B'_i be the *total* transform of B_i by b. Then, with respect to the basis

$$[B'_1], \dots, [B'_6], [E_1], \dots, [E_{16}]$$

of $S_{\tilde{A}}$, the Gram matrix of $S_{\tilde{A}}$ is equal to

$$\texttt{GramSAtilde} := \begin{bmatrix} \texttt{GramSA} & O \\ O & -I_{16} \end{bmatrix}.$$

The list

$$\texttt{KmRatPts}$$

is the list of $\mathbb{F}_{25}$-rational points on $\mathrm{Km}(A)$. We use the coordinates (x_1, y_1) and (x_2, y_2) for the first and the second factor of $A = E \times E$, respectively. Locally around the origin of E, we put

$$\tilde{x} = 1/x, \quad z = y/x^2,$$

so that E is defined by $z^2 = \tilde{x} - \tilde{x}^4$. We also use the coordinates $(\tilde{x}_1, z_1)$ and $(\tilde{x}_2, z_2)$. Note that the singular surface $A/\langle \iota_A \rangle$ is defined by

$$w^2 = (x_1^3 - 1)(x_2^3 - 1), \quad \text{where} \quad w = y_1 y_2.$$

Let

$$\rho : \mathrm{Km}(A) \to A/\langle \iota_A \rangle$$

be the minimal resolution. Suppose that P is an $\mathbb{F}_{25}$-rational point of $\mathrm{Km}(A)$. Let $[\texttt{a1}, \texttt{a2}, \texttt{b}]$ be the (x_1, x_2, w)-coordinates of $\rho(P)$. (When $\texttt{a1} = \infty$ or $\texttt{a2} = \infty$, we put $\texttt{b} = 0$.) If $\rho(P)$ is a smooth point of $A/\langle \iota_A \rangle$, then P is expressed in KmRatPts as $[\texttt{a1}, \texttt{a2}, \texttt{b}]$. Suppose $\rho(P)$ is a singular point of $A/\langle \iota_A \rangle$. Let $\tilde{P} \in A_2$ be the point whose image $\varpi(\tilde{P}) \in A/\langle \iota_A \rangle$ is equal to $\rho(P)$, and let $T_{\tilde{P},A}$ denote the tangent space to A at $\tilde{P}$. Then the (-2)-curve $\rho^{-1}(\rho(P))$ is naturally identified with the projective line $\mathbb{P}_*(T_{\tilde{P},A})$. We express P in `KmRatPts` as $[[\mathtt{a1}, \mathtt{a2}], [\mathtt{c1}, \mathtt{c2}]]$, where $[\mathtt{c1}, \mathtt{c2}]$ is the homogeneous coordinates of $\mathbb{P}_*(T_{\tilde{P},A})$ with respect to the following basis of the linear space $T_{\tilde{P},A}$:

$$\begin{aligned}
T_{\tilde{P},A} &= \langle \partial/\partial y_1, \partial/\partial y_2 \rangle \quad \text{if } \mathtt{a1} \neq \infty \text{ and } \mathtt{a2} \neq \infty, \\
T_{\tilde{P},A} &= \langle \partial/\partial y_1, \partial/\partial z_2 \rangle \quad \text{if } \mathtt{a1} \neq \infty \text{ and } \mathtt{a2} = \infty, \\
T_{\tilde{P},A} &= \langle \partial/\partial z_1, \partial/\partial y_2 \rangle \quad \text{if } \mathtt{a1} = \infty \text{ and } \mathtt{a2} \neq \infty, \\
T_{\tilde{P},A} &= \langle \partial/\partial z_1, \partial/\partial z_2 \rangle \quad \text{if } \mathtt{a1} = \infty \text{ and } \mathtt{a2} = \infty.
\end{aligned}$$

Then `KmRatPts` consists of 1176 points, 760 of which are of type $[\mathtt{a1}, \mathtt{a2}, \mathtt{b}]$ and 416 of which are of type $[[\mathtt{a1}, \mathtt{a2}], [\mathtt{c1}, \mathtt{c2}]]$.

We describe the 96 smooth rational curves on $\mathrm{Km}(A)$ divided into six sets

$$\mathcal{S}_{00}, \mathcal{S}_{01}, \mathcal{S}_{02}, \mathcal{S}_{10}, \mathcal{S}_{11}, \mathcal{S}_{12}.$$

The 16 curves in $\mathcal{S}_{00}$ are the exceptional curves of the minimal resolution $\rho : \mathrm{Km}(A) \to A/\langle \iota_A \rangle$, and hence they are in one-to-one correspondence with A_2. The smooth rational curves in $\mathcal{S}_{00}$ are sorted according to the order of `A2Pts`. We have a finite double covering

$$\pi : \tilde{A} \to \mathrm{Km}(A).$$

Let $R_{00,k}$ denote the kth member of $\mathcal{S}_{00}$. Then we have

$$2E_k = \pi^*(R_{00,k})$$

for $k = 1, \dots, 16$. The other 80 smooth rational curves are obtained from the (hyper-)elliptic curves

$$H = E, \quad F, \quad \text{or} \quad G$$

defined by $\mathtt{defeqE} = 0$, $\mathtt{defeqF} = 0$, $\mathtt{defeqG} = 0$, respectively, where

```
defeqE  :=  v^2+4*u^3+1,
defeqF  :=  v^2+4*u^6+1,
defeqG  :=  v^2+4*sqrt(2)*(u^12+2*u^8+2*u^4+1).
```

Let $\iota_H : H \to H$ denote the involution of H over the u-line. There are 80 embeddings

$$\eta : H \hookrightarrow A$$

satisfying $\iota_A \circ \eta = \eta \circ \iota_H$ such that the strict transforms of $\eta(H)/\langle \iota_A \rangle$ by the minimal resolution $\rho : \mathrm{Km}(A) \to A/\langle \iota_A \rangle$ are the 80 smooth rational curves in $\mathcal{S}_{01}, \mathcal{S}_{02}, \mathcal{S}_{10}, \mathcal{S}_{11}, \mathcal{S}_{12}$. These embeddings

$$\eta := (\psi_1, \psi_2) : H \hookrightarrow E \times E = A, \quad \text{where } \psi_1 = \mathrm{pr}_1 \circ \eta \text{ and } \psi_2 = \mathrm{pr}_2 \circ \eta,$$

are described in the following form:

$$\texttt{LL[i,j,k]} := [\ \texttt{the name of H},\ \ [[\texttt{psi}_{\texttt{1x}},\texttt{psi}_{\texttt{1y}}],[\texttt{psi}_{\texttt{2x}},\texttt{psi}_{\texttt{2y}}]]\ ],$$

for $k=1,\dots,16$, where, for $m=1,2$, the pair

$$[\texttt{psi}_{\texttt{mx}},\texttt{psi}_{\texttt{my}}]$$

is the pair $(\psi_{mx}(u),\psi_{my}(u,v))$ of rational functions of u and v expressing the morphism $\psi_m : H\to E$ given by

$$\psi_m : (u,v)\mapsto (x,y)=(\psi_{mx}(u),\psi_{my}(u,v)).$$

(The constant morphism to the origin of E is denoted by $[\infty,0]$.) The 16 embeddings $\texttt{LL[i,j,1]},\dots,\texttt{LL[i,j,16]}$ yield the 16 smooth rational curves $R_{ij,1},\dots,R_{ij,16}$ in $\mathcal{S}_{ij}$; that is, $\texttt{LL[i,j,k]}$ is the list of the curves $\mathcal{L}_{ij}$ in [KKS].

Remark 3.1. Since $\iota_A\circ\psi=\psi\circ\iota_H$, each $\psi_{my}(u,v)$ is of the form $v\cdot\Psi_m(u)$, where $\Psi_m(u)$ is a rational function of u. If H is defined by $v^2=f_H(u)$, then we have

$$f_H(u)\Psi_m(u)^2=\psi_{mx}(u)^3-1.$$

Remark 3.2. The embeddings $\texttt{LL[i,j,k]}$ are composed from the morphisms

$$\phi_{E,2}:E\to E,\ \ \phi_{F,2}:F\to E,\ \ \phi_{F,3}:F\to E,\ \ \phi_{G,3}:G\to E,\ \ \phi_{G,4}:G\to E,$$

and

$$\gamma:E\to E,\ \ h_F:F\to F,\ \ h'_F:F\to F,\ \ h_G:G\to G,$$

by the translation by the points in A_2 and the automorphism $\tau:(P,Q)\mapsto(Q,\iota_E(P))$ of A. These morphisms are also given in the computational data with the names

`phiE2uv`, `phiF2`, `phiF3`, `phiG3`, `phiG4`, `gammaEuv`, `hF`, `hF2`, `hG`,

respectively. (The morphisms `gammaEuv` and `phiE2uv` are same as `gammaE` and `phiE2`, but are written in variables u and v.) The translation of a morphism to A by the points in A_2 can be calculated from `addE` and `A2Pts`.

The morphism $\eta: H\hookrightarrow A$ given as $\texttt{LL[i,j,k]}$ induces an embedding

$$\bar\eta:\mathbb{P}^1\to\mathrm{Km}(A)$$

from the u-line $\mathbb{P}^1=H/\langle\iota_H\rangle$ into $\mathrm{Km}(A)$. Using $\eta:=\texttt{LL[i,j,k]}$, we make the list

$$\texttt{RatPtsR[i,j,k]}$$

of the $\mathbb{F}_{25}$-rational points of the kth smooth rational curve $R_{ij,k}=\mathrm{Im}\,\bar\eta$ in $\mathcal{S}_{ij}$. Let

```
P1F25 := [infinity, 0, 1, 2, 3, 4,
          sqrt(2), 1 + sqrt(2), 2 + sqrt(2), 3 + sqrt(2), 4 + sqrt(2),
          2 * sqrt(2), 1 + 2 * sqrt(2), 2 + 2 * sqrt(2), 3 + 2 * sqrt(2), 4 + 2 * sqrt(2),
          3 * sqrt(2), 1 + 3 * sqrt(2), 2 + 3 * sqrt(2), 3 + 3 * sqrt(2), 4 + 3 * sqrt(2),
          4 * sqrt(2), 1 + 4 * sqrt(2), 2 + 4 * sqrt(2), 3 + 4 * sqrt(2), 4 + 4 * sqrt(2)]
```

denote the list of $\mathbb{F}_{25}$-rational points of $\mathbb{P}^1$. For $\mathtt{i} = \mathtt{j} \neq 0$, the list $\mathtt{RatPtsR}[\mathtt{i}, \mathtt{j}, \mathtt{k}]$ is sorted according to P1F25; the νth point of P1F25 is mapped to the νth point of $\mathtt{RatPtsR}[\mathtt{i}, \mathtt{j}, \mathtt{k}]$ by the morphism $\bar{\eta} : \mathbb{P}^1 \to \mathrm{Km}(A)$ induced from the $\eta = \mathtt{LL}[\mathtt{i}, \mathtt{j}, \mathtt{k}]$. While for $\mathtt{i} = \mathtt{j} = 0$, RatPtsR[0, 0, k] is sorted according to P1F25 via an isomorphism

$$\eta'_{0,0,k} : \mathbb{P}^1 \xrightarrow{\sim} \rho^{-1}(\varpi(P_k)),$$

where P_k is the kth point in A2Pts, and $\varpi(P_k)$ is the corresponding node of $A/\langle \iota_A \rangle$.

We put

$$\begin{aligned}
\mathtt{P6} &:= [\mathtt{infinity}, 0, 1, 2, 3, 4], \\
\mathtt{P4} &:= [\mathtt{sqrt}(2), 1 + 2 * \mathtt{sqrt}(2), 3 + 3 * \mathtt{sqrt}(2), 4 + 4 * \mathtt{sqrt}(2)], \\
\mathtt{P4conj} &:= [4 * \mathtt{sqrt}(2), 1 + 3 * \mathtt{sqrt}(2), 3 + 2 * \mathtt{sqrt}(2), 4 + \mathtt{sqrt}(2)], \\
\mathtt{P12} &:= [2 * \mathtt{sqrt}(2), 3 * \mathtt{sqrt}(2), 1 + \mathtt{sqrt}(2), 1 + 4 * \mathtt{sqrt}(2), \\
&\quad\ 2 + \mathtt{sqrt}(2), 2 + 2 * \mathtt{sqrt}(2), 2 + 3 * \mathtt{sqrt}(2), 2 + 4 * \mathtt{sqrt}(2), \\
&\quad\ 3 + \mathtt{sqrt}(2), 3 + 4 * \mathtt{sqrt}(2), 4 + 2 * \mathtt{sqrt}(2), 4 + 3 * \mathtt{sqrt}(2)].
\end{aligned}$$

The rational function $\varphi = \varphi_{i,j,k,j'} = \mathtt{varphiCtoP1}[\mathtt{i}, \mathtt{j}, \mathtt{k}, \mathtt{jprime}]$ gives the isomorphism in Corollary 1.3 from the u-line to $\mathbb{P}^1 \otimes \mathbb{F}_{25}$ such that, letting η be the morphism $\mathtt{LL}[\mathtt{i}, \mathtt{j}, \mathtt{k}]$ for the case $\mathtt{i} = \mathtt{j} \neq 0$, or the ismorphism $\eta'_{0,0,k} : \mathbb{P}^1 \xrightarrow{\sim} \rho^{-1}(\varpi(P_k))$ for the case $\mathtt{i} = \mathtt{j} = 0$, we have

$$\begin{aligned}
\varphi^{-1}(\mathtt{P6}) &= \{\, u \mid \text{there is a rational curve in } \mathcal{S}_{ij'} \text{ that passes through } \bar{\eta}(u) \,\}, \\
\varphi^{-1}(\mathtt{P4}) &= \{\, u \mid \text{there is a rational curve in } \mathcal{S}_{i'j'} \text{ that passes through } \bar{\eta}(u) \,\}, \\
\varphi^{-1}(\mathtt{P4conj}) &= \{\, u \mid \text{there is a rational curve in } \mathcal{S}_{i'j''} \text{ that passes through } \bar{\eta}(u) \,\}, \\
\varphi^{-1}(\mathtt{P12}) &= \{\, u \mid \text{there is a rational curve in } \mathcal{S}_{i'j} \text{ that passes through } \bar{\eta}(u) \,\},
\end{aligned}$$

where $i \neq i'$ and $j \neq j' \neq j'' \neq j$.

Let $\widetilde{\Gamma}_{ij,k}$ be the pull-back by the finite morphism $\pi : \tilde{A} \to \mathrm{Km}(A)$ of the kth smooth rational curve $R_{ij,k}$ in $\mathcal{S}_{ij}$; that is, $\widetilde{\Gamma}_{00,k}$ is the divisor $2E_k$, while if $ij \neq 00$, the curve $\widetilde{\Gamma}_{ij,k}$ is the *strict* transform by $\tilde{A} \to A$ of the image $\Gamma_{ij,k}$ of the embedding $\mathtt{LL}[\mathtt{i}, \mathtt{j}, \mathtt{k}]$. Then the class $[\widetilde{\Gamma}_{ij,k}] \in S_{\tilde{A}}$ with respect to the basis

$$[B'_1],\ \dots,\ [B'_6],\ [E_1],\ \dots,\ [E_{16}]$$

of the Néron-Severi lattice $S_{\tilde{A}}$ is given by

$$\mathtt{NSClass}[\mathtt{i}, \mathtt{j}, \mathtt{k}].$$

Since the pull-back by $\pi : \tilde{A} \to \mathrm{Km}(A)$ embeds $S_{\mathrm{Km}(A)}(2)$ into $S_{\tilde{A}}$, we can calculate the intersection numbers of $R_{ij,k}$ by GramSAtilde and $\mathtt{NSClass}[\mathtt{i}, \mathtt{j}, \mathtt{k}]$.

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