15-NODAL QUARTIC SURFACES. PART II:
THE AUTOMORPHISM GROUP:
COMPUTATIONAL DATA

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This note explains the contents of the computational data about the results of
the paper [1] (joint work with Igor Dolgachev). The data is available at
http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3andEnriques.html
in the text file 15nodalcompdata.txt. In this data, we use the Record-format of
GAP [2].

1. Lattices

We use $L_{26}$ to denote the even unimodular hyperbolic lattice $II_{1,25}$ of rank 26.
We fix bases of the lattices $L_{26}$, $S_{16}$ and $S_{15}$. The following data are with respect
to these bases.

- **GramL26.** The Gram matrix of $L_{26}$.
- **GramS16.** The Gram matrix of $S_{16}$.
- **GramS15.** The Gram matrix of $S_{15}$.
- **embS16L26.** The matrix $M$ such that $v \mapsto vM$ is the primitive embedding
  $\epsilon_{16}: S_{16} \hookrightarrow L_{26}$.
- **embS15L26.** The matrix $M$ such that $v \mapsto vM$ is the primitive embedding
  $\epsilon_{15}: S_{15} \hookrightarrow L_{26}$.
- **embS15S16.** The matrix $M$ such that $v \mapsto vM$ is the primitive embedding
  $\epsilon_{15,16}: S_{15} \hookrightarrow S_{16}$.
- **projL26S16.** The matrix $M$ such that $v \mapsto vM$ is the orthogonal projection
  $L_{26} \otimes \mathbb{Q} \to S_{16} \otimes \mathbb{Q}$.
- **projL26S15.** The matrix $M$ such that $v \mapsto vM$ is the orthogonal projection
  $L_{26} \otimes \mathbb{Q} \to S_{15} \otimes \mathbb{Q}$.
- **projS16S15.** The matrix $M$ such that $v \mapsto vM$ is the orthogonal projection
  $S_{16} \otimes \mathbb{Q} \to S_{15} \otimes \mathbb{Q}$.
- **weyl0.** The Weyl vector $w_0 \in L_{26}$.
- **ample16.** The class $\alpha_{16} \in S_{16}$, that is, the image of $w_0$ by the orthogonal
  projection $L_{26} \otimes \mathbb{Q} \to S_{16} \otimes \mathbb{Q}$. This vector is the class of a hyperplane
  section of the $(2,2,2)$-complete intersection model of $Y_{16}$.
- **ample15.** The class $\alpha_{15} \in S_{15} \otimes \mathbb{Q}$, that is, the image of $w_0$ by the orthogonal
  projection $L_{26} \otimes \mathbb{Q} \to S_{15} \otimes \mathbb{Q}$. Note that $\alpha_{15} \notin S_{15}$ but
  $2\alpha_{15} \in S_{15}$.
- **h4X16.** The class $h_4 \in S_{16}$ of the hyperplane section of the quartic surface
  $X_{16}$.
- **h4X16dual.** The class of the hyperplane section of the quartic surface $X_1^\prime_{16}$
  (the dual of $X_{16} \subset \mathbb{P}^3$).
- **h4X15.** The class $h_4 \in S_{15}$ of the hyperplane section of the quartic surface
  $X_{15}$. 

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• $h6Y15$. The class $h_6 \in S_{15}$ of the hyperplane section of the $(2,3)$-complete intersection model $X^{(6)}_{15}$ of $Y_{15}$ (see (5.5) of [1]).

2. $Y_{16}$ AND THE INDUCED CHAMBER $D_{16}$ OF $S_{16}$

2.1. Groups.

• Generators$OS_{16}D_{16}$ is a generating set of the group $O(S_{16}, D_{16})$ of order 23040.
• Generators$AutY_{16}D_{16}$ is a generating set of the group $Aut(Y_{16}, \alpha_{16}) = O(S_{16}, D_{16}) \cap O(S_{16}) \cong (\mathbb{Z}/2\mathbb{Z})^3$.

2.2. Rational curves. The following are the data related with Remark 4.4 of [1].

• RatCurvesOn$Y_{16}deg5$ is the list of classes of smooth rational curves on $Y_{16}$ with degree 5 with respect to $\alpha_{16}$.
• RatCurvesOn$Y_{16}deg7$ is the list of classes of smooth rational curves on $Y_{16}$ with degree 7 with respect to $\alpha_{16}$.

2.3. Walls. The data D16WallRecs is the list of 316 records wallrec, each of which describes a wall $w = D_{16} \cap (v)^\perp$ of $D_{16}$ and consists of the following items.

• no. The number $k$ such that the record wallrec is at the $k$th position of D16WallRecs.
• orbit. The number $i$ of the orbit containing $w$ (see Table 4.2 of [1]).
• innout. "inner" or "outer".
• vector. The primitive defining vector $v$ of $w$.
• n. $n = \langle v, v \rangle$.
• a. $a = \langle v, \alpha_{16} \rangle$.
• adjacentweyl. The Weyl vector $w' \in L_{26}$ that induces the chamber $D'$ adjacent to $D_{16}$ across $w$.
• d. $d = \langle \alpha_{16}, w'_S \rangle$, where $w'_S$ is the image of $w'$ by the orthogonal projection $L_{26} \to S_{16} \otimes \mathbb{Q}$.
• isomL26. An isometry $\tilde{g} \in O(L_{26})$ that preserves $S_{16} \subset L_{26}$ and maps the Conway chamber $D(w_0)$ to the Conway chamber $D'$ such that $\epsilon_{16}(D')$ is the induced chamber $D'$ adjacent to $D_{16}$ across $w$.
• extraaut. An isometry $g_w \in O(S_{16})$ that maps $D_{16}$ to the induced chamber $D'$ adjacent to $D_{16}$ across $w$. When $w$ is inner, this isometry $g_w$ is chosen from $Aut(Y_{16})$.
• index. The combinatorial data of $w$. The 32 lines $N_0, N_{ij}, T_i, T_{ij}$ in the $(2,2,2)$-complete intersection model of $Y_{16}$ are expressed by

["nodal", [0]], ["nodal", [i, j]], ["trope", [i]], ["trope", [i, j]],

respectively.

– When orbit is 1, index indicates the curve whose class defines the wall $w$.

– When orbit is 2, index indicates the Göpel-tetrad.

– When orbit is 3, index indicates the curve that is the exceptional curve over the center of the projection $X_{16} \to \mathbb{P}^2$ or $X'_{16} \to \mathbb{P}^2$ that induces the involution $g_w$.

– When orbit is 4, index indicates the Weber-hexad.
When \( \text{orbit} \) is 1, the record \( \text{wallrec} \) has the following items:

- \text{octad}. The corresponding octad (see Table 4.1 of [1]).
- \text{Leechroot}. The Leech root \( \epsilon_{16}(v) \in L_{26} \).

### 3.2. Walls

The data \( \text{D15WallRecs} \) is the list of 314 records \( \text{wallrec} \), each of which describes a wall \( w = D_{15} \cap (v) \) of \( D_{15} \) and consists of the following items.

- \( \text{no.} \). The number \( k \) such that the record \( \text{wallrec} \) is at the \( k \)th position of \( \text{D15WallRecs} \).
- \( \text{orbit} \). The number \( i \) of the orbit \( O_i \) containing \( w \) (see Table 5.1 of [1]).
- \( \text{innout} \). "inner" or "outer".
- \( \text{vector} \). The primitive defining vector \( v \) of \( w \).
- \( \text{n.} \). \( n = \langle v, v \rangle \).
- \( \text{a.} \). \( a = \langle v, e_{15} \rangle \).
- \( \text{adjacentweyl} \). The Weyl vector \( w' \in L_{26} \) that induces the chamber \( D' \) adjacent to \( D_{15} \) across \( w \).
- \( \text{d.} \). \( d = \langle a_{15}, W_S \rangle \), where \( W_S \) is the image of \( w' \) by the orthogonal projection \( L_{26} \to S_{15} \otimes Q \).
- \( \text{isomL26} \). An isometry \( \tilde{g} \in O(L_{26}) \) that preserves \( S_{15} \subset L_{26} \) and maps the Conway chamber \( D(w_0) \) to the Conway chamber \( D' \) such that \( \epsilon_{15}^{-1}(D') \) is the induced chamber \( D' \) adjacent to \( D_{15} \) across \( w \).
- \( \text{extraaut} \). An isometry \( g_w \in O(S_{15}) \) that maps \( D_{15} \) to the induced chamber \( D' \) adjacent to \( D_{15} \) across \( w \). When \( w \) is inner, this isometry \( g_w \) is the unique extra-automorphism \( g_w \in \text{Aut}(X_{15}) \).
- \( \text{index} \). The combinatorial data of \( w \).
  - When \( \text{orbit} \) is 1, then \( \text{index} \) is a double trio \( (ijk)(lmn) = [i, j, k, [l, m, n]] \).
  - When \( \text{orbit} \) is 2, then \( \text{index} \) is a duad \( (ij) = [i, j] \).
  - When \( \text{orbit} \) is 3, then \( \text{index} \) is a syntheme \( (ij)(kl)(mn) = [[i, j], [k, l], [m, n]] \).
  - When \( \text{orbit} \) is 4, then \( \text{index} \) is a double trio \( (ijk)(lmn) = [i, j, k, [l, m, n]] \).
  - When \( \text{orbit} \) is 5, then \( \text{index} \) is a number \( \nu \in \{1, \ldots, 6\} \).
  - When \( \text{orbit} \) is 6, then \( \text{index} \) is a pair of double trios
    \[
    \{ (i_1j_1k_1)(l_1m_1n_1), (i_2j_2k_2)(l_2m_2n_2), \}
    = \{ [i_1, j_1, k_1], [l_1, m_1, n_1], [i_2, j_2, k_2], [l_2, m_2, n_2] \}
    \]
  - When \( \text{orbit} \) is 7, then \( \text{index} \) is a number \( \nu \in \{1, \ldots, 6\} \).
  - When \( \text{orbit} \) is 8, then \( \text{index} \) is a duad \( (ij) = [i, j] \).
  - When \( \text{orbit} \) is 9, then \( \text{index} \) is an index \( [t(a), t(b), \ldots, t(f)] \) of \( I_{\text{tripod}} \) (see Figure 5.3 of [1]).
  - When \( \text{orbit} \) is 10, then \( \text{index} \) is an index \([p(a), p(b), \ldots, p(e)] \) of \( I_{\text{penta}} \) (see Figure 5.4 of [1]).
Remark 3.1. When there exist several choices of representatives of a combinatorial data, we choose the minimal one. For example, a double trio \((123)(456)\) can be written in 72 ways

\[
[[1, 2, 3], [4, 5, 6]], [[1, 3, 2], [4, 5, 6]], \ldots, [[6, 5, 4], [3, 2, 1]],
\]

and we choose \([[1, 2, 3], [4, 5, 6]]\) as a representative.

3.3. Involutions. Let \(\iota \in \text{Aut}(Y_{15})\) be an involution that is obtained from a rational double covering \(Y_{15} \to \mathbb{P}^2\) (in several ways). Then \(\iota\) is described by a record \(\text{involrec}\) with the following items:

- \(\text{invol}\) is the matrix representation of the action of \(\iota\) in \(S_{15}\).
- \(\text{degree}\) is the \(\alpha_{15}\)-degree \(\langle \alpha_{15}, \alpha_{15}' \rangle\).
- \(\text{index}\) indicates a combinatorial data that specifies \(\iota\). The content of \(\text{index}\) depends on the type of \(\iota\).
- \(\text{h2recs}\) is a list of records \(\text{h2rec}\) that describe polarizations \(h_2\) of degree 2 such that \(|h_2|\) gives a rational double covering \(Y_{15} \to \mathbb{P}^2\) inducing \(\iota\). Each \(\text{h2rec}\) has the following items:
  - \(h_2\) is the vector \(h_2 \in S_{15}\).
  - \(\text{sing}\) describes the singular points \(P\) of the branch curve \(B\) of the covering \(Y_{15} \to \mathbb{P}^2\). Each singular point \(P\) of \(B\) is given by a pair of an ADE-type such as "A1", "A2", "D4", \ldots and the list of classes of smooth rational curves contracted to \(P\) by \(Y_{15} \to \mathbb{P}^2\).

We have the following lists of involutions of \(Y_{15}\).

3.3.1. Sigmas. The list of six involutions \(\sigma^{(\nu)}\) that make \(Y_{15}\) the focal surface of a congruence of bi-degree \((2, 3)\). See Section 5.2.1 of [1]. The index \(\text{involrec.index}\) is the number \(\nu \in \{1, \ldots, 6\}\). Each \(\text{involrec.h2recs}\) consists of five records.

3.3.2. X15Projections. The list of 15 involutions obtained by the projection of \(X_{15}\) with the center being a node \(p_\delta\) of \(X_{15}\). See Example 5.6 of [1]. The index \(\text{involrec.index}\) is the duad \(\delta\) corresponding to the center \(p_\delta\). Each \(\text{involrec.h2recs}\) consists of a single record.

3.3.3. SevenNodals. The list of 360 involutions obtained by the linear system on \(Y_{15}\) cut out by quadric surfaces passing through a set of 7 nodes of \(X_{15}\) obtained from the 7 edges of the graph in Figure 5.2 of [1]. See Example 5.7 of [1]. The index \(\text{involrec.index}\) is the list of 7 duads corresponding to the 7 nodes of \(X_{15}\). Each \(\text{involrec.h2recs}\) consists of a single record.

3.3.4. Pentagons. The list of 72 involutions obtained by the linear system on \(Y_{15}\) cut out by cubic surfaces passing through a certain set of 5 nodes of \(X_{15}\) with given multiplicities. See Example 5.8 of [1]. The index \(\text{involrec.index}\) is the list of 5 duads corresponding to the 5 nodes of \(X_{15}\). Each \(\text{involrec.h2recs}\) consists of five records.

3.3.5. X6ModelProjections. The list of 45 involutions obtained from the projections of the \((2, 3)\)-complete intersection \(X_{15}^{(6)}\) with the center being the line passing through two ordinary nodes of \(X_{15}^{(6)} \subset \mathbb{P}^4\). See Section 5.2.3 of [1]. The index \(\text{involrec.index}\) is the pair of double trios corresponding the two nodes of \(X_{15}^{(6)}\). Each \(\text{involrec.h2recs}\) consists of a single record.
3.4. **Faces.** The data \( D_{15}^{\text{InnFaceRecs}} \) is the list of 5235 records \( \text{facerec} \), each of which describes a face \( f = w_1 \cap w_2 \) of \( D_{15} \) with codimension 2 and consists of the following items.

- **no.** The number \( k \) such that the record \( \text{facerec} \) is at the \( k \)th position of \( D_{15}^{\text{InnFaceRecs}} \).
- **orbit.** The number \( i \) of the orbit containing \( f \) (see Table in Theorem 5.11 of [1]).
- **walls.** The pair of \( \text{wallrec}1.\text{no} \) and \( \text{wallrec}2.\text{no} \), where \( \text{wallrec}1 \) and \( \text{wallrec}2 \) are the records that describe the walls \( w_1 \) and \( w_2 \) containing \( f \), respectively.
- **relation.** Let \( (D_0, \ldots, D_m) \) be one of the two simple chamber loops around \( f \) from \( D_0 \) to \( D_m = D_0 \), and let \( g_1, \ldots, g_m \) be the extra-automorphisms such that
  \[
  D_i = D_0^{g_i \cdots g_1}
  \]
  for \( i = 1, \ldots, m \). Then we have a relation
  \[
  g_m \cdots g_1 = 1.
  \]
  The item \( \text{facerec}\.\text{relation} \) is the list \([\nu_m, \ldots, \nu_1]\) of numbers, where \( \nu_i \) is the number \( \text{wallrec}\.\text{no} \) of the record \( \text{wallrec} \) such that \( \text{wallrec}\.\text{extraaut} \) is equal to \( g_i \).

**References**


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