AUTOMORPHISM GROUPS OF THREE SINGULAR K3 SURFACES: EXPLANATION OF THE DATA

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In this note, we explain the computational data related to Section 10 of the paper

[Algo] An algorithm to compute automorphism groups of K3 surfaces
and an application to singular K3 surfaces
given in the author’s web-page


In each of the three text files
disc11.txt, disc15.txt, disc16.txt,
we present a finite set of generators \( \Gamma \) of the image of \( \varphi_X : \text{Aut}(X) \to O^+(S_X) \), where \( X \) is a singular K3 surface whose transcendental lattice \( T_X \) is given by

\[
\begin{bmatrix}
2 & 1 \\
1 & 6
\end{bmatrix}, \quad \begin{bmatrix}
4 & 1 \\
1 & 4
\end{bmatrix}, \quad \begin{bmatrix}
2 & 0 \\
0 & 8
\end{bmatrix},
\]

respectively.

More precisely, we present the following data in each of these files. We fix bases of the lattices \( L, S_X \) and \( R \). To indicate elements of \( S_X' \) or \( R' \), we use the dual bases of these bases.

- **GramMatL.** A Gram matrix of \( L = U \oplus E_8 \oplus E_8 \oplus E_8 \).
- **GramMatS.** A Gram matrix of \( S_X = U_6 \oplus T_X^{-} \oplus E_8 \oplus E_8 \). (The order of basis is different from \( S_X = L_{18}(\phi) \oplus T_X^{-} \) given in the paper.)
- **GramMatR.** A Gram matrix of the orthogonal complement \( R \) of \( S_X \) in \( L \).
- **embMatS.** The \( 20 \times 26 \) matrix \( M \) such that \( x \mapsto xM \) gives the primitive embedding of \( S_X \) into \( L \).
- **embMatR.** The \( 6 \times 26 \) matrix \( M \) such that \( x \mapsto xM \) gives the primitive embedding of \( R \) into \( L \).
- **prMatSdual.** The \( 26 \times 20 \) matrix \( M \) such that \( x \mapsto xM \) gives the projection \( L \to S_X' \), where a vector \( S_X' \) is expressed by the dual basis.
- **prMatRdual.** The \( 26 \times 6 \) matrix \( M \) such that \( x \mapsto xM \) gives the projection \( L \to R' \), where a vector \( R' \) is expressed by the dual basis.
• **discgrS.** The discriminant group $S_X^*/S_X$. If $\text{discgrS} = [a]$, then $S_X^*/S_X \cong \mathbb{Z}/a\mathbb{Z}$ (in this case, we put $\nu = 1$), while if $\text{discgrS} = [a, b]$, then $S_X^*/S_X \cong \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ (in this case, we put $\nu = 2$).

• **etaS1 and etaS1.** etaS1 is a $\nu \times 20$ matrix, and etaS2 is a $20 \times \nu$ matrix. These two data describe the homomorphism $\eta_{S_X} : \text{O}(S_X) \to \text{O}(q_{S_X})$. Suppose that we are given a $20 \times 20$ matrix $M$ representing an element $g$ of $\text{O}(S_X)$ with respect to the fixed basis of $S_X$. Then the action of $g$ on $S_X^*$ is given by the matrix $M' := (\text{GramMatS})^{-1} \cdot M \cdot (\text{GramMatS})$ with respect to the dual basis. (Note that the action of $\text{O}(S_X)$ on $S_X$ is from the right.) Then $\eta_{S_X}(g)$ is given by the $\nu \times \nu$ matrix etaS1 $\cdot M' \cdot$ etaS2 under an isomorphism $S_X^*/S_X \cong \mathbb{Z}/a\mathbb{Z}$ or $S_X^*/S_X \cong \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$.

• **fphi.** The vector $f_\phi \in S_X$ of the class of a fiber of $\phi : X \to \mathbb{P}^1$.

• **u0.** The vector $u_0 \in S_X$. We can confirm that $D_0 \subset N(X)$ by fphi, u0 and the data chamberdata[0][wallorbit] below.

• **Gamma.** The finite set $\Gamma$ of generators of the image of $\varphi_X : \text{Aut}(X) \to \text{O}^+(S_X)$ obtained by Algorithm 6.1.

• **B.** The set $B$ of $(-2)$-vectors obtained by Algorithm 6.1.

• **maxchamnumb.** The maximum of the index $i$ of the chambers $D_i$ in the complete set $\mathbb{D}$ of representatives of $G_X$-equivalence classes of $\mathcal{R}^*_{L_{LIS}}$-chambers contained in $N(X)$ obtained by Algorithm 6.1.

For $i = 0, \ldots, \text{maxchamnumb}$, we give the following data of the $\mathcal{R}^*_{L_{LIS}}$-chamber $D_i$.

• **chamberdata[i][weyl].** A Weyl vector $w$ of $D_i$.

• **chamberdata[i][Deltaw].** The set $\Delta_w$, which is a subset of $L$. Note that the set $\text{pr}_S(\Delta_w) \subset S_X^*$ calculated by prMatSdual and chamberdata[i][Deltaw] is a defining set of the chamber $D_i$. In general, $\text{pr}_S(\Delta_w)$ contains many redundant vectors (vectors that do not define walls of $D_i$).

• **chamberdata[i][orderaut].** The order of $\text{Aut}_{G_X}(D_i)$.

• **chamberdata[i][generatorsaut].** A finite set of generators of $\text{Aut}_{G_X}(D_i)$.

• **chamberdata[i][numbordinwalls].** The number $t$ of the orbits of the action of $\text{Aut}_{G_X}(D_i)$ on the primitively minimal defining set $\Delta_{S_X}(D_i)$ of $D_i$. For $k = 0, \ldots, t - 1$, we present the following data of the $k$th orbit $\alpha_k$.

  - **chamberdata[i][wallorbit][k].** The list of elements of $\alpha_k$ written in terms of the dual basis of $S_X^*$. The first member $v$ of chamberdata[i][wallorbit][k] is used as the representative in Steps 2-3 and 2-4 of Algorithm 6.1.

  - **chamberdata[i][adjacent][k].** The description of the $\mathcal{R}^*_{L_{LIS}}$-chamber $D'$ adjacent to $D_i$ along the wall $(v)^\perp \in \alpha_k$.

    * If it is $[-\text{minustwo}]$, the orbit $\alpha_k$ satisfies $\alpha_k \subset \mathcal{R}^*_{L_{LIS}}$. Hence $D'$ is not calculated. Instead, $r = \alpha v$ with $\alpha \in \mathbb{Z}_{>0}$ and $\alpha^2 v^2 = -2$ is in the set $B$. 


If it is \[\text{\_backto, j}\], then \(D'\) is equal to the previously found \(R_{L|S}^*\)-chamber \(D_j\). (The chamber \(D_i\) is calculated as an chamber adjacent to \(D_j\).)

* If it is \[\text{\_newcham, j}\], then \(D'\) is equal to a new representative \(D_j \in \mathcal{D}\) of \(G_X\)-congruence class.

* Suppose that it is \[\text{\_isom, j, \_via, M}\]. Then \(M\) is a \(20 \times 20\) matrix representing an element \(h \in G_X\) with respect to the fixed basis of \(S_X\), and we have \(D' = D_j^h\). In this case, we also present the following:

  * \text{chamberdata[i][adjacentweyl][k]}. A Weyl vector \(w'\) of \(D'\).
  * \text{chamberdata[i][adjacentDeltaw][k]}. The set \(\Delta_{w'}\).

The action of \(h\) on \(S_X'\) is given by the matrix \(M' := (\text{GramMatS})^{-1} \cdot M \cdot (\text{GramMatS})\) with respect to the dual basis. Let \(w_j\) be the Weyl vector \text{chamberdata[j][weyl]} of \(D_j\). Using \text{prMatSdual}, we obtain \(\text{pr}_S(\Delta_{w_j})\) from \text{chamberdata[j][Deltaw]}, and \(\text{pr}_S(\Delta_{w'})\) from \text{chamberdata[i][adjacentDeltaw][k]}. Then \(M'\) maps \(\text{pr}_S(\Delta_{w_j})\) maps \(\text{pr}_S(\Delta_{w'})\) bijectively.

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