AUTOMORPHISM GROUPS OF THREE SINGULAR K3 SURFACES: EXPLANATION OF THE DATA

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In this note, we explain the computational data related to Section 10 of the paper

[Algo] An algorithm to compute automorphism groups of $K3$ surfaces
and an application to singular $K3$ surfaces

given in the author's web-page

`http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3.html` .

In each of the three text files

`disc11.txt`, `disc15.txt`, `disc16.txt`,

we present a finite set of generators `Gamma` of the image of $\varphi_X : \mathrm{Aut}(X) \to \mathrm{O}^+(S_X)$, where X is a singular $K3$ surface whose transcendental lattice T_X is given by

$$\begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix},$$

respectively.

More precisely, we present the following data in each of these files. We fix bases of the lattices $\mathbf{L}$, S_X and R. To indicate elements of $S_X^\vee$ or $R^\vee$, we use the dual bases of these bases.

- `GramMatL`. A Gram matrix of $\mathbf{L} = U \oplus E_8 \oplus E_8 \oplus E_8$.
- `GramMatS`. A Gram matrix of $S_X = U_\phi \oplus T_X^- \oplus E_8 \oplus E_8$. (The order of basis is different from $S_X = \mathbf{L}_{18}(\phi) \oplus T_X^-$ given in the paper.)
- `GramMatR`. A Gram matrix of the orthogonal complement R of S_X in $\mathbf{L}$.
- `embMatS`. The 20×26 matrix M such that $x \mapsto xM$ gives the primitive embedding of S_X into $\mathbf{L}$.
- `embMatR`. The 6×26 matrix M such that $x \mapsto xM$ gives the primitive embedding of R into $\mathbf{L}$.
- `prMatSdual`. The 26×20 matrix M such that $x \mapsto xM$ gives the projection $\mathbf{L} \to S_X^\vee$, where a vector $S_X^\vee$ is expressed by the dual basis.
- `prMatRdual`. The 26×6 matrix M such that $x \mapsto xM$ gives the projection $\mathbf{L} \to R^\vee$, where a vector $R^\vee$ is expressed by the dual basis.

- discgrS. The discriminant group $S_X^\vee/S_X$. If $\texttt{discgrS} = [a]$, then $S_X^\vee/S_X \cong \mathbb{Z}/a\mathbb{Z}$ (in this case, we put $\nu = 1$), while if $\texttt{discgrS} = [a, b]$, then $S_X^\vee/S_X \cong \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ (in this case, we put $\nu = 2$).
- etaS1 and etaS1. etaS1 is a $\nu \times 20$ matrix, and etaS2 is a $20 \times \nu$ matrix. These two data describe the homomorphism $\eta_{S_X} : \mathrm{O}(S_X) \to \mathrm{O}(q_{S_X})$. Suppose that we are given a 20×20 matrix M representing an element g of $\mathrm{O}(S_X)$ with respect to the fixed basis of S_X. Then the action of g on $S_X^\vee$ is given by the matrix $M^\vee := (\texttt{GramMatS})^{-1} \cdot M \cdot (\texttt{GramMatS})$ with respect to the dual basis. (Note that the action of $\mathrm{O}(S_X)$ on S_X is from the right.) Then $\eta_{S_X}(g)$ is given by the $\nu \times \nu$ matrix $\texttt{etaS1} \cdot M^\vee \cdot \texttt{etaS2}$ under an isomorphism $S_X^\vee/S_X \cong \mathbb{Z}/a\mathbb{Z}$ or $S_X^\vee/S_X \cong \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$. Since $C_X = \{\pm 1\}$, we can determine whether $g \in G_X$ or not by etaS1 and etaS2.
- fphi. The vector $f_\phi \in S_X$ of the class of a fiber of $\phi : X \to \mathbb{P}^1$.
- u0. The vector $u_0 \in S_X$. We can confirm that $D_0 \subset N(X)$ by fphi, u0 and the data chamberdata[0][wallorbit] below.
- Gamma. The finite set Γ of generators of the image of $\varphi_X : \mathrm{Aut}(X) \to \mathrm{O}^+(S_X)$ obtained by Algorithm 6.1.
- B. The set $\mathcal{B}$ of (-2)-vectors obtained by Algorithm 6.1.
- maxchamnumb. The maximum of the index i of the chambers D_i in the complete set $\mathbb{D}$ of representatives of G_X-equivalence classes of $\mathcal{R}^*_{\mathbf{L}|S}$-chambers contained in $N(X)$ obtained by Algorithm 6.1.

For $\texttt{i} = 0, \dots, \texttt{maxchamnumb}$, we give the following data of the $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D_i.

- chamberdata[i][weyl]. A Weyl vector w of D_i.
- chamberdata[i][Deltaw]. The set Δ_w, which is a subset of $\mathbf{L}$. Note that the set $\mathrm{pr}_S(\Delta_w) \subset S_X^\vee$ calculated by prMatSdual and chamberdata[i][Deltaw] is a defining set of the chamber D_i. In general, $\mathrm{pr}_S(\Delta_w)$ contains many redundant vectors (vectors that do not define walls of D_i).
- chamberdata[i][orderaut]. The order of $\mathrm{Aut}_{G_X}(D_i)$.
- chamberdata[i][generatorsaut]. A finite set of generators of $\mathrm{Aut}_{G_X}(D_i)$.
- chamberdata[i][numborbitswalls]. The number t of the orbits of the action of $\mathrm{Aut}_{G_X}(D_i)$ on the primitively minimal defining set $\Delta_{S_X^\vee}(D_i)$ of D_i. For $\texttt{k} = 0, \dots, \texttt{t} - 1$, we present the following data of the kth orbit o_k.
  - chamberdata[i][wallorbit][k]. The list of elements of o_k written in terms of the dual basis of $S_X^\vee$. The first member v of chamberdata[i][wallorbit][k] is used as the representative in Steps 2-3 and 2-4 of Algorithm 6.1.
  - chamberdata[i][adjacent][k]. The description of the $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D' adjacent to D_i along the wall $(v)^\perp \in o_k^*$.
    - If it is [_minustwo], the orbit o_k satisfies $o_k^* \subset \mathcal{R}^*_{S_X}$. Hence D' is not calculated. Instead, $r = \alpha v$ with $\alpha \in \mathbb{Z}_{>0}$ and $\alpha^2 v^2 = -2$ is in the set B.

- If it is `[_backto, j]`, then D' is equal to the previously found $\mathcal{R}^*_{\mathbf{L}|S}$-chamber D_j. (The chamber D_i is calculated as an chamber adjacent to D_j.)
- If it is `[_newcham, j]`, then D' is equal to a new representative $D_j \in \mathbb{D}$ of G_X-congruence class.
- Suppose that it is `[_isom, j, _via, M]`. Then M is a 20×20 matrix representing an element $h \in G_X$ with respect to the fixed basis of S_X, and we have $D' = D_j^h$. In this case, we also present the following:
  - `chamberdata[i][adjacentweyl][k]`. A Weyl vector w' of D'.
  - `chamberdata[i][adjacentDeltaw][k]`. The set $\Delta_{w'}$.

  The action of h on $S_X^\vee$ is given by the matrix $M^\vee := (\texttt{GramMatS})^{-1} \cdot M \cdot (\texttt{GramMatS})$ with respect to the dual basis. Let w_j be the Weyl vector `chamberdata[j][weyl]` of D_j. Using `prMatSdual`, we obtain $\mathrm{pr}_S(\Delta_{w_j})$ from `chamberdata[j][Deltaw]`, and $\mathrm{pr}_S(\Delta_{w'})$ from `chamberdata[i][adjacentDeltaw][k]`. Then $M^\vee$ maps $\mathrm{pr}_S(\Delta_{w_j})$ maps $\mathrm{pr}_S(\Delta_{w'})$ bijectively.

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