ON THE TOPOLOGY OF PROJECTIVE SUBSPACES IN COMPLEX FERMAT VARIETIES

ALEX DEGTYAREV AND ICHIRO SHIMADA

Abstract. Let X be the complex Fermat variety of dimension $n = 2d$ and degree $m > 2$. We investigate the submodule of the middle homology group of X with integer coefficients generated by the classes of standard d-dimensional subspaces contained in X, and give an algebraic (or rather combinatorial) criterion for the primitivity of this submodule.

1. Introduction

Unless specified otherwise, all (co-)homology groups are with coefficients in $\mathbb{Z}$.

Let X be the complex Fermat variety

$$z_0^m + \cdots + z_{n+1}^m = 0$$

of dimension n and degree $m > 2$ in a projective space $\mathbb{P}^{n+1}$ with homogeneous coordinates $(z_0 : \cdots : z_{n+1})$. Suppose that $n = 2d$ is even. Let $\mathcal{J}$ be the set of all unordered partitions of the index set $\overline{n+1} := \{0, 1, \ldots, n+1\}$ into unordered pairs, *i.e.*, lists

$$J := [[j_0, k_0], \ldots, [j_d, k_d]]$$

of pairs of indices such that

$$\{j_0, k_0, \ldots, j_d, k_d\} = \overline{n+1}, \quad j_i < k_i \ (i = 0, \ldots, d), \quad j_0 < \cdots < j_d, \tag{1.1}$$

and let $\mathcal{B}$ be the set of $(d+1)$-tuples $\beta = (\beta_0, \ldots, \beta_d)$ of complex numbers β_i such that $\beta_i^m = -1$. (Note that we always have $j_0 = 0$.) For $J \in \mathcal{J}$ and $\beta \in \mathcal{B}$, we define the *standard d-space* $L_{J,\beta}$ to be the projective subspace of $\mathbb{P}^{n+1}$ defined by the equations

$$z_{k_i} = \beta_i z_{j_i} \quad (i = 0, \ldots, d). \tag{1.2}$$

The number of these spaces equals $(2d+1)!!\, m^{d+1}$, where $(2d+1)!!$ is the product of all *odd* numbers from 1 to $(2d+1)$. Each standard d-space $L_{J,\beta}$ is contained in X, and hence we have its class $[L_{J,\beta}]$ in the middle homology group $H_n(X)$ of X. Let $\mathcal{L}(X)$ denote the $\mathbb{Z}$-submodule of $H_n(X)$ generated by the classes $[L_{J,\beta}]$ of all standard d-spaces.

In the case $n = 2$, the problem to determine whether $\mathcal{L}(X)$ is primitive in $H_n(X)$ or not was raised by Aoki and Shioda [1] in the study of the Picard groups of Fermat surfaces. In degrees m prime to 6, the primitivity of $\mathcal{L}(X)$ implies that the Picard group of X is generated by the classes of the lines contained in X. Schütt, Shioda and van Luijk [7] studied this problem using the reduction of X at supersingular

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primes. Recently, the first author of the present article solved in [3] this problem affirmatively by means of the Galois covering $X \to \mathbb{P}^2$ and the method of Alexander modules.

The purpose of this paper is to study the subgroup $\mathcal{L}(X) \subset H_n(X)$ for higher-dimensional Fermat varieties. For a non-empty subset $\mathcal{K}$ of $\mathcal{J}$, we denote by $\mathcal{L}_{\mathcal{K}}(X)$ the $\mathbb{Z}$-submodule of $H_n(X)$ generated by the classes $[L_{J,\beta}]$, where $J \in \mathcal{K}$ and $\beta \in \mathcal{B}$.

To state our results, we prepare several polynomials in $\mathbb{Z}[t_1, \dots, t_{n+1}]$, rings, and modules. We put

$$\phi(t) := t^{m-1} + \dots + t + 1, \quad \rho(x, y) := \sum_{\mu=0}^{m-2} x^{\mu} \left(\sum_{\nu=0}^{\mu} y^{\nu} \right).$$

For $J = [[j_0, k_0], \dots, [j_d, k_d]] \in \mathcal{J}$, we put

$$\begin{aligned} \tau_J &:= (t_{k_0} - 1) \cdots (t_{k_d} - 1), \\ \psi_J &:= \tau_J \cdot \phi(t_{j_1} t_{k_1}) \cdots \phi(t_{j_d} t_{k_d}), \\ \rho_J &:= \rho(t_{j_1}, t_{k_1}) \cdots \rho(t_{j_d}, t_{k_d}). \end{aligned}$$

Consider the ring

$$\Lambda := \mathbb{Z}[t_0^{\pm 1}, \dots, t_{n+1}^{\pm 1}]/(t_0 \dots t_{n+1} - 1) = \mathbb{Z}[t_1^{\pm 1}, \dots, t_{n+1}^{\pm 1}]$$

of Laurent polynomials and let

$$\begin{aligned} R &:= \Lambda/(t_0^m - 1, \dots, t_{n+1}^m - 1) = \mathbb{Z}[t_1, \dots, t_{n+1}]/(t_1^m - 1, \dots, t_{n+1}^m - 1), \\ \bar{R} &:= R/(\phi(t_0), \dots, \phi(t_{n+1})) = \mathbb{Z}[t_1, \dots, t_{n+1}]/(\phi(t_1), \dots, \phi(t_{n+1})). \end{aligned}$$

For $J = [[j_0, k_0], \dots, [j_d, k_d]] \in \mathcal{J}$, we put

$$\begin{aligned} R_J &:= R/(t_{j_1} t_{k_1} - 1, \dots, t_{j_d} t_{k_d} - 1), \\ \bar{R}_J &:= \bar{R}/(t_{j_1} t_{k_1} - 1, \dots, t_{j_d} t_{k_d} - 1). \end{aligned}$$

Note that we always have $t_{j_0} t_{k_0} - 1 = 0$ in R_J and $\bar{R}_J$. The multiplicative identities of these rings, *i.e.*, the images of $1 \in \Lambda$ under the quotient projection, are denoted by 1_J.

Our primary concern is the structure of the *abelian group* $H_n(X)/\mathcal{L}_{\mathcal{K}}(X)$. For this reason, whenever speaking about the *torsion* of an abelian group A, we always mean its $\mathbb{Z}$-torsion $\operatorname{Tors} A := \operatorname{Tors}_{\mathbb{Z}} A$, even if A happens to be an R- or $\bar{R}$-module. (Over R, almost all our modules have torsion.) Respectively, A is said to be *torsion free* if its $\mathbb{Z}$-torsion $\operatorname{Tors}_{\mathbb{Z}} A$ is trivial.

Our main results are as follows.

Theorem 1.1 (see Section 4). *Let $\mathcal{K}$ be a non-empty subset of $\mathcal{J}$. Then the torsion of the quotient module $H_n(X)/\mathcal{L}_{\mathcal{K}}(X)$ is isomorphic to the torsions of any of the following modules:*

(a) *the ring $R/(\psi_J \mid J \in \mathcal{K})$, where $(\psi_J \mid J \in \mathcal{K})$ is the ideal of R generated by ψ_J with J running through $\mathcal{K}$,*
(b) *the ring $\bar{R}/(\rho_J \mid J \in \mathcal{K})$, where $(\rho_J \mid J \in \mathcal{K})$ is the ideal of $\bar{R}$ generated by ρ_J with J running through $\mathcal{K}$,*
(c) *the R-module*

$$\mathcal{C}_{\mathcal{K}} := \left(\bigoplus_{J \in \mathcal{K}} R_J \right) / \mathcal{M},$$

where $\mathcal{M}$ is the R-submodule of $\bigoplus_{J \in \mathcal{K}} R_J$ generated by $\sum_{J \in \mathcal{K}} \tau_J 1_J$,

(d) *the $\overline{R}$-module*

$$\bar{\mathcal{C}}_{\mathcal{K}} := \left(\bigoplus_{J\in\mathcal{K}} \overline{R}_J\right)/\overline{\mathcal{M}},$$

where $\overline{\mathcal{M}}$ is the $\overline{R}$-submodule of $\bigoplus_{J\in\mathcal{K}} \overline{R}_J$ generated by $\sum_{J\in\mathcal{K}} 1_J$.

In particular, we assert that the torsion parts of all four modules listed in Theorem 1.1 are isomorphic, although not always canonically: sometimes, we use the abstract isomorphism $A \cong \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ for a finite abelian group A, see Section 4.5 for details. It is worth mentioning that, according to [4], in the case $d = 2$ of Fermat surfaces, the *a priori* more complicated module dealt with in [3] (which was found by means of a completely different approach) is isomorphic to the one that is given in Theorem 1.1(c).

Conjecture 1.2. If $\mathcal{K} = \mathcal{J}$, the group $H_n(X)/\mathcal{L}_{\mathcal{K}}(X)$ is torsion free.

This conjecture is supported by some numerical evidence (see Section 5 for details) and by the fact that it holds in the cases $d = 0$ (obvious) and $d = 1$ (see [3]). Theorem 1.1 reduces Conjecture 1.2 to a purely algebraic (or even combinatorial) question. However, for the moment it remains open.

Definition 1.3. Let μ_m be the subgroup $\{z \in \mathbb{C} \mid z^m = 1\}$ of $\mathbb{C}^\times$. Denote by $\Gamma_{\mathcal{K}}$ the subset of $\mu_m^{n+1} = \mathrm{Spec}(R\otimes\mathbb{C})$ consisting of the elements $(a_1, \dots, a_{n+1}) \in \mu_m^{n+1}$ such that $a_i \neq 1$ for $i = 1, \dots, n + 1$ and that there exists $J = [[j_0, k_0], \dots, [j_d, k_d]] \in \mathcal{K}$ such that $a_{j_i} a_{k_i} = 1$ hold for $i = 1, \dots, d$.

Theorem 1.4 (see Section 4.3)**.** *For any non-empty subset $\mathcal{K}$ of $\mathcal{J}$, the rank of the group $\mathcal{L}_{\mathcal{K}}(X)$ is equal to $|\Gamma_{\mathcal{K}}| + 1$.*

As a corollary, we obtain the following statement, which is a higher-dimensional generalization of Corollary 4.4 of [7]:

Corollary 1.5 (see Section 4.3)**.** *For any non-empty subset $\mathcal{K}$ of $\mathcal{J}$, the order of the torsion of $H_n(X)/\mathcal{L}_{\mathcal{K}}(X)$ may be divisible only by those primes that divide m.*

Applying Theorem 1.1 to a subset $\mathcal{K}$ consisting of a single element and using a deformation from X, we also prove the following generalization of Theorem 1.4 of [3]. Let $f_i(x, y)$ be a homogeneous binary form of degree m for $i = 0, \dots, d$. Suppose that the hypersurface W in $\mathbb{P}^{n+1}$ defined by

$$f_0(z_0, z_1) + f_1(z_2, z_3) + \cdots + f_d(z_n, z_{n+1}) = 0 \tag{1.3}$$

is smooth. Then each $f_i(x, y) = 0$ has m distinct zeros $(\alpha_1^{(i)} : \beta_1^{(i)}), \dots, (\alpha_m^{(i)} : \beta_m^{(i)})$ on $\mathbb{P}^1$. Consider the points

$$P_\nu^{(i)} := (0 : \cdots : \underset{(2i)}{\alpha_\nu^{(i)}} : \underset{(2i+1)}{\beta_\nu^{(i)}} : \cdots : 0)$$

of $\mathbb{P}^{n+1}$. Then, for each $(d + 1)$-tuple $(\nu_0, \dots, \nu_d)$ of integers ν_i with $1 \le \nu_i \le m$, the d-space $L'_{(\nu_0,\dots,\nu_d)}$ spanned by $P_{\nu_0}^{(0)}, \dots, P_{\nu_d}^{(d)}$ is contained in W.

Corollary 1.6 (see Section 4.6)**.** *The submodule of $H_n(W)$ generated by the classes $[L'_{(\nu_0,\dots,\nu_d)}]$ of the m^{d+1} subspaces $L'_{(\nu_0,\dots,\nu_d)}$ contained in W is of rank $(m-1)^{d+1}+1$ and is primitive in $H_n(W)$.*

The last statement can further be extended to what we call a *partial Fermat variety*, *i.e.*, a hypersurface $W_s \subset \mathbb{P}^{n+1}$ given by equation (1.3) with

$$f_0(x, y) = \cdots = f_s(x, y) = x^m + y^m$$

and the remaining polynomials distinct (pairwise and from $x^m + y^m$) and generic. Such a variety contains $(2s+1)!!\, m^{d+1}$ projective linear subspaces L'_* of dimension d: each subspace can be obtained as the projective span of one of the s-spaces in the Fermat variety

$$X(2s) := W_s \cap \{z_{2s+2} = \cdots = z_{n+1} = 0\} \subset \mathbb{P}^{2s+1}$$

and one of the $(d-s)$-tuples of points $P^{(s+1)}_{\nu_{s+1}}, \ldots, P^{(d)}_{\nu_d}$ as above. Then, we have the following conditional statement.

Corollary 1.7 (see Section 4.6). *Assume that the statement of Conjecture 1.2 holds for Fermat varieties of dimension $2s \geq 0$. Then, for any $d \geq s$, the submodule of $H_n(W_s)$ generated by the classes $[L'_*]$ of the subspaces L'_* contained in W_s is primitive in $H_n(W_s)$. In particular, this submodule is primitive for $s = 0$ or 1.*

We conclude this introductory section with a very brief outline of the other developments related to the subject.

In [10] and [12], the $\mathbb{Q}$-Hodge structure on the rational cohomology $H^n(X, \mathbb{Q})$ was intensively investigated. Letting $\zeta := e^{2\pi\sqrt{-1}/m}$, the tensor product $H^n(X) \otimes \mathbb{Q}(\zeta)$ decomposes into simple representations of a certain abelian group G (see Section 2 below), which are all of dimension 1 and pairwise distinct. This decomposition is compatible with the Hodge filtration, and the Hodge indices of the summands are computed explicitly. As a by-product of this computation, one concludes that, at least if the degree m is a prime, the space of rational Hodge classes $H^{d,d}(X) \cap H^n(X, \mathbb{Q})$ is generated by the classes of the standard d-spaces. (See also Ran [6].) (In the special case $d = 1$ of surfaces, this rational generation property holds for all degrees prime to 6.) It is this fact that motivates our work and makes the study of the torsion of the quotient $H_n(X)/\mathcal{L}_{\mathcal{J}}(X)$ particularly important: if this torsion is trivial, the classes of the standard d-spaces generate the $\mathbb{Z}$-module of *integral* Hodge classes $H^{d,d}(X) \cap H^n(X, \mathbb{Z})$.

In [8], we investigated the Fermat variety X_{q+1} of even dimension and degree $q + 1$ in characteristic $p > 0$, where q is a power of p. By considering the middle-dimensional subspaces contained in X_{q+1}, we showed that the discriminant of the lattice of numerical equivalence classes of middle-dimensional algebraic cycles of X_{q+1} is a power of p. Note that the rank of this lattice is equal to the middle Betti number of X_{q+1}, that is, X_{q+1} is supersingular.

In [9], we suggested a general method to calculate the primitive closure in $H^2(Y)$ of the lattice generated by the classes of given curves on a complex algebraic surface Y. As an example, we applied this method to certain branched covers of the complex projective plane.

In [4], the method of [3] was generalized to the calculation of the Picard groups of the so-called *Delsarte surfaces* Y. More precisely, the computation of the Picard rank was suggested in [11], and [4] deals with the (im-)primitivity of the subgroup $\mathcal{L}(Y) \subset H_2(Y)$ generated by the classes of certain "obvious" divisors. In a few cases, this subgroup is primitive, but as a rule the quotient $H_2(Y)/\mathcal{L}(Y)$ does have a certain controlled torsion.

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Notation. By $(a, \dots, \underset{(i)}{b}, \dots a)$, we denote a vector whose ith coordinate is b and other coordinates are a. The hat $\hat{}$ means omission of an element; for example, by $(a_1, \dots, \hat{a_i}, \dots, a_N)$, we denote the vector $(a_1, \dots, a_{i-1}, a_{i+1} \dots, a_N)$.

2. An outline of the proof

To avoid confusion, let us denote by $\mathbf{P}^{n+1}$ another copy of the projective space, the one with homogeneous coordinates $(w_0 : \cdots : w_{n+1})$. (Below, we will also use $\mathbf{C}^{n+1}$ for an affine chart of $\mathbf{P}^{n+1}$.) In $\mathbf{P}^{n+1}$, consider the hyperplane Π defined by

$$w_0 + \cdots + w_{n+1} = 0.$$

Then we have the Galois covering $\pi \colon X \to \Pi$ defined by

$$(z_0 : \cdots : z_{n+1}) \mapsto (z_0^m : \cdots : z_{n+1}^m).$$

We put $\zeta := e^{2\pi\sqrt{-1}/m}$. Then the Galois group G of π is generated by

$$\gamma_i \colon (z_0 : \cdots : z_i : \cdots : z_{n+1}) \mapsto (z_0 : \cdots : \zeta z_i : \cdots : z_{n+1})$$

for $i = 0, \dots, n+1$. Since $\gamma_0 \cdots \gamma_{n+1} = 1$, this group G is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^{n+1}$. Throughout this paper, we regard R as the group ring $\mathbb{Z}[G]$ by corresponding $\gamma_i \in G$ to the variable t_i for $i = 1, \dots, n+1$, and $\gamma_0 \in G$ to $t_0 = t_1^{-1} \cdots t_{n+1}^{-1}$. Then we can regard $H_n(X)$ as an R-module. Note that, for any subset $\mathcal{K}$ of $\mathcal{J}$, the subgroup $\mathcal{L}_{\mathcal{K}}(X)$ of $H_n(X)$ is in fact an R-submodule, because, for any $J \in \mathcal{J}$, $g \in G$, and $\beta \in \mathcal{B}$, there exists $\beta' \in \mathcal{B}$ such that $g(L_{J,\beta}) = L_{J,\beta'}$.

Let Y_0 be the hyperplane section of X defined by $\{z_0 = 0\}$, which is G-invariant. Since the fundamental classes $[X] \in H_{2n}(X)$ and $[Y_0] \in H_{2n-2}(Y_0)$ are also fixed by G, the Poincaré–Lefschetz duality isomorphisms

$$H_n(X \setminus Y_0) = H^n(X, Y_0), \quad H_{2n-i}(X) = H^i(X), \quad H_{2n-2-i}(Y_0) = H^i(Y_0)$$

are R-linear; hence, they convert the cohomology exact sequence of the pair (X, Y) into a long exact sequence *of R-modules*

$$\cdots \longrightarrow H_{n-1}(Y_0) \xrightarrow{\partial} H_n(X \setminus Y_0) \xrightarrow{\iota_*} H_n(X) \longrightarrow H_{n-2}(Y_0) \longrightarrow \cdots, \tag{2.1}$$

where $\iota \colon X \setminus Y_0 \hookrightarrow X$ is the inclusion. We then put

$$V_n(X) := \operatorname{Im}(\iota_* \colon H_n(X \setminus Y_0) \to H_n(X)).$$

Since the group $H_{n-2}(Y_0)$ is torsion free, the R-submodule $V_n(X)$ of $H_n(X)$ is primitive in $H_n(X)$ as a $\mathbb{Z}$-submodule.

The structure of the R-module $V_n(X)$ is given by the theory of *Pham polyhedron* developed in [5]. Let $z_0 = 1$ and regard $(z_1, \dots, z_{n+1})$ as affine coordinates on the affine space $\mathbb{C}^{n+1} := \mathbb{P}^{n+1} \setminus \{z_0 = 0\}$, in which $X \setminus Y_0$ is defined by

$$1 + z_1^m + \cdots + z_{n+1}^m = 0.$$

Fix the m-th root $\eta := e^{\pi\sqrt{-1}/m}$ of -1, and consider the (topological) n-simplex

$$D := \{\ (s_1\eta, \dots, s_{n+1}\eta) \ \mid\ s_i \in \mathbb{R},\ \ s_1^m + \cdots + s_{n+1}^m = 1,\ \ 0 \le s_i \le 1\ \}$$

in $X \setminus Y_0$, oriented so that that, if we consider $(s_1, \ldots, s_n)$ as local real coordinates of D at an interior point of D, then

$$(-\partial/\partial s_1, \ldots, -\partial/\partial s_n)$$

is a positively-oriented basis of the real tangent space of D at this point. Then it is easy to see that the chain

$$S := (1-\gamma_1^{-1})\ldots(1-\gamma_{n+1}^{-1})D$$

is a cycle; moreover, it is homeomorphic to the join of $(n+1)$ copies of the two-point space $\{\eta, \zeta\eta\}$, i.e., to the n-sphere. (Here and below, we do not distinguish between "simple" singular chains in X and the corresponding geometric objects, *viz.* unions of simplices with the orientation taken into account and the common parts of the boundary identified. For this reason, we freely apply the module notation to simplices.) Hence, we have the class $[S] \in H_n(X \setminus Y_0)$ and its image $[S] \in V_n(X)$ by ι_*. Pham [5] proved the following:

Theorem 2.1 (see [5]). *The homomorphism $1 \mapsto [S]$ from R to $H_n(X \setminus Y_0)$ induces an isomorphism $\overline{R} \cong H_n(X \setminus Y_0)$ of R-modules, and hence a surjective homomorphism $R \twoheadrightarrow V_n(X)$ of R-modules.*

The Poincaré duality gives rise to symmetric bilinear pairings $\langle\ ,\ \rangle$ on the groups $H_n(X \setminus Y_0)$, $V_n(X)$, and $H_n(X)$, which is interpreted geometrically as the signed intersection of n-cycles brought to a general position. We emphasize that these pairings are $\mathbb{Z}$-bilinear and G-invariant (as so is the fundamental class $[X]$). The homomorphisms $H_n(X \setminus Y_0) \twoheadrightarrow V_n(X) \hookrightarrow H_n(X)$ preserve $\langle\ ,\ \rangle$. Note that $\langle\ ,\ \rangle$ is non-degenerate on $H_n(X)$, but not on $H_n(X \setminus Y_0)$. Later, we will see that $\langle\ ,\ \rangle$ is also nondegenerate on $V_n(X)$.

The main ingredient of the proof of Theorems 1.1 and 1.4 is the following:

Theorem 2.2 (see Section 3). *For $\beta_i \in \mathbb{C}^\times$ with $\beta_i^m = -1$, we put*

$$s(\beta_i) := \begin{cases} 1 & \text{if } \beta_i = \eta, \\ -1 & \text{if } \beta_i = \eta^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

(Recall that we fixed $\eta := e^{\pi\sqrt{-1}/m}$.) For $J = [[j_0, k_0], \ldots, [j_d, k_d]] \in \mathcal{J}$ ordered as in (1.1)*, let σ_J be the permutation*

$$\begin{pmatrix} 0 & 1 & \ldots & \ldots & n & n+1 \\ j_0 & k_0 & \ldots & \ldots & j_d & k_d \end{pmatrix}.$$

Then we have

$$\langle L_{J,\beta}, S\rangle = \operatorname{sgn}(\sigma_J)s(\beta_0)\cdots s(\beta_d),$$

where $\beta = (\beta_0, \ldots, \beta_d) \in \mathcal{B}$.

We use Theorem 2.2 and the fact that the pairing on $H_n(X)$ is nondegenerate to compute the subgroup $\mathcal{L}_{\mathcal{K}}(X) \subset H_n(X)$. Various stages of this computation result in most principal statements of the paper.

3. Intersection of S and the standard d-spaces

In this section, we prove Theorem 2.2. The affine part $X \setminus Y_0$ of X is defined by $1 + z_1^m + \cdots + z_{n+1}^m = 0$ in the affine space $\mathbb{C}^{n+1}$ with coordinates $(z_1, \ldots, z_{n+1})$. We put

$$\mathbf{C}^{n+1} := \mathbf{P}^{n+1} \setminus \{w_0 = 0\},$$

and setting $w_0 = 1$, we regard $(w_1, \ldots, w_{n+1})$ as affine coordinates of $\mathbf{C}^{n+1}$. We put

$$z_i = x_i + \sqrt{-1}y_i, \quad w_i = u_i + \sqrt{-1}v_i,$$

where x_i, y_i, u_i, v_i are real coordinates. Consider the affine hyperplane

$$\Pi^\circ := \Pi \cap \mathbf{C}^{n+1} = \{1 + w_1 + \cdots + w_{n+1} = 0\}$$

of $\mathbf{C}^{n+1}$. In the real part

$$\Pi^\circ \cap \{v_1 = \cdots = v_{n+1} = 0\} = \{\ (u_1, \ldots, u_{n+1}) \in \mathbf{R}^{n+1} \ \mid\ 1 + u_1 + \cdots + u_{n+1} = 0\ \}$$

of Π°, we have an n-simplex Δ defined by

$$1 + u_1 + \cdots + u_{n+1} = 0 \quad \text{and} \quad -1 \le u_i \le 0 \ \text{ for } \ i = 1, \ldots, n+1.$$

Then $\pi\colon X \to \Pi$ induces a homeomorphism $\pi|_D\colon D \xrightarrow{\sim} \Delta$. We put

$$p_i := (0, \ldots, \underset{(i)}{\eta}, \ldots 0) \in D,$$

and put $\bar{p}_i := \pi(p_i) = (0, \ldots, \underset{(i)}{-1}, \ldots, 0)$. Then $\bar{p}_1, \ldots, \bar{p}_{n+1}$ are the vertices of Δ.

Remark 3.1. Note that $S \subset \pi^{-1}(\Delta)$, and that

$$S \cap \pi^{-1}(\{\bar{p}_1, \ldots, \bar{p}_{n+1}\}) = \{p_1, \gamma_1^{-1}(p_1), \ldots, p_{n+1}, \gamma_{n+1}^{-1}(p_{n+1})\}.$$

Remark 3.2. By the definition of the orientation of D given in Section 2, we see that, locally at p_i, the n-chain D is identified with the product

$$(-1)^{i+1}\ \overrightarrow{p_ip_1} \times \cdots \times \overrightarrow{p_ip_{i-1}} \times \overrightarrow{p_ip_{i+1}} \times \cdots \times \overrightarrow{p_ip_{n+1}}$$

of 1-chains, where $\overrightarrow{p_ip_k}$ is the 1-dimensional edge of D connecting p_i and p_k and oriented from p_i to p_k.

By the condition (1.1) on $\mathcal{J}$, we always have $j_0 = 0$. Let b_0 be an element of $\mathbb{Z}/m\mathbb{Z}$ such that

$$\beta_0 = \eta^{1+2b_0} = \zeta^{b_0}\eta.$$

In the affine coordinates $(z_1, \ldots, z_{n+1})$ of $\mathbb{C}^{n+1}$, the equations (1.2) of $L_{J,\beta}$ are written as

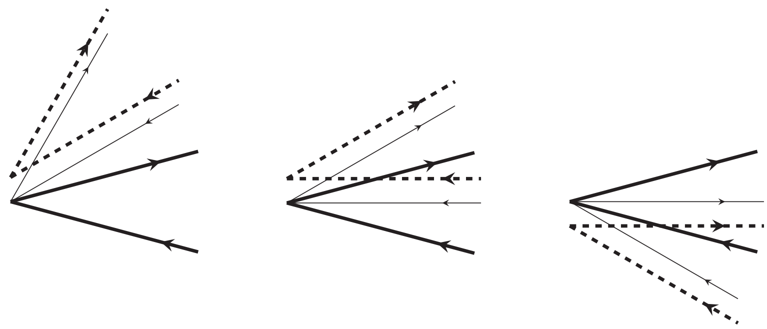
$$z_{k_0} = \beta_0, \quad z_{k_i} = \beta_i z_{j_i} \quad (i = 1, \ldots, d). \tag{3.1}$$

If (3.1) holds, then we have $z_{k_i}^m = -z_{j_i}^m$ for $i = 1, \ldots, d$, and hence $L_{J,\beta} \cap \pi^{-1}(\Delta)$ consists of a single point

$$(0, \ldots, \underset{(k_0)}{\beta_0}, \ldots 0) = \gamma_{k_0}^{b_0}(p_{k_0})$$

by Remark 3.1. Therefore, we have

$$L_{J,\beta} \cap S = \begin{cases} \emptyset & \text{if } \beta_0 \ne \eta \text{ and } \beta_0 \ne \eta^{-1}, \\ \{p_{k_0}\} & \text{if } \beta_0 = \eta, \\ \{\gamma_{k_0}^{-1}(p_{k_0})\} & \text{if } \beta_0 = \eta^{-1}. \end{cases}$$



$\beta = \eta^3$ $\beta = \eta$ $\beta = \eta^{-1}$

FIGURE 3.1. $W(\pi/m)$, $f(W(\pi/m))$ and $\tilde{f}(W(\pi/m))$

In particular, we have

$$\langle L_{J,\beta}, S\rangle = 0 \quad \text{if } \beta_0 \neq \eta \text{ and } \beta_0 \neq \eta^{-1}. \tag{3.2}$$

In order to calculate $\langle L_{J,\beta}, S\rangle$ in the cases where $\beta_0 = \eta^{\pm 1}$, we need the following lemma. For an angle θ, we consider the oriented real semi-line

$$H(\theta) := \mathbb{R}_{\geq 0}\, e^{\sqrt{-1}\theta} \quad \text{with the orientation from 0 to } e^{\sqrt{-1}\theta}$$

on the complex plane $\mathbb{C}$, and define the chain (with closed support)

$$W(\theta) := H(\theta) - H(\theta - 2\pi/m) = (1 - \gamma^{-1})H(\theta),$$

where $\gamma\colon \mathbb{C} \to \mathbb{C}$ is the multiplication by $\zeta = e^{2\pi\sqrt{-1}/m}$. Note that $W(\pi/m) = H(\pi/m) - H(-\pi/m)$. Let $\mathbb{C}^2$ be equipped with coordinates (z, z'). For $\beta_i \in \mathbb{C}$ with $\beta_i^m = -1$, we denote by Λ_{β_i} the linear subspace of $\mathbb{C}^2$ defined by $z' = \beta_i z$.

Lemma 3.3. *The local intersection number $\ell(\beta_i)$ at the origin in $\mathbb{C}^2$ of the chains $W(\pi/m) \times W(\pi/m)$ and Λ_{β_i} is equal to $s(\beta_i)$.*

Proof. The linear subspace Λ_{β_i} is the graph of the function $f\colon z \mapsto z' = \beta_i z$, and hence $f(W(\pi/m))$ is obtained by rotating $W(\pi/m)$ by $\beta_i \in \mathbb{C}^\times$. Let ε and ε' be sufficiently small positive real numbers. We perturb Λ_{β_i} locally at the origin to the graph $\tilde{\Lambda}_{\beta_i}$ of the function

$$\tilde{f}\colon z \mapsto z' = \beta_i z + \varepsilon e^{\sqrt{-1}\tau}\rho(|z|),$$

where $\rho\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the function

$$\rho(x) = \begin{cases} 1 & \text{if } x \leq \varepsilon', \\ 2 - x/\varepsilon' & \text{if } \varepsilon' \leq x \leq 2\varepsilon', \\ 0 & \text{if } 2\varepsilon' \leq x. \end{cases}$$

The direction τ of the perturbation is given as in Figure 3.1, where $W(\pi/m)$ are drawn by thick arrows, $f(W(\pi/m))$ are drawn by thin arrows and $\tilde{f}(W(\pi/m))$ are drawn by broken arrows.

Suppose that $\beta_i \neq \eta$ and $\beta_i \neq \eta^{-1}$. As Figure 3.1 illustrates in the case $\beta_i = \eta^3$, we see that $\tilde{f}(W(\pi/m))$ and $W(\pi/m)$ are disjoint, and hence

$$\tilde{\Lambda}_{\beta_i} \cap (W(\pi/m) \times W(\pi/m)) = \emptyset.$$

Therefore $\ell(\beta_i) = 0$.

Suppose that $\beta_i = \eta$. Then the intersection of $\tilde{\Lambda}_\eta$ and $W(\pi/m) \times W(\pi/m)$ consists of a single point $(Q, \tilde{f}(Q))$, where $Q \in H(-\pi/m)$ and $\tilde{f}(Q) \in H(\pi/m)$. We choose a positively-oriented basis of the real tangent space of $\mathbb{C}^2$ at this point as

$$(\partial/\partial x, \partial/\partial y, \partial/\partial x', \partial/\partial y'), \quad \text{where} \quad z = x + \sqrt{-1}y, \quad z' = x' + \sqrt{-1}y'.$$

The positively-oriented basis of the tangent space of $\tilde{\Lambda}_\eta$ at $(Q, \tilde{f}(Q))$ is

$$(1, 0, \cos(\pi/m), \sin(\pi/m)), \quad (0, 1, -\sin(\pi/m), \cos(\pi/m)),$$

while the positively-oriented basis of the tangent space of $W(\pi/m) \times W(\pi/m)$ at $(Q, \tilde{f}(Q)) \in H(-\pi/m) \times H(\pi/m)$ is

$$(-\cos(-\pi/m), -\sin(-\pi/m), 0, 0), \quad (0, 0, \cos(\pi/m), \sin(\pi/m)).$$

(Note that $W(\pi/m)$ is oriented *toward* the origin on $H(-\pi/m)$.) Calculating the sign of the determinant of the 4×4 matrix with row vectors being the four vectors above in this order, we see that $\ell(\eta) = 1$.

Suppose that $\beta_i = \eta^{-1}$. Then $\tilde{\Lambda}_{\eta^{-1}} \cap W(\pi/m) \times W(\pi/m)$ consists of a single point $(Q, \tilde{f}(Q))$, where $Q \in H(\pi/m)$ and $\tilde{f}(Q) \in H(-\pi/m)$. The positively-oriented basis of the tangent space of $\tilde{\Lambda}_{\eta^{-1}}$ at $(Q, \tilde{f}(Q))$ is

$$(1, 0, \cos(-\pi/m), \sin(-\pi/m)), \quad (0, 1, -\sin(-\pi/m), \cos(-\pi/m)),$$

while that of $W(\pi/m) \times W(\pi/m)$ at $(Q, \tilde{f}(Q)) \in H(\pi/m) \times H(-\pi/m)$ is

$$(\cos(\pi/m), \sin(\pi/m), 0, 0), \quad (0, 0, -\cos(-\pi/m), -\sin(-\pi/m)).$$

Calculating the determinant, we see that $\ell(\eta^{-1}) = -1$. $\square$

Let p be p_i or $\gamma_i^{-1}(p_i)$. In a small neighborhood $\mathcal{U}_p$ of p in $X \setminus Y_0$, we have local coordinates $(z_1, \dots, \hat{z}_i, \dots, z_{n+1})$ of $X \setminus Y_0$. Let

$$\iota_p\colon\ \mathcal{U}_p \hookrightarrow \mathbb{C} \times \cdots \times \mathbb{C} \quad (n \text{ factors})$$

be the open immersion defined by $(z_1, \dots, \hat{z}_i, \dots, z_{n+1})$. We consider an element

$$g := \gamma_1^{\nu_1} \cdots \gamma_{n+1}^{\nu_{n+1}} \ \in\ G,$$

and give a local description of $g(D)$ at $p = p_i$ and $p = \gamma_i^{-1}(p_i)$ via ι_p.

(1) Locally around $p = p_i$. If $\nu_i \neq 0$, then $p_i \notin g(D)$ and hence $\mathcal{U}_p \cap g(D) = \emptyset$. Suppose that $\nu_i = 0$. Using Remark 3.2 and the fact that g preserves the orientation, we see that $g(D)$ is identified with

$$\begin{aligned}(-1)^{i+1}\ & H((2\nu_1+1)\pi/m) \times \cdots \times H((2\nu_{i-1}+1)\pi/m) \times \\ & \qquad H((2\nu_{i+1}+1)\pi)/m) \times \cdots \times H((2\nu_{n+1}+1)\pi)/m).\end{aligned} \tag{3.3}$$

(2) Locally around $p = \gamma^{-1}(p_i)$. If $\nu_i \neq -1$, then $\gamma^{-1}(p_i) \notin g(D)$ and hence $\mathcal{U}_p \cap g(D)$ is empty. Suppose that $\nu_i = -1$. Then $g(D)$ is identified with (3.3) because the action of γ_i maps the local descriptions of $g(D)$ at $\gamma_i^{-1}(p_i)$ to that of $\gamma_i g(D)$ at p_i.

We put

$$S_i := (1 - \gamma_1^{-1}) \cdots (1 - \gamma_{i-1}^{-1})(1 - \gamma_{i+1}^{-1}) \cdots (1 - \gamma_{n+1}^{-1}) D$$

(note that γ_i is missing), which is a hemisphere of the n-sphere S containing p_i. The other hemisphere is $\gamma_i^{-1}(S_i)$, and we have $S = S_i - \gamma_i^{-1}(S_i)$. Since $p_i \in S_i$ and $p_i \notin \gamma_i^{-1}(S_i)$, S is identified with

$$(-1)^{i+1}\, W(\pi/m) \times \cdots \times W(\pi/m)$$

locally at p_i by ι_{p_i}; while since $\gamma_i^{-1}(p_i) \notin S_i$ and $\gamma_i^{-1}(p_i) \in \gamma_i^{-1}(S_i)$, S is identified with

$$-(-1)^{i+1}\, W(\pi/m) \times \cdots \times W(\pi/m)$$

locally at $\gamma_i^{-1}(p_i)$ by $\iota_{\gamma_i^{-1}(p_i)}$.

Suppose that $\beta_0 = \eta$. We calculate the local intersection number of $L_{J,\beta}$ and S at $p := p_{k_0}$. As was shown above, the topological n-cycle S is identified locally at p with

$$(-1)^{k_0+1} W(\pi/m) \times \cdots \times W(\pi/m)$$

by the local coordinates $(z_1, \ldots, \hat{z_{k_0}}, \ldots, z_{n+1})$ of $X \setminus Y_0$ with the origin p. Note that $\{1, \ldots, \hat{k_0}, \ldots, n+1\}$ is equal to $\{j_1, k_1, \ldots, j_d, k_d\}$. We permute the coordinate system $(z_1, \ldots, \hat{z_{k_0}}, \ldots, z_{n+1})$ to

$$(z_{j_1}, z_{k_1}, \ldots, z_{j_d}, z_{k_d}),$$

and define a new open immersion

$$\iota'_p\colon\ \mathcal{U}_p \hookrightarrow \overbrace{\mathbb{C} \times \cdots \times \mathbb{C}}^{n \text{ times}} = \overbrace{\mathbb{C}^2 \times \cdots \times \mathbb{C}^2}^{d \text{ times}}$$

by this new coordinate system. By ι'_p, the topological n-cycle S is identified locally at p with

$$(-1)^{k_0+1}\,\mathrm{sgn}(\sigma'_J)\, W(\pi/m) \times \cdots \times W(\pi/m),$$

where σ'_J is the permutation

$$\begin{pmatrix} 1 & & \ldots & \hat{k_0} & \ldots & n & n+1 \\ j_1 & k_1 & \ldots & & \ldots & j_d & k_d \end{pmatrix}.$$

On the other hand, $L_{J,\beta}$ is identified by ι'_p with

$$\Lambda_{\beta_1} \times \cdots \times \Lambda_{\beta_d}$$

locally at p. By Lemma 3.3, we have

$$\langle L_{J,\beta}, S\rangle = (-1)^{k_0+1}\mathrm{sgn}(\sigma'_J) s(\beta_1)\cdots s(\beta_d) \quad \text{if } \beta_0 = \eta. \tag{3.4}$$

Suppose that $\beta_0 = \eta^{-1}$. We calculate the local intersection number of $L_{J,\beta}$ and S at $p := \gamma_{k_0}^{-1}(p_{k_0})$. As was shown above, the new open immersion ι'_p identifies S with

$$-(-1)^{k_0+1}\mathrm{sgn}(\sigma'_J) W(\pi/m) \times \cdots \times W(\pi/m),$$

locally at p. Calculating as above, we have

$$\langle L_{J,\beta}, S\rangle = -(-1)^{k_0+1}\mathrm{sgn}(\sigma'_J) s(\beta_1)\cdots s(\beta_d) \quad \text{if } \beta_0 = \eta^{-1}. \tag{3.5}$$

The dependence on β_0 in the right-hand sides of (3.2), (3.4), (3.5) can be expressed by the extra factor $s(\beta_0)$. Observing that $(-1)^{k_0+1}\mathrm{sgn}(\sigma'_J) = \mathrm{sgn}(\sigma_J)$, we complete the proof of Theorem 2.2. $\square$

4. The R-submodule $\mathcal{L}_{\mathcal{K}}(X)$

4.1. **Preliminaries.** For an R-module M, we put $M^\vee := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, which is regarded as an R-module *via* the contragredient action of G on $M^\vee$.

Let M be a finitely generated $\mathbb{Z}$-module. We put $d_M := \operatorname{rank} M = \dim_{\mathbb{Q}} M \otimes \mathbb{Q}$. Note that M is torsion free if and only if it can be generated by d_M elements.

Lemma 4.1. *Let $x_1, \dots, x_N$ be variables. We put*

$$A := \mathbb{Z}[x_1, \dots, x_N]/(x_1^m - 1, \dots, x_N^m - 1),$$

and $\theta := (x_1 - 1) \cdots (x_N - 1)$. Then $A/(\theta)$ is torsion free as a $\mathbb{Z}$-module. Moreover the annihilator ideal of θ in A is generated by $\phi(x_1), \dots, \phi(x_N)$.

Proof. We fix the monomial order `grevlex` on $\mathbb{Z}[x_1, \dots, x_N]$ (see [2, Chapter 2]). Since the leading coefficients of $x_1^m - 1, \dots, x_N^m - 1$ and θ are 1, the division algorithm by the set of these polynomials can be carried out over $\mathbb{Z}$. Then we see that $A/(\theta)$ is generated as a $\mathbb{Z}$-module by

$$x_1^{\nu_1} \cdots x_N^{\nu_N} \quad \text{with } 0 \le \nu_i < m \text{ for all } i \text{ and } \nu_i = 0 \text{ for at least one } i. \tag{4.1}$$

On the other hand, the reduced 0-dimensional scheme $\mathrm{Spec}(A/(\theta) \otimes \mathbb{C})$ consists of the closed points

$$(a_1, \dots, a_N) \in \mu_m^N \quad \text{with } a_i = 1 \text{ for at least one } i. \tag{4.2}$$

The number of monomials in (4.1) is equal to the number of points in (4.2), and the latter is equal to $d_{A/(\theta)}$. Hence, by the observation above, we see that $A/(\theta)$ is torsion free. The second part also follows from the division algorithm over $\mathbb{Z}$ by $\{\phi(x_1), \dots, \phi(x_N)\}$ of monic polynomials of degree $m - 1$. □

4.2. **Proof of Part (a) of Theorem 1.1.** We define a non-degenerate symmetric bilinear form $[\ ,\ ]\colon R \times R \to \mathbb{Z}$ by

$$[\, t_1^{\nu_1} \cdots t_{n+1}^{\nu_{n+1}} \,,\, t_1^{\nu_1'} \cdots t_{n+1}^{\nu_{n+1}'} \,] := \delta_{\nu_1 \nu_1'} \dots \delta_{\nu_{n+1} \nu_{n+1}'},$$

where δ_{ij} is the Kronecker delta on $\mathbb{Z}/m\mathbb{Z}$. Since $[\ ,\ ]$ obviously is unimodular and satisfies $[gf, gf'] = [f, f']$ for $f, f' \in R$ and $g \in G$, it induces an isomorphism $R \cong R^\vee$ of R-modules. Note that the image of the dual homomorphism $f^\vee\colon M^\vee \to R$ of an R-linear homomorphism $f\colon R \to M$ is an ideal of R, and the cokernel of $f^\vee$ is always torsion free, because

$$\mathrm{Im}\, f^\vee = \{\, x \in R \mid [x, y] = 0 \text{ for any } y \in \mathrm{Ker}\, f \,\}.$$

In particular, the surjective homomorphism $R \twoheadrightarrow V_n(X)$ in Theorem 2.1 defines an ideal $V_n(X)^\vee \hookrightarrow R$ of R such that $R/V_n(X)^\vee$ is torsion free as a $\mathbb{Z}$-module. On the other hand, the G-invariant intersection pairing $\langle\ ,\ \rangle$ defines an isomorphism $H_n(X) \cong H_n(X)^\vee$ of R-modules. Hence we obtain the dual homomorphism $H_n(X) \to V_n(X)^\vee$ of $V_n(X) \hookrightarrow H_n(X)$, which is surjective because $V_n(X)$ is primitive in $H_n(X)$ (see (2.1)). By construction, the composite $H_n(X) \to R$ of the two homomorphisms $H_n(X) \twoheadrightarrow V_n(X)^\vee$ and $V_n(X)^\vee \hookrightarrow R$ maps $\tau \in H_n(X)$ to

$$\sum_{\nu_1, \dots, \nu_{n+1} \in \mathbb{Z}/m\mathbb{Z}} \langle\, \tau, \gamma_1^{\nu_1} \cdots \gamma_{n+1}^{\nu_{n+1}}(S) \,\rangle \cdot t_1^{\nu_1} \cdots t_{n+1}^{\nu_{n+1}} \quad \in \quad R.$$

Consider the composite

$$\mathcal{L}_{\mathcal{K}}(X) \hookrightarrow H_n(X) \twoheadrightarrow V_n(X)^\vee,$$

where the second homomorphism is the dual of $V_n(X) \hookrightarrow H_n(X)$. Let $\mathcal{L}'_{\mathcal{K}}(X)$ be the image of this composite. We have the following:

Claim 4.2. One has $\operatorname{rank} \mathcal{L}_{\mathcal{K}}(X) = \operatorname{rank} \mathcal{L}'_{\mathcal{K}}(X) + 1$, and

$$H_n(X)/\mathcal{L}_{\mathcal{K}}(X) \cong V_n(X)^\vee/\mathcal{L}'_{\mathcal{K}}(X).$$

Proof. Let $P_X \in H_n(X)$ denote the class of the intersection of X and a $(d+1)$-dimensional subspace of $\mathbb{P}^{n+1}$. By the Lefschetz hyperplane section theorem, the kernel of $H_n(X) \twoheadrightarrow V_n(X)^\vee$ is $\mathbb{Z}P_X$. Therefore it is enough to show that $\mathcal{L}_{\mathcal{K}}(X)$ contains P_X. Since $\mathcal{K}$ is non-empty, we can assume by a permutation of coordinates that $J_0 := [[0,1],[2,3],\dots,[n,n+1]]$ is an element of $\mathcal{K}$. Consider the $(d+1)$-dimensional subspace of $\mathbb{P}^{n+1}$ defined by

$$z_2 - \eta z_3 = z_4 - \eta z_5 = \cdots = z_{2d} - \eta z_{2d+1} = 0.$$

Then its intersection with X is defined in $\mathbb{P}^{n+1}$ by

$$z_0^m + z_1^m = z_2 - \eta z_3 = z_4 - \eta z_5 = \cdots = z_n - \eta z_{n+1} = 0,$$

which is the union of m standard d-spaces $L_{[J_0,(\eta\zeta^\nu,\eta,\dots,\eta)]}$ for $\nu = 0,\dots,m-1$ in X. Thus we have $P_X \in \mathcal{L}_{\mathcal{K}}(X)$ and Claim 4.2 is proved. □

Since $\mathcal{L}'_{\mathcal{K}}(X)$ is an R-submodule of the ideal $V_n(X)^\vee$ of R and $R/V_n(X)^\vee$ is torsion free, the torsion of $H_n(X)/\mathcal{L}_{\mathcal{K}}(X) \cong V_n(X)^\vee/\mathcal{L}'_{\mathcal{K}}(X)$ is isomorphic to the torsion of $R/\mathcal{L}'_{\mathcal{K}}(X)$. Therefore, in order to prove Part (a) of Theorem 1.1, it is enough to show that the ideal $\mathcal{L}'_{\mathcal{K}}(X)$ of R is generated by the polynomials ψ_J, where J runs through $\mathcal{K}$.

For each $J = [[j_0,k_0],\dots,[j_d,k_d]] \in \mathcal{J}$, we let G acts on the set $\mathcal{B}$ by

$$[g,J](\beta) := (\ \zeta^{-\nu_{k_0}}\beta_0,\ \ \zeta^{\nu_{j_1}-\nu_{k_1}}\beta_1,\dots,\ \ \zeta^{\nu_{j_d}-\nu_{k_d}}\beta_d\ ). \tag{4.3}$$

Then we have

$$g^{-1}(L_{J,\beta}) = L_{J,[g,J](\beta)}.$$

Moreover, for any $\beta,\beta' \in \mathcal{B}$ and $J \in \mathcal{J}$, there exists $g \in G$ such that $\beta' = [g,J](\beta)$. Hence, for a fixed $J \in \mathcal{J}$, the $\mathbb{Z}$-submodule $\mathcal{L}_{\{J\}}(X)$ of $H_n(X)$ generated by the classes $[L_{J,\beta}]$ of $L_{J,\beta}$ ($\beta \in \mathcal{B}$) is the R-submodule generated by a single element $[L_{J,(\eta,\dots,\eta)}]$. It is therefore enough to show that the image ψ'_J of $[L_{J,(\eta,\dots,\eta)}]$ by the homomorphism $\mathcal{L}_{\mathcal{K}}(X) \hookrightarrow H_n(X) \twoheadrightarrow V_n(X)^\vee \hookrightarrow R$ is equal to ψ_J up to sign.

Suppose that

$$\psi'_J = \sum a_{\nu_1\dots\nu_{n+1}} t_1^{\nu_1}\cdots t_{n+1}^{\nu_{n+1}},$$

where the summation is taken over all $(n+1)$-tuples $(\nu_1,\dots,\nu_{n+1}) \in (\mathbb{Z}/m\mathbb{Z})^{n+1}$, and $a_{\nu_1\dots\nu_{n+1}} \in \mathbb{Z}$. For simplicity, we put

$$e(\nu) := s(\zeta^{-\nu}\eta) = \begin{cases} 1 & \text{if } \nu = 0, \\ -1 & \text{if } \nu = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, writing $\gamma_1^{\nu_1}\cdots\gamma_{n+1}^{\nu_{n+1}}$ by g, we have

$$\begin{aligned} a_{\nu_1\dots\nu_{n+1}} &= \langle L_{J,(\eta,\dots,\eta)}, g(S)\rangle \\ &= \langle g^{-1}(L_{J,(\eta,\dots,\eta)}), S\rangle \\ &= \langle L_{J,[g,J](\eta,\dots,\eta)}, S\rangle \\ &= \operatorname{sgn}(\sigma_J) e(\nu_{k_0}) e(\nu_{k_1}-\nu_{j_1})\cdots e(\nu_{k_d}-\nu_{j_d}). \end{aligned}$$

where the last equality follows from Theorem 2.2. It remains to notice that

$$\sum_{\nu\in\mathbb{Z}/m\mathbb{Z}} e(\nu)t^{\nu} = 1-t \quad \text{and} \quad \sum_{\nu,\nu'\in\mathbb{Z}/m\mathbb{Z}} e(\nu-\nu')t_1^{\nu}t_2^{\nu'} = (1-t_1)\phi(t_1t_2).$$

Therefore we do have $\psi'_J = \pm\psi_J$. □

4.3. **Proof of Theorem 1.4 and Corollary 1.5.** We put

$$A_{\mathcal{K}} := R/(\psi_J \mid J\in\mathcal{K}).$$

Let K_p be an algebraically closed field of characteristic $p\geq 0$. Since

$$\dim_{K_p}(R\otimes K_p) = m^{n+1}$$

does not depend on p, the $\mathbb{Z}$-module $A_{\mathcal{K}}$ has a torsion element of order p if and only if

$$\dim_{K_p}(A_{\mathcal{K}}\otimes K_p) > \dim_{\mathbb{C}}(A_{\mathcal{K}}\otimes\mathbb{C}).$$

On the other hand, by Claim 4.2 and $\mathcal{L}'_{\mathcal{K}}(X) = (\psi_J \mid J\in\mathcal{K})$ in R, we have

$$\operatorname{rank}\mathcal{L}_{\mathcal{K}}(X) = m^{n+1} - \dim_{\mathbb{C}}(A_{\mathcal{K}}\otimes\mathbb{C}) + 1.$$

Therefore it is enough to prove the following:

Claim 4.3. If $p=0$ or $(p,m)=1$, then

$$\dim_{K_p}(A_{\mathcal{K}}\otimes K_p) = m^{n+1} - |\Gamma_{\mathcal{K}}|.$$

Thus, from now on we assume that $p=0$ or $(p,m)=1$. Then $R\otimes K_p$ is a semisimple ring, and all its simple modules have dimension one over K_p: they correspond to the multi-eigenvalues of $(t_1,\dots,t_{n+1})$, which are all m-th roots of unity (*cf.* Definition 1.3 in the case $K_p=\mathbb{C}$). In other words,

$$M := \operatorname{Spec}(R\otimes K_p)$$

is a reduced scheme of dimension zero consisting of m^{n+1} closed points. Then $\operatorname{Spec}(A_{\mathcal{K}}\otimes K_p)$ is a closed subscheme $M_{\mathcal{K}}$ of M, and $\dim_{K_p}(A_{\mathcal{K}}\otimes K_p)$ is the number of closed points of $M_{\mathcal{K}}$. Let $\Gamma_{\mathcal{K}}$ be the subset of M defined by Definition 1.3 with $\mathbb{C}$ replaced by K_p. Note that, for $a\in K_p^{\times}$ with $a^m=1$, we have

$$\phi(a)=0 \iff a\neq 1.$$

Therefore, for $P=(a_1,\dots,a_{n+1})\in M$, we have

$$\begin{aligned}
P\notin M_{\mathcal{K}} &\iff \psi_J(a_1,\dots,a_{n+1})\neq 0 \ \text{ for some } J\in\mathcal{K}\\
&\iff a_{k_0}\neq 1,\dots,a_{k_d}\neq 1 \text{ and } a_{j_1}a_{k_1}=\cdots=a_{j_d}a_{k_d}=1\\
&\qquad\qquad\qquad\qquad \text{for some } J=[[j_0,k_0],\dots,[j_d,k_d]]\in\mathcal{K}\\
&\iff a_i\neq 1 \ \text{ for } i=1,\dots,n+1 \quad\text{and}\\
&\qquad a_{j_1}a_{k_1}=\cdots=a_{j_d}a_{k_d}=1 \ \text{ for some } J=[[j_0,k_0],\dots,[j_d,k_d]]\in\mathcal{K}\\
&\iff P\in\Gamma_{\mathcal{K}}.
\end{aligned}$$

Therefore we have $\dim_{K_p}(A_{\mathcal{K}}\otimes K_p) = |M_{\mathcal{K}}| = |M|-|\Gamma_{\mathcal{K}}|$. This concludes the proof of Claim 4.3 and, hence, that of Theorem 1.4 and Corollary 1.5. □

Remark 4.4. The rank of $\mathcal{L}(X) = \mathcal{L}_{\mathcal{J}}(X) = 1 + |\Gamma_{\mathcal{J}}|$ is equal to the constant term of the expansion of

$$\begin{cases} 1 + (x_1 + \cdots + x_{h-1} + 1 + x_{h-1}^{-1} + \cdots + x_1^{-1})^{n+2} & \text{if } m = 2h \text{ is even,} \\ 1 + (x_1 + \cdots + x_h + x_h^{-1} + \cdots + x_1^{-1})^{n+2} & \text{if } m = 2h+1 \text{ is odd.} \end{cases}$$

For small dimensions n, we have

$$\operatorname{rank} \mathcal{L}(X) = \begin{cases} 3m^2 - 9m + 6 + \delta_m & \text{for } n = 2, \\ 15m^3 - 90m^2 + 175m - 100 + (15m - 39)\delta_m & \text{for } n = 4, \\ 105m^4 - 1050m^3 + 3955m^2 - 6335m + 3325 + & \\ \qquad + (210m^2 - 1302m + 2010)\delta_m & \text{for } n = 6, \end{cases}$$

where $\delta_m \in \{0, 1\}$ satisfies $\delta_m \equiv m - 1 \bmod 2$.

4.4. **Proof of Part (b) of Theorem 1.1.** The following lemma is immediate:

Lemma 4.5. *In* $\mathbb{Z}[x, y]/(x^m - 1, y^m - 1)$*, we have*

$$(y - 1)\phi(xy) = -(x - 1)(y - 1)\rho(x, y).$$

We put

$$\lambda := (t_1 - 1) \cdots (t_{n+1} - 1).$$

By Lemma 4.5, we have

$$\psi_J := \pm\lambda\rho_J.$$

Hence $R/(\psi_J \mid J \in \mathcal{K})$ in Part (a) of Theorem 1.1 is equal to $R/(\lambda\rho_J \mid J \in \mathcal{K})$. Consider the natural exact sequence

$$0 \;\to\; (\lambda)/(\lambda\rho_J \mid J \in \mathcal{K}) \;\to\; R/(\lambda\rho_J \mid J \in \mathcal{K}) \;\to\; R/(\lambda) \;\to\; 0.$$

Since $R/(\lambda)$ is a free $\mathbb{Z}$-module by Lemma 4.1, the torsion of $R/(\psi_J \mid J \in \mathcal{K})$ is isomorphic to the torsion of $(\lambda)/(\lambda\rho_J \mid J \in \mathcal{K})$. The homomorphism $R \twoheadrightarrow (\lambda)$ given by $f \mapsto f\lambda$ identifies (λ) with $\bar{R}$ by Lemma 4.1, and under this identification, the submodule $(\lambda\rho_J \mid J \in \mathcal{K})$ of (λ) coincides with the ideal $(\rho_J \mid J \in \mathcal{K})$ of $\bar{R}$. Therefore we have $(\lambda)/(\lambda\rho_J \mid J \in \mathcal{K}) \cong \bar{R}/(\rho_J \mid J \in \mathcal{K})$. □

4.5. **Proof of Parts (c) and (d) of Theorem 1.1.** Part (c) and Part (d) are dual to Part (a) and Part (b), respectively. We use the following simple observation. Let $\varphi\colon M_1 \to M_2$ be a homomorphism of free $\mathbb{Z}$-modules, and let $\varphi^\vee\colon M_2^\vee \to M_1^\vee$ be the dual of φ. Then there exist canonical isomorphisms

$$\operatorname{Tors}\operatorname{Coker}(\varphi) = \operatorname{Ext}_{\mathbb{Z}}(\operatorname{Tors}\operatorname{Coker}(\varphi^\vee), \mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Tors}\operatorname{Coker}(\varphi^\vee), \mathbb{Q}/\mathbb{Z}),$$

where $\operatorname{Tors} M$ denotes the torsion of a $\mathbb{Z}$-module M. Hence, there also exists a non-canonical isomorphism $\operatorname{Tors}\operatorname{Coker}(\varphi) \cong \operatorname{Tors}\operatorname{Coker}(\varphi^\vee)$.

We put

$$L_{\mathcal{K}} := \bigcup_{J \in \mathcal{K},\ \beta \in \mathcal{B}} L_{J,\beta},$$

and consider the groups

$$H_n(L_{\mathcal{K}}) = \bigoplus_{J \in \mathcal{K},\ \beta \in \mathcal{B}} \mathbb{Z}[L_{J,\beta}], \quad H^n(L_{\mathcal{K}}) = \bigoplus_{J \in \mathcal{K},\ \beta \in \mathcal{B}} \mathbb{Z}[L_{J,\beta}]^\vee,$$

each of which has a natural structure of the R-modules (see (4.3)). The inclusion $L_{\mathcal{K}} \hookrightarrow X$ induces an R-linear homomorphism

$$\varphi\colon H_n(L_{\mathcal{K}}) \to H_n(X).$$

Then $H_n(X)/\mathcal{L}_{\mathcal{K}}(X) = \mathrm{Coker}(\varphi)$. Note that $\langle\ ,\ \rangle$ defines an isomorphism $H_n(X) \cong H_n(X)^\vee$ (the Poincaré duality), and hence we obtain the dual homomorphism

$$\varphi^\vee\colon H_n(X) \to H^n(L_{\mathcal{K}}).$$

By the observation above, the torsion in question is the dual of the torsion of $\mathrm{Coker}(\varphi^\vee)$, and hence these torsions are isomorphic. Consider the composite

$$\varphi_V^\vee\colon R \twoheadrightarrow V_n(X) \hookrightarrow H_n(X) \to H^n(L_{\mathcal{K}}),$$

where the first surjection is given by Theorem 2.1. Since $V_n(X)$ is primitive in $H_n(X)$ (see (2.1)), the torsion of $H_n(X)/\mathcal{L}_{\mathcal{K}}(X)$ is isomorphic to the torsion of $\mathrm{Coker}(\varphi_V^\vee)$. Recall that we regard $H^n(L_{\mathcal{K}})$ as an R-module *via*

$$g([L_{J,\beta}]^\vee) = [L_{J,[g^{-1},J]\beta}]^\vee.$$

For $J = [[j_0,k_0],\dots,[j_d,k_d]] \in \mathcal{K}$, the natural homomorphism

$$R \twoheadrightarrow R[L_{J,(\eta,\dots,\eta)}]^\vee = \bigoplus_{\beta\in\mathcal{B}} \mathbb{Z}[L_{J,\beta}]^\vee \tag{4.4}$$

given by $1 \mapsto [L_{J,(\eta,\dots,\eta)}]^\vee$ identifies $R[L_{J,(\eta,\dots,\eta)}]^\vee$ with

$$R_J = R/(t_{j_1}t_{k_1}-1,\dots,t_{j_d}t_{k_d}-1) = \mathbb{Z}[t_{k_0},\dots,t_{k_d}]/(t_{k_0}^m-1,\dots,t_{k_d}^m-1), \tag{4.5}$$

where the second equality follows from the relations $t_{j_\nu} = t_{k_\nu}^{m-1}$ ($\nu = 1,\dots,d$) in R_J. Indeed, each $t_{j_i}t_{k_i}-1$ is contained in the kernel of (4.4) by the definition (4.3) of the action of G, and both $\mathbb{Z}$-modules R_J and $R[L_{J,(\eta,\dots,\eta)}]^\vee$ are free of rank $m^{d+1} = |\mathcal{B}|$. Hence we have

$$H^n(L_{\mathcal{K}}) = \bigoplus_{J\in\mathcal{K}} R_J.$$

The homomorphism $\varphi_V^\vee$ is given by

$$1 \mapsto \sum_{J\in\mathcal{K}}\sum_{\beta\in\mathcal{B}} \langle S, L_{J,\beta}\rangle [L_{J,\beta}]^\vee.$$

For $J = [[j_0,k_0],\dots,[j_d,k_d]] \in \mathcal{K}$, we have

$$[(\gamma_{k_0}^{-\alpha_0}\cdots\gamma_{k_d}^{-\alpha_d})^{-1}, J](\eta,\dots,\eta) = (\zeta^{-\alpha_0}\eta,\dots,\zeta^{-\alpha_d}\eta),$$

and hence, by Theorem 2.2, we obtain

$$\begin{aligned}
&\sum_{\beta\in\mathcal{B}} \langle S, L_{J,\beta}\rangle [L_{J,\beta}]^\vee \\
= \;& \mathrm{sgn}(\sigma_J) \sum_{\alpha_0\in\mathbb{Z}/m\mathbb{Z}} \cdots \sum_{\alpha_d\in\mathbb{Z}/m\mathbb{Z}} e(\alpha_0)\cdots e(\alpha_d)[L_{J,(\zeta^{-\alpha_0}\eta,\dots,\zeta^{-\alpha_d}\eta)}]^\vee \\
= \;& \mathrm{sgn}(\sigma_J) \sum_{\alpha_0=0}^{1} \cdots \sum_{\alpha_d=0}^{1} e(\alpha_0)\cdots e(\alpha_d)\gamma_{k_0}^{-\alpha_0}\cdots\gamma_{k_d}^{-\alpha_d}[L_{J,(\eta,\dots,\eta)}]^\vee \\
= \;& \mathrm{sgn}(\sigma_J)(1-t_{k_0}^{-1})\cdots(1-t_{k_d}^{-1})[L_{J,(\eta,\dots,\eta)}]^\vee \\
= \;& \mathrm{sgn}(\sigma_J)(t_{k_0}-1)\cdots(t_{k_d}-1)t_{k_0}^{-1}\cdots t_{k_d}^{-1}[L_{J,(\eta,\dots,\eta)}]^\vee \\
= \;& \tau_J c_J,
\end{aligned}$$

where $c_J := \operatorname{sgn}(\sigma_J)t_{k_0}^{-1}\cdots t_{k_d}^{-1}[L_{J,(\eta,\dots,\eta)}]^\vee$. Note that $\operatorname{sgn}(\sigma_J)t_{k_0}^{-1}\cdots t_{k_d}^{-1}$ is a unit in R_J. Replacing the generator $[L_{J,(\eta,\dots,\eta)}]^\vee$ of each factor of $H^n(\mathcal{L}_\mathcal{K}) = \bigoplus_{J\in\mathcal{K}} R_J$ by c_J, the image of $\varphi_V^\vee$ is the R-submodule $\mathcal{M}$ generated by

$$s := \sum_{J\in\mathcal{K}} \tau_J 1_J.$$

Thus Part (c) is proved.

For $J = [[j_0,k_0],\dots,[j_d,k_d]] \in \mathcal{K}$, let (τ_J) be the ideal of R_J generated by τ_J. Then $s \in L_\mathcal{K}^\vee = \bigoplus_{J\in\mathcal{K}} R_J$ is contained in $\bigoplus_{J\in\mathcal{K}}(\tau_J)$. We consider the exact sequence

$$0 \;\to\; \Bigl(\bigoplus_{J\in\mathcal{K}}(\tau_J)\Bigr)/Rs \;\to\; \Bigl(\bigoplus_{J\in\mathcal{K}} R_J\Bigr)/Rs \;\to\; \bigoplus_{J\in\mathcal{K}}(R_J/(\tau_J)) \;\to\; 0.$$

Since

$$R_J/(\tau_J) = \mathbb{Z}[t_{k_0},\dots,t_{k_d}]/(t_{k_0}^m-1,\dots,t_{k_d}^m-1,\tau_J)$$

is a free $\mathbb{Z}$-module by the second equality of (4.5) and Lemma 4.1, the torsion of $\bigoplus_{J\in\mathcal{K}} R_J/Rs$ is isomorphic to the torsion of $\bigoplus_{J\in\mathcal{K}}(\tau_J)/Rs$. On the other hand, the homomorphism $R_J \twoheadrightarrow (\tau_J)$ given by $f\mapsto f\tau_J$ identifies (τ_J) with

$$\overline{R}_J = \mathbb{Z}[t_{k_0},\dots,t_{k_d}]/(\phi(t_{k_0}),\dots,\phi(t_{k_d}))$$

by Lemma 4.1, and under this identification, the element $\tau_J \in (\tau_J)$ corresponds to the multiplicative unit 1_J of $\overline{R}_J$. Therefore, by $\bigoplus_{J\in\mathcal{K}}(\tau_J) \cong \bigoplus_{J\in\mathcal{K}}\overline{R}_J$, the element $s\in\bigoplus_{J\in\mathcal{K}} R_J$ corresponds to $\sum_{J\in\mathcal{K}} 1_J \in \bigoplus_{J\in\mathcal{K}}\overline{R}_J$. Hence $(\bigoplus_{J\in\mathcal{K}}(\tau_J))/Rs$ is isomorphic to $(\bigoplus_{J\in\mathcal{K}}\overline{R}_J)/\overline{\mathcal{M}}$. □

4.6. **Proof of Corollaries 1.6 and 1.7.** To prove Corollary 1.6, we merely put $J_0 := [[0,1],[2,3],\dots,[n,n+1]]$, and apply Part (d) of Theorem 1.1 to the case $\mathcal{K} = \{J_0\}$. We immediately see that $\mathcal{L}_{\{J_0\}}(X)$ is primitive in $H_n(X)$. Let $\mathcal{W} = \{W_t\}_{t\in U}$ be the family of smooth hypersurfaces defined by the equations of the form (1.3). The parameter space U of this family is connected, and hence there exists a path $\gamma\colon [0,1]\to U$ from the Fermat variety $X = W_{\gamma(0)}$ to an arbitrary member $W = W_{\gamma(1)}$ of $\mathcal{W}$. Along the family $W_{\gamma(t)}$, the subspaces $L_{J_0,\beta}$ ($\beta\in\mathcal{B}$) in X deform to subspaces of $W_{\gamma(t)}$ defined by equations of the form

$$\beta_\nu^{(i)}(t)z_{2i} = \alpha_\nu^{(i)}(t)z_{2i+1} \quad (i=0,\dots,d,\;\; \nu=1,\dots,m).$$

Thus, along the constant (with respect to the Gauss–Manin connection) family $H_n(W_{\gamma(t)})$ of $\mathbb{Z}$-modules over γ, the submodule $\mathcal{L}_{\{J_0\}}(X)$ of $H_n(X)$ is transported to the submodule of $H_n(W)$ generated by the classes $[L'_{(\nu_0,\dots,\nu_d)}]$ of subspaces $L'_{(\nu_0,\dots,\nu_d)}$ in W. The rank and the primitivity are preserved during the transport.

For Corollary 1.7, we use the same continuity argument, deforming W_s to the Fermat variety and representing the submodule in question as $\mathcal{L}_{\mathcal{J}_s}(X)$, where $\mathcal{J}_s$ is the set of all partitions "identical beyond s", *i.e.*, those of the form

$$[[j_0,k_0],\dots,[j_s,k_s],[2s+2,2s+3],\dots,[n,n+1]], \quad 0\le j_i,k_i\le 2s+1.$$

The restriction of $\mathcal{J}_s$ to the index set $\overline{2s+1}$ is well-defined and coincides with the full set $\mathcal{J}(2s)$ of partitions of $\overline{2s+1}$. Then, denoting by $(\,\cdot\,)$ the dependence on the dimension (or the number of variables in the polynomial rings), it is easy to see that the module $\bar{\mathcal{C}}_{\mathcal{J}_s}(2d)$ given by Part (d) of Theorem 1.1 can be represented in the form

$$\bar{\mathcal{C}}_{\mathcal{J}_s}(2d) = \bar{\mathcal{C}}_{\mathcal{J}(2s)}(2s) \otimes_{\mathbb{Z}} \bar{S}(s,d),$$

where

$$\bar{S}(s,d) := \mathbb{Z}[t_{2s+2}, t_{2s+4}, \ldots, t_{2d}]/(\phi(t_{2s+2}), \phi(t_{2s+4}), \ldots, \phi(t_{2d})).$$

(Since the tail of each partition is fixed, we have the "constant" relations

$$t_{2s+2}t_{2s+3} = \cdots = t_{2d}t_{2d+1} = 1;$$

hence, we can retain the even index variables only and take these variables out.) Thus, this module is free (as an abelian group) if and only if so is $\bar{\mathcal{C}}_{\mathcal{J}(2s)}(2s)$, *i.e.*, if and only if Conjecture 1.2 holds for Fermat varieties of dimension $2s$ in $\mathbb{P}^{2s+1}$.

For the last assertion of Corollary 1.7, we observe that Conjecture 1.2 does hold for the Fermat varieties of dimension 0 (obvious) and 2 (see [3]). □

5. Computational criterion

In this section, we focus on the description of the torsion of $H_n(X)/\mathcal{L}_{\mathcal{K}}(X)$ given by Part (b) of Theorem 1.1. We put

$$B_{\mathcal{K}} := \bar{R}/(\rho_J \mid J \in \mathcal{K}).$$

By Lemma 4.5, the ideal $(\rho_J \mid J \in \mathcal{K})$ defines the closed subscheme $\Gamma_{\mathcal{K}}$ in the reduced 0-dimensional scheme $\operatorname{Spec}(\bar{R} \otimes \mathbb{C}) = (\mu_m \setminus \{1\})^{n+1}$, and hence we can calculate $d_0 := \dim_{\mathbb{C}}(B_{\mathcal{K}} \otimes \mathbb{C}) = |\Gamma_{\mathcal{K}}|$. On the other hand, for each prime divisor p of m, we can calculate $d_p := \dim_{\mathbb{F}_p}(B_{\mathcal{K}} \otimes \mathbb{F}_p)$ by calculating a Gröbner basis of the ideal

$$(\phi(t_1), \ldots, \phi(t_{n+1})) + (\rho_J \mid J \in \mathcal{K}) \tag{5.1}$$

in the polynomial ring $\mathbb{F}_p[t_1, \ldots, t_{n+1}]$. By Corollary 1.5, we see that $\mathcal{L}_{\mathcal{K}}(X)$ is primitive in $H_n(X)$ if and only if $d_0 = d_p$ holds for any prime divisor p of m.

Using this method, we have confirmed the primitivity of $\mathcal{L}(X) = \mathcal{L}_{\mathcal{J}}(X)$ in $H_n(X)$ by the computer-aided calculation in the following cases:

$$(n,m) = (4,m) \text{ where } 3 \le m \le 12, \quad (6,3), (6,4), (6,5), (8,3).$$

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Bilkent University, Department of Mathematics, 06800 Ankara, TURKEY
E-mail address: degt@fen.bilkent.edu.tr

Department of Mathematics, Graduate School of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526 JAPAN
E-mail address: shimada@math.sci.hiroshima-u.ac.jp