CONNECTED COMPONENTS OF THE MODULI OF ELLIPTIC $K3$ SURFACES

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Abstract. The combinatorial type of an elliptic $K3$ surface with a zero section is the pair of the ADE-type of the singular fibers and the torsion part of the Mordell-Weil group. We determine the set of connected components of the moduli of elliptic $K3$ surfaces with fixed combinatorial type. Our method relies on the theory of Miranda and Morrison on the structure of a genus of even indefinite lattices, and computer-aided calculations of p-adic quadratic forms.

1. Introduction

Elliptic $K3$ surfaces have been intensively studied by many authors from various points of view, not only in algebraic and arithmetic geometry, but also in theoretical physics of string theory. In this paper, we investigate certain moduli of complex elliptic $K3$ surfaces, and determine the connected components of the moduli.

An *elliptic* $K3$ *surface* is a triple (X, f, s), where X is a complex $K3$ surface, $f\colon X \to \mathbb{P}^1$ is a fibration whose general fiber is a curve of genus 1, and $s\colon \mathbb{P}^1 \to X$ is a section of f. An elliptic $K3$ surface (X, f, s) is sometimes denoted simply by f with X and s being understood.

Let (X, f, s) be an elliptic $K3$ surface. Then the set of sections of f has a natural structure of abelian group with zero element s. This group is called the *Mordell-Weil group*. We denote by A_f the torsion part of the Mordell-Weil group of (X, f, s). If an irreducible curve C on X is contained in a fiber of f and is disjoint from the zero section s, then C is a smooth rational curve. The set Φ_f of the classes of these smooth rational curves form an ADE-configuration of vectors of square-norm -2 in $H^2(X, \mathbb{Z})$. (See Section 2.4 for the definition of an ADE-configuration.) The *combinatorial type* of an elliptic $K3$ surface (X, f, s) is the pair (Φ_f, A_f). Let Φ be an ADE-configuration, and A a finite abelian group. We say that an elliptic $K3$ surface (X, f, s) is *of type* (Φ, A) if $\Phi \cong \Phi_f$ and $A \cong A_f$.

In our previous papers [25], [20], we made the complete list of (Φ, A) that can be realized as combinatorial type of elliptic $K3$ surfaces. The cardinality of this list is 3693. In this paper, we refine this result to the following:

Theorem 1.1. *The moduli of elliptic* $K3$ *surfaces of type* (Φ, A) *has more than one connected component if and only if* (Φ, A) *appears in Tables* I *and* II *in Section 7.*

See Section 3.1 for the precise definition of the connected components of the moduli of elliptic $K3$ surfaces of fixed type (Φ, A). In Tables I and II, the ADE-configuration Φ is presented by the ADE-type of the configuration. The finite abelian group $\mathbb{Z}/a\mathbb{Z}$ is denoted by $[a]$, and $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}$ is denoted by $[a, b]$.

2010 *Mathematics Subject Classification.* 14J28, 11E81.

This work was supported by JSPS KAKENHI Grant Number 16H03926, 16K13749.

Tables I and II are obtained by machine-computation. The purpose of this paper is to explain the algorithm to calculate the set of connected components of the moduli.

The non-connectedness of the moduli is caused by two totally different reasons; one is algebraic and the other is transcendental.

For a $K3$ surface X, we denote by $\mathrm{NS}(X) := H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ the *Néron-Severi lattice of* X, and by $T(X)$ the *transcendental lattice of* X; that is, $T(X)$ is the orthogonal complement of $\mathrm{NS}(X)$ in $H^2(X, \mathbb{Z})$.

Let (X, f, s) be an elliptic $K3$ surface. We denote by $U_f \subset H^2(X, \mathbb{Z})$ the sublattice generated by the class of a fiber of f and the class of the section s, by $L(\Phi_f)$ the sublattice of $H^2(X, \mathbb{Z})$ generated by $\Phi_f \subset H^2(X, \mathbb{Z})$, and by $M(\Phi_f)$ the primitive closure of $L(\Phi_f)$ in $H^2(X, \mathbb{Z})$. It is well-known that A_f is isomorphic to $M(\Phi_f)/L(\Phi_f)$. We then denote by T_f the orthogonal complement of $U_f \oplus M(\Phi_f)$ in $H^2(X, \mathbb{Z})$. We obviously have $\mathrm{NS}(X) \supset U_f \oplus M(\Phi_f)$ and $T(X) \subset T_f$.

Definition 1.2. Let $\mathcal{C}$ be a connected component of the moduli of elliptic $K3$ surfaces of type (Φ, A). Suppose that an elliptic $K3$ surface (X, f, s) corresponds to a point of $\mathcal{C}$. The *Néron-Severi lattice of* $\mathcal{C}$ is defined to be the isomorphism class of the lattice $U_f \oplus M(\Phi_f)$, and the *transcendental lattice of* $\mathcal{C}$ is defined to be the isomorphism class of the lattice T_f.

It is obvious that the Néron-Severi lattice and the transcendental lattice of a connected component $\mathcal{C}$ do not depend on the choice of the member (X, f, s) of $\mathcal{C}$. It will be seen that, if (X, f, s) is chosen generally in $\mathcal{C}$, then the Néron-Severi lattice of $\mathcal{C}$ is isomorphic to $\mathrm{NS}(X)$, and the transcendental lattice of $\mathcal{C}$ is isomorphic to $T(X)$. (See the proof of Theorem 3.5.)

Definition 1.3. We say that two elliptic $K3$ surfaces (X, f, s) and (X', f', s') of the same type (Φ, A) are *algebraically equivalent* if there exists an isomorphism $\Phi_f \cong \Phi_{f'}$ of ADE-configurations such that the induced isometry $L(\Phi_f) \cong L(\Phi_{f'})$ maps the even overlattice $M(\Phi_f)$ of $L(\Phi_f)$ to the even overlattice $M(\Phi_{f'})$ of $L(\Phi_{f'})$. If there exist no such isomorphisms $\Phi_f \cong \Phi_{f'}$, we say that (X, f, s) and (X', f', s') are *algebraically distinguished.*

If (X, f, s) and (X', f', s') are algebraically distinguished, then the intersection patterns of the torsion sections and the irreducible components of the reducible fibers for (X, f, s) and for (X', f', s') are different, and hence they cannot be in the same connected component of the moduli.

Definition 1.4. We say that two connected components $\mathcal{C}_1$ and $\mathcal{C}_2$ are *algebraically distinguished* if an elliptic $K3$ surface belonging to $\mathcal{C}_1$ and an elliptic $K3$ surface belonging to $\mathcal{C}_2$ are algebraically distinguished. Otherwise, we say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *algebraically equivalent.*

By definition, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are algebraically equivalent, then their Néron-Severi lattices are isomorphic, but their transcendental lattices may be non-isomorphic.

An elliptic $K3$ surface (X, f, s) is called *extremal* if the rank of $L(\Phi_f)$ attains the possible maximum 18. Suppose that (X, f, s) is extremal. Then the transcendental lattice $T(X)$ of X is an even positive definite lattice of rank 2, and $T(X)$ is equal to the transcendental lattice of the connected component containing (X, f, s).

Explanation of the entries of Tables I and II. Table I is the list of non-connected moduli of extremal elliptic $K3$ surfaces. The horizontal line in the 4th-5th columns separates the connected components that are algebraically distinguished. (This separating line appears only in nos. 27 and 64. See Section 6.1 for the detail of example no. 64.) The 4th column shows the list of components $[a, b, c]$ of the transcendental lattice

$$T = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

written in the reduced form in the sense of Gauss (see [25]). The 5th column displays $[r, c]$, where r (resp. c) is the number of connected components that are (resp. are *not*) invariant under complex conjugation. In particular, the number c is always even.

Table II is the list of non-connected moduli of non-extremal elliptic $K3$ surfaces. The 2nd column shows the rank of $L(\Phi_f)$. The list $[c_1, \dots, c_k]$ in the 5th column indicates that there exist exactly k algebraic equivalence classes of connected components, and that each algebraic equivalence class has exactly c_i connected components. Examining Table II and investigating the set of connected components further (see Remark 4.16), we obtain the following:

Corollary 1.5. *The moduli of non-extremal elliptic $K3$ surfaces of type (Φ, A) has more than one connected component that can not be algebraically distinguished if and only if A is trivial and Φ is one of the following:*

$$\begin{gathered} E_7 + 2A_5, \;\; E_6 + A_{11}, \;\; E_6 + A_6 + A_5, \;\; E_6 + 2A_5 + A_1, \\ D_5 + 2A_6, \;\; D_4 + 2A_6 + A_1, \;\; A_{11} + A_5 + A_1, \;\; A_7 + 2A_5, \\ 2A_6 + A_3 + 2A_1, \;\; A_6 + 2A_5 + A_1, \;\; E_6 + 2A_5, \;\; 3A_5 + A_1. \end{gathered}$$

For each of these types (Φ, A), the moduli has exactly two connected components, and they are complex conjugate to each other.

If (X, f, s) is extremal, then the $K3$ surface X is *singular* in the sense of [26]. It is known that a pair of singular $K3$ surfaces with isomorphic Néron-Severi lattices and non-isomorphic transcendental lattices has some interesting properties. See [19], [23] for arithmetic properties, and [2], [7], and [22] for topological properties. On the other hand, for non-extremal elliptic $K3$ surfaces, Corollary 1.5 implies the following:

Corollary 1.6. *The transcendental lattice of a connected component of the moduli of non-extremal elliptic $K3$ surfaces of fixed type is determined by the algebraic equivalence class of the connected component.*

In fact, we present an algorithm to calculate the set $\mathfrak{C}(\Phi, A, G)$ of G-connected components of the moduli of *marked* elliptic $K3$ surfaces of type (Φ, A), where G is a subgroup of the automorphism group $\mathrm{Aut}(\Phi)$ of the ADE-configuration Φ. (See Section 3.1 for the definition of the set $\mathfrak{C}(\Phi, A, G)$.) Theorem 1.1 and Corollaries 1.5, 1.6 are the statements for the case where G is the full automorphism group $\mathrm{Aut}(\Phi)$, which means that elliptic $K3$ surfaces are not marked. See Section 6.2.

Torelli theorem for the period map of complex $K3$ surfaces [18] enables us to study moduli of $K3$ surfaces by lattice-theoretic tools. In order to investigate moduli of lattice polarized $K3$ surfaces, we have to determine the set of primitive embeddings of the polarizing lattice into the $K3$ lattice. This task is easy when the $K3$ surfaces are singular, because the transcendental lattices are positive definite of rank 2 in this case. When the transcendental lattices are indefinite of rank ≥ 3, we use Miranda-Morrison theory [15, 16, 14]. Let L be an even indefinite lattice of rank ≥ 3, let $\mathcal{G}$ be the genus containing L, and let $\mathrm{O}(L) \to \mathrm{O}(D_L, q_L)$ be the natural homomorphism from the orthogonal group of L to the automorphism group of the discriminant form (D_L, q_L) of L. Miranda and Morrison defined a certain finite abelian group $\mathcal{M}$ that fits in an exact sequence

$$0 \longrightarrow \mathrm{Coker}(\mathrm{O}(L) \to \mathrm{O}(D_L, q_L)) \longrightarrow \mathcal{M} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Then they gave a method to calculate this exact sequence in terms of the spinor norms of certain isometries of the p-adic lattices $L \otimes \mathbb{Z}_p$. When we apply this theory to the study of moduli of elliptic $K3$ surfaces, the genus $\mathcal{G}$ is the genus containing the transcendental lattices of algebraic equivalence classes of connected components of the moduli. We have to incorporate the positive sign structures of L in the theory, and to calculate the action on $\mathcal{M}$ of a subgroup of $\mathrm{Aut}(\Phi)$ explicitly. The flipping of positive sign structures corresponds to the complex conjugation, and the action of a subgroup of $\mathrm{Aut}(\Phi)$ corresponds to changing the marking.

Miranda-Morrison theory was first applied to the study of moduli of $K3$ surfaces by Akyol and Degtyarev [1] in their study of equisingular family of irreducible plane sextics. Recently, Güneş Aktaş [10] used it to the study of certain classes of quartic surfaces. In these works, the calculation of isometries of p-adic lattices and their spinor norms was not fully-automated, and a case-by-case method was employed at several points. The complete list of ADE-types of singularities of these normal $K3$ surfaces had been obtained by Yang ([28], [29]).

A new ingredient of this paper is a refinement of the Miranda-Morrison group $\mathcal{M}$, which enables us to treat the positive sign structures in a simplified way. Another new ingredient is an algorithm to lift a given automorphism of the discriminant form $(D_{L\otimes\mathbb{Z}_p}, q_{L\otimes\mathbb{Z}_p})$ of a p-adic lattice $L \otimes \mathbb{Z}_p$ to an isometry of $L \otimes \mathbb{Z}_p$, and to calculate the spinor norm of this isometry. Our method employs approximate calculations in p-adic topology. To obtain precise results, the estimation of approximation errors is in need. Using this algorithm, we can compute the set $\mathfrak{C}(\Phi, A, G)$ of connected components of our moduli by computer.

This paper is organized as follows. In Section 2, we collect preliminaries about lattices. In particular, we recall the theory of discriminant forms due to Nikulin [17] and its application to the genus theory. In Section 3, we define the set $\mathfrak{C}(\Phi, A, G)$ of G-connected components of the moduli of marked elliptic $K3$ surfaces of fixed type (Φ, A), where G is a subgroup of $\mathrm{Aut}(\Phi)$. We then introduce a set $\mathcal{Q}(\Phi, A)/\!\sim_G$, which is defined in purely lattice-theoretic terms. Using the theory of refined period map of marked $K3$ surfaces [3, Chapter VIII], we show that there exists a natural bijection between $\mathfrak{C}(\Phi, A, G)$ and $\mathcal{Q}(\Phi, A)/\!\sim_G$. In Section 4, we formulate a refinement of Miranda-Morrison theory, and interpret the set $\mathcal{Q}(\Phi, A)/\!\sim_G$ as a finite disjoint union of certain finite dimensional $\mathbb{F}_2$-vector spaces $\mathcal{T}_{\mathcal{G}}/\!\sim_{\bar{G}}$, which are closely related to the Miranda-Morrison group $\mathcal{M}$. Section 5 is the technical core of our algorithm. We present an algorithm to calculate the spinor norm of an isometry of a p-adic lattice that induces a given automorphism of the discriminant form. The results in Sections 4 and 5 establish an algorithm to calculate the $\mathbb{F}_2$-vector spaces $\mathcal{T}_{\mathcal{G}}/\!\sim_{\bar{G}}$. Using this algorithm combined with the results in Section 3, we can compute the set $\mathfrak{C}(\Phi, A, G)$. Applying this calculation to the case $G = \mathrm{Aut}(\Phi)$, we obtain Tables I and II in Section 7. In Section 6, we investigate some examples in detail.

The data computed in this paper is available from the author's web-page [24]. For the computation, we used `GAP` [9].

Thanks are due to Alex Degtyarev, Klaus Hulek and Michael Lönne for many discussions. The author also thanks the referee for many comments and suggestions on the first version of this paper.

Convention. In this paper, a homomorphism $f\colon M \to M'$ of abelian groups is written as $v \mapsto v^f$. In particular, we denote the composite of $f\colon M \to M'$ and $f'\colon M' \to M''$ by ff' or $f \cdot f'$.

2. Lattices

We fix notions and notation about lattices, and recall some classical results. Let R be either $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, or $\mathbb{R}$, and let k be the quotient field of R.

2.1. Gram matrix.

An *R-lattice* is a free R-module L of finite rank equipped with a non-degenerate symmetric bilinear form

$$\langle\ ,\ \rangle\colon L \times L \to R.$$

Let L be an R-lattice of rank n. By choosing a basis $e_1, \dots, e_n$ of the free R-module L, the form $\langle\ ,\ \rangle$ is expressed by a symmetric matrix M of size n whose (i,j)-component is $\langle e_i, e_j\rangle$. This matrix M is called the *Gram matrix* of L with respect to the basis $e_1, \dots, e_n$. The *discriminant* $\mathrm{disc}(L)$ of L is defined by

$$\mathrm{disc}(L) := \det(M) \bmod (R^\times)^2 \quad \in \quad (R \setminus \{0\})/(R^\times)^2.$$

We denote by $\mathrm{O}(L)$ the group of isometries of L. By our convention, the group $\mathrm{O}(L)$ acts on L from the *right.* The determinant of matrices representing isometries of L gives rise to a homomorphism

$$\det\colon \mathrm{O}(L) \to \mathrm{Det} := \{\pm 1\}.$$

Note that $L \otimes k$ has a natural structure of k-lattice, and $\mathrm{O}(L)$ is naturally embedded in $\mathrm{O}(L \otimes k)$.

2.2. Positive sign structure.

Let L be an $\mathbb{R}$-lattice. It is well-known that L has a diagonal Gram matrix M whose diagonal components are ± 1, and that the number s_+ of $+1$ (resp. s_- of -1) on the diagonal is independent of the choice of M. The *signature* $\mathrm{sign}(L)$ of L is (s_+, s_-). We say that L is *indefinite* if $s_+ > 0$ and $s_- > 0$, whereas L is *positive* or *negative definite* if $s_- = 0$ or $s_+ = 0$, respectively. We say that L is *hyperbolic* if $s_+ = 1$.

According to [13], we define a *positive sign structure* of L to be a choice of one of the connected components of the manifold parametrizing *oriented* s_+-dimensional subspaces Π of L such that the restriction $\langle\ ,\ \rangle|_\Pi$ of $\langle\ ,\ \rangle$ to Π is positive-definite. Unless L is negative-definite, L has exactly two positive sign structures.

The signature and the positive sign structures of a $\mathbb{Z}$- or a $\mathbb{Q}$-lattice L are defined to be those of $L \otimes \mathbb{R}$. The orthogonal group $\mathrm{O}(L)$ acts on the set of positive sign structures of L in a natural way.

2.3. **Discriminant form.** The theory of discriminant forms was developed by Nikulin in [17]. A *finite quadratic form* is a quadratic form

$$q\colon D \to \mathbb{Q}/2\mathbb{Z},$$

where D is a finite abelian group. The *length* $\operatorname{leng}(D)$ of D is the minimal number of generators of D. Let (D,q) be a finite quadratic form. We say that (D,q) is *non-degenerate* if the associated symmetric bilinear form $b\colon D\times D \to \mathbb{Q}/\mathbb{Z}$ is non-degenerate, where

$$b(x,y) := \frac{1}{2}(q(x+y)-q(x)-q(y)).$$

We denote by $\mathrm{O}(D,q)$ the automorphism group of (D,q). Note again that $\mathrm{O}(D,q)$ acts on (D,q) from the *right*.

Suppose that R is either $\mathbb{Z}$ or $\mathbb{Z}_p$, and let L be an R-lattice of rank n. The *dual lattice* $L^\vee$ of L is defined by

$$L^\vee := \{\, x \in L\otimes k \;\mid\; \langle x, v\rangle \in R \text{ for all } v\in L \,\}, \tag{2.1}$$

which is a free R-module of rank n containing L as a submodule of finite index. The dual lattice $L^\vee$ has a natural k-valued non-degenerate symmetric bilinear form that extends the R-valued form $\langle\ ,\ \rangle$ on L. We put

$$D_L := L^\vee/L,$$

and call it the *discriminant group* of L. We say that L is *unimodular* if D_L is trivial. When R is $\mathbb{Z}$, the order of D_L is equal to $|\operatorname{disc}(L)|$.

We say that L is *even* if $\langle x,x\rangle \in 2R$ holds for all $x\in L$. (When $R=\mathbb{Z}_p$ with p odd, every R-lattice is even. A $\mathbb{Z}$-lattice L is even if and only if the $\mathbb{Z}_2$-lattice $L\otimes\mathbb{Z}_2$ is even.) Note that we have a natural isomorphism

$$\mathbb{Q}/2\mathbb{Z} \;\cong\; \bigoplus_p \mathbb{Q}_p/2\mathbb{Z}_p.$$

Hence, when $R=\mathbb{Z}_p$, we can regard $k/2R$ as a submodule of $\mathbb{Q}/2\mathbb{Z}$. Suppose that L is even. Then the *discriminant form*

$$q_L\colon D_L \to \mathbb{Q}/2\mathbb{Z}$$

of L is a finite quadratic form defined by $q_L(\bar{x}) := \langle x,x\rangle \bmod 2R$, where $\bar{x}\in D_L$ denotes $x \bmod L$ for $x\in L^\vee$. Since $\langle\ ,\ \rangle$ is non-degenerate, the finite quadratic form (D_L,q_L) is non-degenerate. If $\varphi\colon L \xrightarrow{\sim} L'$ is an isometry of even R-lattices, then φ induces an isomorphism $L^\vee \xrightarrow{\sim} L'^\vee$ and hence an isomorphism

$$q_\varphi\colon (D_L,q_L) \xrightarrow{\sim} (D_{L'},q_{L'})$$

of their discriminant forms. In particular, we have a natural homomorphism

$$\mathrm{O}(L)\to\mathrm{O}(D_L,q_L).$$

Remark 2.1. If we adapt $L^\vee := \operatorname{Hom}(L,R)$ as the definition of the dual lattice, it is natural to say that an isometry $\varphi\colon L\xrightarrow{\sim} L'$ induces contravariantly an isomorphism $(D_{L'},q_{L'}) \xrightarrow{\sim} (D_L,q_L)$. Under the present definition (2.1), however, the functor $\varphi\mapsto q_\varphi$ is covariant.

2.4. **Roots.** Let L be an even $\mathbb{Z}$-lattice. A vector $r \in L$ is said to be a *root* of L if $\langle r, r\rangle = -2$. We put

$$\mathrm{Roots}(L) := \{\ r \in L \ \mid\ \langle r, r\rangle = -2\ \}.$$

Let $\Phi = \{r_1, \dots, r_m\}$ be a set of roots of L. Suppose that $\langle r_i, r_j\rangle \in \{0, 1\}$ holds for any $i \neq j$. The *dual graph* of Φ is the graph whose set of vertices is Φ and whose set of edges is the set of pairs $\{r_i, r_j\}$ such that $\langle r_i, r_j\rangle = 1$. We say that Φ is an *ADE-configuration* if each connected component of the dual graph of Φ is a Dynkin diagram of type A_l ($l \geq 1$), D_m ($m \geq 4$), or E_n ($n = 6, 7, 8$). (See [8, Figure 1.7] for the definition of these Dynkin diagrams). Let Φ be an ADE-configuration. The formal sum of the types A_l, D_m, E_n of the connected components of the dual graph is called the *ADE-type of* Φ. An isomorphism of ADE-configurations Φ and Φ' is a bijection $\gamma\colon \Phi \xrightarrow{\sim} \Phi'$ such that $\langle r^\gamma, r'^\gamma\rangle = \langle r, r'\rangle$ holds for all $r, r' \in \Phi$. An isomorphism class of ADE-configurations is determined by the ADE-type. The automorphism group $\mathrm{Aut}(\Phi)$ of an ADE-configuration Φ is just the automorphism group of the dual graph of Φ.

A negative definite even $\mathbb{Z}$-lattice L is said to be a *root lattice* if L is generated by $\mathrm{Roots}(L)$. We have the following classical result. See [8, Theorem 1.2], for example.

Proposition 2.2. *Let L be a root lattice. Then there exists an ADE-configuration $\Phi \subset \mathrm{Roots}(L)$ that forms a basis of L.*

The ADE-configuration Φ in this proposition is called a *fundamental root system* of the root lattice L. When an ADE-configuration Φ is given, we denote by $L(\Phi)$ the root lattice with a fundamental root system Φ.

2.5. **Even $\mathbb{Z}_p$-lattices.** Let p be a prime integer. The isomorphism classes of even $\mathbb{Z}_p$-lattices and their discriminant forms are well-understood. See [17] or [14, Chapter IV] for details.

We say that a finite quadratic form $q\colon D \to \mathbb{Q}/2\mathbb{Z}$ is *p-adic* if the order of D is a power of p. If (D, q) is p-adic, then the image of q is included in the subgroup $\mathbb{Q}_p/2\mathbb{Z}_p \subset \mathbb{Q}/2\mathbb{Z}$. It is obvious that the discriminant form (D_L, q_L) of an even $\mathbb{Z}_p$-lattice L is p-adic. We have the *normal form theorems* for non-degenerate p-adic finite quadratic forms and for even $\mathbb{Z}_p$-lattices.

Proposition 2.3. *A non-degenerate p-adic finite quadratic form is isomorphic to an orthogonal direct-sum of indecomposable p-adic finite quadratic forms listed in Table 2.1.*

More precisely, we have an algorithm to decompose a given non-degenerate p-adic finite quadratic form into an orthogonal direct-sum of indecomposable ones. See [14, Chapter IV].

Proposition 2.4. *An even $\mathbb{Z}_p$-lattice is isomorphic to an orthogonal direct-sum of indecomposable even $\mathbb{Z}_p$-lattices whose Gram matrices are listed below.*

When p is odd:

$$[2p^\nu] \quad or \quad [2p^\nu n_p],$$

where ν runs through $\mathbb{Z}_{\geq 0}$, and $n_p \in \mathbb{Z}$ represents a non-square residue in $\mathbb{F}_p^\times$.

When $p = 2$:

$$2^\nu[2\varepsilon], \ \ or \ \ 2^\nu \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \ or \ \ 2^\nu \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

When p is odd:

Name	(D, q)	Brown invariant
$w^{1}_{p,\nu}$	$\left(\mathbb{Z}/p^\nu\mathbb{Z}, \left[\dfrac{2}{p^\nu}\right]\right)$	$\begin{cases} 0 & \text{if } \nu \text{ is even,} \\ 1-(-1)^{(p-1)/2} & \text{if } \nu \text{ is odd,} \end{cases}$
$w^{-1}_{p,\nu}$	$\left(\mathbb{Z}/p^\nu\mathbb{Z}, \left[\dfrac{2n_p}{p^\nu}\right]\right)$	$\begin{cases} 0 & \text{if } \nu \text{ is even,} \\ -3-(-1)^{(p-1)/2} & \text{if } \nu \text{ is odd.} \end{cases}$

When $p = 2$:

Name	(D, q)	Brown invariant
$w^{\varepsilon}_{2,\nu}$	$\left(\mathbb{Z}/2^\nu\mathbb{Z}, \left[\dfrac{\varepsilon}{2^\nu}\right]\right)$	$\varepsilon + \nu(\varepsilon^2 - 1)/2$
u_ν	$\left(\mathbb{Z}/2^\nu\mathbb{Z} \times \mathbb{Z}/2^\nu\mathbb{Z}, \dfrac{1}{2^\nu}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$	0
v_ν	$\left(\mathbb{Z}/2^\nu\mathbb{Z} \times \mathbb{Z}/2^\nu\mathbb{Z}, \dfrac{1}{2^\nu}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$	4ν

In this table, ν runs through $\mathbb{Z}_{>0}$. When p is odd, $n_p \in \mathbb{Z}$ represents a non-square residue in $\mathbb{F}_p^\times$. When $p = 2$, $\varepsilon \in \{1, 3, 5, 7\}$ if $\nu > 1$, whereas $\varepsilon \in \{1, 3\}$ if $\nu = 1$.

Table 2.1. Indecomposable p-adic finite quadratic forms

where ν runs through $\mathbb{Z}_{\geq 0}$, and $\varepsilon \in \{1, 3, 5, 7\}$.

An indecomposable even $\mathbb{Z}_p$-lattice L in the list above is unimodular if and only if $\nu = 0$ and $L \not\cong [2\varepsilon]$.

Moreover, we have an algorithm to determine whether two given orthogonal direct-sums of these indecomposable objects are isomorphic or not. (See [14, Chapter IV].) In particular, for a given positive integer r, a given element $d \in (\mathbb{Z}_p \setminus \{0\})/(\mathbb{Z}_p^\times)^2$, and a given non-degenerate p-adic finite quadratic form (D, q), we can easily determine whether there exists an even $\mathbb{Z}_p$-lattice L such that $\operatorname{rank}(L) = r$, $\operatorname{disc}(L) = d$, and $(D_L, q_L) \cong (D, q)$, and if it exists, we can write a Gram matrix of such an even $\mathbb{Z}_p$-lattice explicitly (see Section 5.2). As corollaries, we obtain the following:

Proposition 2.5. *The isomorphism class of an even $\mathbb{Z}_p$-lattice L is determined by* $\operatorname{rank}(L)$, $\operatorname{disc}(L) \in (\mathbb{Z}_p \setminus \{0\})/(\mathbb{Z}_p^\times)^2$, *and the isomorphism class of the discriminant form (D_L, q_L).*

Proposition 2.6. *If L is an even $\mathbb{Z}_p$-lattice, then the natural homomorphism* $\mathrm{O}(L) \to \mathrm{O}(D_L, q_L)$ *is surjective.*

2.6. **Even overlattices.** Suppose that L is an even $\mathbb{Z}$-lattice. An *even overlattice* of L is a $\mathbb{Z}$-submodule M of $L^\vee$ containing L such that the natural $\mathbb{Q}$-valued symmetric bilinear form on $L^\vee$ takes values in $\mathbb{Z}$ on M, and that M is an even $\mathbb{Z}$-lattice by this $\mathbb{Z}$-valued form. The following theorem is due to Nikulin [17].

Proposition 2.7. *Let L be an even $\mathbb{Z}$-lattice, and let $\mathrm{pr}_L\colon L^\vee \to D_L$ denote the natural projection. Then the mapping $K \mapsto \mathrm{pr}_L^{-1}(K)$ gives rise to a bijection from the set of totally isotropic subgroups $K \subset D_L$ of (D_L, q_L) to the set of even overlattices of L.*

A submodule A of a free $\mathbb{Z}$-module M is said to be *primitive* if M/A is torsion free. The *primitive closure* of A in M is the primitive submodule $(A \otimes \mathbb{Q}) \cap M$ of M. As a corollary of Proposition 2.7, we obtain the following:

Corollary 2.8. *Let S and T be even $\mathbb{Z}$-lattices. Then there exists a canonical bijective correspondence between the set of even unimodular overlattices H of the orthogonal direct sum $S \oplus T$ such that S and T are primitive in H, and the set of anti-isometries $(D_S, -q_S) \xrightarrow{\sim} (D_T, q_T)$.*

The correspondence is given as follows. Let $\gamma\colon (D_S, -q_S) \xrightarrow{\sim} (D_T, q_T)$ be an anti-isometry of the discriminant forms. Then the pull-back of the graph of γ in $D_S \oplus D_T$ by the natural projection $S^\vee \oplus T^\vee \to D_S \oplus D_T$ is an even unimodular overlattice of $S \oplus T$.

2.7. **Genus of even $\mathbb{Z}$-lattices.** Let (D, q) be a non-degenerate finite quadratic form. For each prime divisor p of $d := |D|$, let D_p denote the p-part

$$\{\, x \in D \;\mid\; p^\nu x = 0 \;\text{ for some integer }\; \nu \geq 0 \,\}$$

of D, and let q_p denote the restriction of q to D_p. Then (D_p, q_p) is a non-degenerate p-adic finite quadratic form. We say that (D_p, q_p) is the *p-part* of (D, q). If $p \neq p'$, then D_p and $D_{p'}$ are orthogonal with respect to the bilinear form b of (D, q). Hence we obtain a canonical orthogonal direct-sum decomposition

$$(D, q) = \bigoplus_{p|d} (D_p, q_p). \tag{2.2}$$

Suppose that L is an even $\mathbb{Z}$-lattice. Then the even $\mathbb{Z}_p$-lattice $L \otimes \mathbb{Z}_p$ is not unimodular if and only if p divides the order $|D_L|$ of the discriminant group, and the p-part of the discriminant form (D_L, q_L) is isomorphic to $(D_{L\otimes\mathbb{Z}_p}, q_{L\otimes\mathbb{Z}_p})$. Moreover, the discriminant $\mathrm{disc}(L \otimes \mathbb{Z}_p)$ is equal to $\mathrm{disc}(L) \bmod (\mathbb{Z}_p^\times)^2$. If $\mathrm{sign}(L) = (s_+, s_-)$, we have $\mathrm{disc}(L) = (-1)^{s_-}|D_L|$. Hence, by the results we have stated so far, we obtain the following:

Proposition 2.9. *Let L and L' be even $\mathbb{Z}$-lattices. Then the following conditions are equivalent:*

(i) $\mathrm{sign}(L) = \mathrm{sign}(L')$ *and* $(D_L, q_L) \cong (D_{L'}, q_{L'})$.
(ii) $L \otimes \mathbb{R} \cong L' \otimes \mathbb{R}$, *and* $L \otimes \mathbb{Z}_p \cong L' \otimes \mathbb{Z}_p$ *for all* p.

Definition 2.10. We say that even $\mathbb{Z}$-lattices L and L' are *in the same genus* if the two conditions in Proposition 2.9 are satisfied.

Definition 2.11. Let (s_+, s_-) be a pair of non-negative integers such that $r := s_+ + s_- > 0$, and let (D, q) be a non-degenerate finite quadratic form. The *genus determined by* (s_+, s_-) *and* (D, q) is the set of isomorphism classes of even $\mathbb{Z}$-lattices L of rank r such that $\mathrm{sign}(L) = (s_+, s_-)$ and $(D_L, q_L) \cong (D, q)$.

We have the following criterion, due to Nikulin [17], for the genus determined by (s_+, s_-) and (D, q) to be non-empty. (See also Theorem 5.2 in [14, Chapter V].)

The *Brown invariant* $\mathrm{Br}(D,q)$ of a non-degenerate finite quadratic form (D,q) is defined to be the element of $\mathbb{Z}/8\mathbb{Z}$ that satisfies

$$\exp\left(\frac{2\pi\sqrt{-1}}{8}\,\mathrm{Br}(D,q)\right)=\frac{1}{\sqrt{|D|}}\sum_{x\in D}\exp(\sqrt{-1}\pi\,q(x)).$$

(See [14, Chapter III] for the existence of the Brown invariant.) The Brown invariant is additive under the operation of orthogonal direct-sum of non-degenerate finite quadratic forms, and the values of this invariant for the indecomposable non-degenerate p-adic finite quadratic forms are given in Table 2.1. Hence, using the decomposition (2.2) and Proposition 2.3, we can easily calculate $\mathrm{Br}(D,q)$ for any (D,q).

Theorem 2.12. *Let s_+ and s_- be non-negative integers such that $s_+ + s_- > 0$, and let (D,q) be a non-degenerate finite quadratic form. We put $r := s_+ + s_-$ and $d := (-1)^{s_-}|D|$. Then the genus determined by (s_+, s_-) and (D,q) is non-empty if and only if the following hold:*

(i) $r \ge \mathrm{leng}(D)$,
(ii) $\mathrm{Br}(D,q) \equiv s_+ - s_- \bmod 8$, *and*
(iii) *for each prime divisor p of d, there exists an even $\mathbb{Z}_p$-lattice of rank r, discriminant $d \bmod (\mathbb{Z}_p^\times)^2$, and with the discriminant form isomorphic to the p-part (D_p, q_p) of (D,q).*

Remark 2.13. Another formulation of the criterion by means of p-excess is given by Conway and Sloan [6, Chapter 15]. In our previous papers [20, 21], we used this p-excess version.

By the weak Hasse principle, we obtain the following proposition (Theorem 1.1 in [14, Chapter VIII]).

Proposition 2.14. *If even $\mathbb{Z}$-lattices L and L' are in the same genus, then the $\mathbb{Q}$-lattices $L\otimes\mathbb{Q}$ and $L'\otimes\mathbb{Q}$ are isomorphic.*

3. Connected components of moduli

In this section, we fix an ADE-configuration Φ, a finite abelian group A, and a subgroup G of the automorphism group $\mathrm{Aut}(\Phi)$ of Φ.

3.1. **Definition of connected components.** Let (X,f,s) be an elliptic $K3$ surface. We consider the second cohomology group $H^2(X,\mathbb{Z})$ as a $\mathbb{Z}$-lattice by the cup-product. It is well-known that $H^2(X,\mathbb{Z})$ is a $K3$-*lattice*; that is, $H^2(X,\mathbb{Z})$ is an even unimodular $\mathbb{Z}$-lattice of signature $(3,19)$, which is unique up to isomorphism. Recall that $\Phi_f \subset H^2(X,\mathbb{Z})$ is the set of classes of smooth rational curves on X that are contracted by f and are disjoint from s, and that A_f is the torsion part of the Mordell-Weil group. By the classical theory of elliptic surfaces (see [11]), we know that Φ_f is an ADE-configuration. Recall that $L(\Phi_f)$ denotes the root sublattice of $H^2(X,\mathbb{Z})$ generated by Φ_f.

Suppose that (X,f,s) is of type (Φ,A); that is, $\Phi_f\cong\Phi$ and $A_f\cong A$. A *marking* of (X,f,s) is an isomorphism $\phi\colon \Phi \xrightarrow{\sim} \Phi_f$ of ADE-configurations. If ϕ is a marking of (X,f,s), we denote by (X,f,s,ϕ) the marked elliptic $K3$ surface. We say that two markings ϕ and ϕ' of (X,f,s) are G-*isomorphic* if there exists an element $g\in G$ such that $\phi = g\cdot\phi'$ holds. More generally, we say that two marked elliptic

$K3$ surfaces (X, f, s, ϕ) and (X', f', s', ϕ') of type (Φ, A) are *G-isomorphic* if there exist an isomorphism $\psi\colon X' \xrightarrow{\sim} X$ of $K3$ surfaces and an element g of G that satisfy the following:

- We have $f \circ \psi = f'$ and $\psi \circ s' = s$, so that ψ induces an isomorphism of elliptic $K3$ surfaces $(X', f', s') \xrightarrow{\sim} (X, f, s)$. Hence the pull-back by ψ induces an isomorphism
$$\Phi_\psi\colon \Phi_f \xrightarrow{\sim} \Phi_{f'}$$
of ADE-configurations.
- The diagram
$$\begin{array}{ccc} \Phi & \xrightarrow{g} & \Phi \\ \phi \downarrow & & \downarrow \phi' \\ \Phi_f & \xrightarrow[\Phi_\psi]{} & \Phi_{f'} \end{array}$$
commutes.

We consider the moduli space that parameterizes the G-isomorphism classes of marked elliptic $K3$ surfaces of type (Φ, A), and define the set $\mathfrak{C}(\Phi, A, G)$ of connected components of this moduli space.

Remark 3.1. Theorem 1.1 and Corollaries 1.5, 1.6 stated in Introduction are for the case where $G = \mathrm{Aut}(\Phi)$.

A *connected family* $(\mathcal{X}, F, S)/B$ *of elliptic* $K3$ *surfaces of type* (Φ, A) is a commutative diagram
$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{F} & \mathbb{P}^1_B \\ {}_\pi\searrow & & \swarrow{}_{\pi_P} \\ & B & \end{array}$$
with a section $S\colon \mathbb{P}^1_B \to \mathcal{X}$ of F such that the following hold:

- B is a connected analytic variety, $\pi\colon \mathcal{X} \to B$ is a family of $K3$ surfaces, $\pi_P\colon \mathbb{P}^1_B \to B$ is a $\mathbb{P}^1$-fibration, and
- for any point $t \in B$, the pullback (X_t, f_t, s_t) of $(\mathcal{X}, F, S)$ by $\{t\} \hookrightarrow B$ is an elliptic $K3$ surface of type (Φ, A).

Let $(\mathcal{X}, F, S)/B$ be a connected family as above. For a point $t \in B$, we denote by Φ_t the ADE-configuration Φ_{f_t} of the elliptic $K3$ surface (X_t, f_t, s_t). The family $\{\Phi_t \mid t \in B\}$ defines a locally constant system
$$\Phi_B \to B$$
of ADE-configurations. A *marking* of a connected family $(\mathcal{X}, F, S)/B$ is a choice of a base point $o \in B$ and a marking $\phi_o\colon \Phi \xrightarrow{\sim} \Phi_o$ of (X_o, f_o, s_o). We say that a marked connected family $((\mathcal{X}, F, S)/B, \phi_o)$ is *G-connected* if the image of the monodromy representation
$$m_B\colon \pi_1(B, o) \to \mathrm{Aut}(\Phi)$$
obtained from the locally constant system $\Phi_B \to B$ and the marking $\phi_o\colon \Phi \xrightarrow{\sim} \Phi_o$ is contained in G. Suppose that $((\mathcal{X}, F, S)/B, \phi_o)$ is G-connected, and let t be a point of B. Since B is connected, there exists a path $\gamma\colon [0, 1] \to B$ from the base point o to t. The composite of $\phi_o\colon \Phi \xrightarrow{\sim} \Phi_o$ and the transportation $\Phi_o \xrightarrow{\sim} \Phi_t$ in the locally constant system $\Phi_B \to B$ along γ gives rise to a marking
$$\phi_t\colon \Phi \xrightarrow{\sim} \Phi_t$$

of (X_t, f_t, s_t). This marking depends on the choice of the path γ, but the G-isomorphism class of ϕ_t is independent of the choice of γ. Therefore a G-connected family parametrizes a family of G-isomorphism classes of marked elliptic $K3$ surfaces.

We say that two marked elliptic $K3$ surfaces (X, f, s, ϕ) and (X', f', s', ϕ') of type (Φ, A) are *G-connected* if there exists a marked G-connected family $((\mathcal{X}, F, S)/B, \phi_o)$ of elliptic $K3$ surfaces of type (Φ, A) with two fibers G-isomorphic to (X, f, s, ϕ) and (X', f', s', ϕ'), respectively. This relation of G-connectedness is an equivalence relation. The transitivity is proved as follows. Suppose that (X_1, f_1, s_1, ϕ_1) and (X_2, f_2, s_2, ϕ_2) are G-isomorphic to the fibers of a marked G-connected family $((\mathcal{X}, F, S)/B, \phi_o)$ over $t_1 \in B$ and $t_2 \in B$, respectively, and that (X_2, f_2, s_2, ϕ_2) and (X_3, f_3, s_3, ϕ_3) are G-isomorphic to the fibers of $((\mathcal{X}', F', S')/B', \phi_{o'})$ over $t'_2 \in B'$ and $t'_3 \in B'$, respectively. Let B'' be the connected analytic space obtained by gluing B and B' at $t_2 \in B$ and $t'_2 \in B'$, and let $(\mathcal{X}'', F'', S'')/B''$ be the family obtained by gluing $(\mathcal{X}, F, S)/B$ and $(\mathcal{X}', F', S')/B'$ along the fibers over $t_2 \in B$ and $t'_2 \in B'$, both of which are isomorphic to (X_2, f_2, s_2). Then $((\mathcal{X}'', F'', S'')/B'', \phi_o)$ is a marked G-connected family, and hence (X_1, f_1, s_1, ϕ_1) and (X_3, f_3, s_3, ϕ_3) are G-connected.

We define a *G-connected component of the moduli of elliptic $K3$ surfaces of type* (Φ, A) to be an equivalence class of the relation of G-connectedness of marked elliptic $K3$ surfaces of type (Φ, A). We denote by $\mathfrak{C}(\Phi, A, G)$ the set of G-connected components of this moduli.

3.2. **Lattice invariant of connected components.** In this section, we define a set $\mathcal{Q}(\Phi, A)/{\sim_G}$ in purely lattice-theoretic terms, and establish a bijection

$$\bar{\zeta}\colon \mathfrak{C}(\Phi, A, G) \xrightarrow{\sim} \mathcal{Q}(\Phi, A)/{\sim_G}.$$

We denote by $L(\Phi)$ the root lattice with a fundamental root system Φ, and put

$$r_\Phi := \operatorname{rank} L(\Phi).$$

Let (X, f, s) be an elliptic $K3$ surface of type (Φ, A). Recall that $M(\Phi_f)$ denotes the primitive closure of $L(\Phi_f)$ in $H^2(X, \mathbb{Z})$, that U_f denotes the sublattice of $H^2(X, \mathbb{Z})$ generated by the class of a fiber of f and the class of the section s. Then U_f is an even unimodular hyperbolic $\mathbb{Z}$-lattice of rank 2, and is orthogonal to $L(\Phi_f)$ in $H^2(X, \mathbb{Z})$. Hence $U_f \oplus M(\Phi_f)$ is a primitive sublattice of $H^2(X, \mathbb{Z})$.

Proposition 3.2. *We have $M(\Phi_f)/L(\Phi_f) \cong A$ and $\operatorname{Roots}(M(\Phi_f)) = \operatorname{Roots}(L(\Phi_f))$.*

Proof. The isomorphism $M(\Phi_f)/L(\Phi_f) \cong A_f \cong A$ is classically known. See [27], for example. Note that $\operatorname{Roots}(M(\Phi_f)) \supset \operatorname{Roots}(L(\Phi_f))$. Suppose that $\operatorname{Roots}(M(\Phi_f))$ were strictly larger than $\operatorname{Roots}(L(\Phi_f))$. Then there would be a smooth rational curve on X whose class is orthogonal to U_f but does not belong to Φ_f, which is a contradiction. □

Since U_f is unimodular, we have a canonical isomorphism

$$(D_{U_f \oplus M(\Phi_f)}, q_{U_f \oplus M(\Phi_f)}) \xrightarrow{\sim} (D_{M(\Phi_f)}, q_{M(\Phi_f)}).$$

Recall that T_f denotes the orthogonal complement of $U_f \oplus M(\Phi_f)$ in $H^2(X, \mathbb{Z})$. Then T_f is an even $\mathbb{Z}$-lattice of signature $(2, 18 - r_\Phi)$. Moreover, Corollary 2.8 implies that we have a unique anti-isomorphism

$$\alpha_f\colon (D_{M(\Phi_f)}, -q_{M(\Phi_f)}) \xrightarrow{\sim} (D_{T_f}, q_{T_f})$$

of discriminant forms that gives rise to the even unimodular overlattice $H^2(X,\mathbb{Z})$ of $(U_f \oplus M(\Phi_f)) \oplus T_f$. Hence T_f belongs to the genus determined by the signature $(2, 18 - r_\Phi)$ and the finite quadratic form $(D_{M(\Phi_f)}, -q_{M(\Phi_f)})$. Let $\omega_X \in H^2(X,\mathbb{C})$ denote the class of a nowhere vanishing holomorphic 2-form on X, which is unique up to a non-zero multiplicative constant. We have $\omega_X \in T_f \otimes \mathbb{C}$, $\langle \omega_X, \omega_X \rangle = 0$, and $\langle \omega_X, \bar{\omega}_X \rangle > 0$. Let $H^{1,1}(X,\mathbb{R})^\perp$ denote the orthogonal complement of $H^{1,1}(X,\mathbb{R})$ in $H^2(X,\mathbb{R})$. Then $H^{1,1}(X,\mathbb{R})^\perp$ is a positive definite 2-dimensional $\mathbb{R}$-lattice. The two real vectors $\operatorname{Re}\omega_X$ and $\operatorname{Im}\omega_X$ in this order form an oriented orthogonal basis of the real subspace $H^{1,1}(X,\mathbb{R})^\perp$ of $T_f \otimes \mathbb{R}$. Thus the Hodge structure of $H^2(X)$ canonically defines a positive sign structure θ_f of T_f.

These geometric objects $M(\Phi_f)$, T_f, α_f, and θ_f motivate the following lattice-theoretic definitions.

Definition 3.3. For an even overlattice M of $L(\Phi)$, let $\mathcal{G}(M)$ denote the genus of even $\mathbb{Z}$-lattices determined by the signature $(2, 18 - r_\Phi)$ and the discriminant form $(D_M, -q_M)$. Let $\mathcal{E}(\Phi, A)$ denote the set of even overlattices M of $L(\Phi)$ such that

- $M/L(\Phi) \cong A$ and $\operatorname{Roots}(M) = \operatorname{Roots}(L(\Phi))$, and
- $\mathcal{G}(M)$ is non-empty.

We define $\mathcal{Q}(\Phi, A)$ to be the set of quartets (M, T, α, θ) of the following objects; M is an element of $\mathcal{E}(\Phi, A)$, T is an even $\mathbb{Z}$-lattice belonging to the genus $\mathcal{G}(M)$, α is an isomorphism $(D_M, -q_M) \xrightarrow{\sim} (D_T, q_T)$, and θ is a positive sign structure of T.

We define an equivalence relation $\sim_G$ on the set $\mathcal{Q}(\Phi, A)$. Since we have a natural homomorphism $\operatorname{Aut}(\Phi) \to \mathrm{O}(L(\Phi))$, the subgroup G of $\operatorname{Aut}(\Phi)$ acts on the set $\mathcal{E}(\Phi, A)$. Note that this action is from the right. If $g \in G$ maps $M \in \mathcal{E}(\Phi, A)$ to $M' \in \mathcal{E}(\Phi, A)$, then g induces an isometry $g|M \colon M \xrightarrow{\sim} M'$, and hence an isomorphism $q_{g|M} \colon (D_M, q_M) \xrightarrow{\sim} (D_{M'}, q_{M'})$.

Definition 3.4. Let (M, T, α, θ) and $(M', T', \alpha', \theta')$ be elements of $\mathcal{Q}(\Phi, A)$. We put $(M, T, \alpha, \theta) \sim_G (M', T', \alpha', \theta')$ if there exist an automorphism $g \in G$ and an isometry $\psi \colon T \xrightarrow{\sim} T'$ with the following properties.

- g maps M to M',
- ψ maps θ to θ', and
- the following diagram is commutative:

$$\begin{array}{ccc} (D_M, q_M) & \xrightarrow{q_{g|M}} & (D_{M'}, q_{M'}) \\ \alpha \downarrow & & \downarrow \alpha' \\ (D_T, q_T) & \xrightarrow[q_\psi]{} & (D_{T'}, q_{T'}). \end{array}$$

Next we define a map $\bar{\zeta}$ from $\mathfrak{C}(\Phi, A, G)$ to $\mathcal{Q}(\Phi, A)/\!\sim_G$. Let (X, f, s, ϕ) be a marked elliptic $K3$ surface of type (Φ, A). The marking $\phi \colon \Phi \xrightarrow{\sim} \Phi_f$ induces an isometry $\phi_L \colon L(\Phi) \xrightarrow{\sim} L(\Phi_f)$. By Proposition 3.2 and the existence of T_f, there exists a unique element $M_{f,\phi}$ of $\mathcal{E}(\Phi, A)$ such that the isometry ϕ_L induces an isometry $\phi_M \colon M_{f,\phi} \xrightarrow{\sim} M(\Phi_f)$. The composite of the isomorphism $(D_{M_{f,\phi}}, q_{M_{f,\phi}}) \xrightarrow{\sim} (D_{M(\Phi_f)}, q_{M(\Phi_f)})$ induced by ϕ_M and the isomorphism α_f yields an isomorphism

$$\alpha_{f,\phi} \colon (D_{M_{f,\phi}}, -q_{M_{f,\phi}}) \xrightarrow{\sim} (D_{T_f}, q_{T_f}).$$

Thus we obtain a quartet

$$\zeta(X, f, s, \phi) := (\, M_{f,\phi},\ T_f,\ \alpha_{f,\phi},\ \theta_f \,) \in \mathcal{Q}(\Phi, A).$$

Suppose that marked elliptic $K3$ surfaces (X, f, s, ϕ) and (X', f', s', ϕ') are G-isomorphic. Then we obviously have $\zeta(X, f, s, \phi) \sim_G \zeta(X', f', s', \phi')$ by definitions. Let $((\mathcal{X}, F, S)/B, \phi_o)$ be a marked G-connected family of elliptic $K3$ surfaces of type (Φ, A). For a point $t \in B$, let (X_t, f_t, s_t) denote the fiber of $(\mathcal{X}, F, S)/B$ over t, and let Φ_t, U_t, $L(\Phi_t)$, $M(\Phi_t)$, T_t, α_t, and θ_t be the geometric objects associated with (X_t, f_t, s_t) defined above. Let $\phi_t \colon \Phi \xrightarrow{\sim} \Phi_t$ be the marking of (X_t, f_t, s_t) induced by a path $\gamma\colon [0,1] \to B$ connecting o and t. The transportation along γ induces an isometry $H^2(X_o, \mathbb{Z}) \xrightarrow{\sim} H^2(X_t, \mathbb{Z})$, and this isometry induces isometries of sublattices $U_o \xrightarrow{\sim} U_t$, $L(\Phi_o) \xrightarrow{\sim} L(\Phi_t)$, $M(\Phi_o) \xrightarrow{\sim} M(\Phi_t)$, and $T_o \xrightarrow{\sim} T_t$. Hence the anti-isometries α_o and α_t are compatible with the isomorphisms $(D_{M_o}, q_{M_o}) \xrightarrow{\sim} (D_{M_t}, q_{M_t})$ and $(D_{T_o}, q_{T_o}) \xrightarrow{\sim} (D_{T_t}, q_{T_t})$ obtained by the transportation along γ. Since θ_t is defined by the Hodge structure of X_t, the analytic structure of $F\colon \mathcal{X} \to B$ implies that the isometry $T_o \xrightarrow{\sim} T_t$ along γ maps θ_o to θ_t. Hence we have $\zeta(X_o, f_o, s_o, \phi_o) \sim_G \zeta(X_t, f_t, s_t, \phi_t)$. In other words, the map ζ induces a map

$$\bar{\zeta}\colon \mathfrak{C}(\Phi, A, G) \to \mathcal{Q}(\Phi, A)/\!\sim_G.$$

Theorem 3.5. *The map $\bar{\zeta}$ is a bijection.*

Proof. Let (M, T, α, θ) be an element of $\mathcal{Q}(\Phi, A)$. Let U denote the even unimodular hyperbolic $\mathbb{Z}$-lattice of rank 2 with a basis $v_{\mathrm{fib}}, v_{\mathrm{zero}}$ and the Gram matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}.$$

We define H to be the even unimodular overlattice of $(U \oplus M) \oplus T$ defined by the anti-isometry

$$(D_{U\oplus M}, -q_{U\oplus M}) = (D_M, -q_M) \underset{\alpha}{\xrightarrow{\sim}} (D_T, q_T)$$

of the discriminant forms given by α. Then H is a $K3$-lattice. An *H-marking* of a $K3$ surface X is an isometry $\mu\colon H \xrightarrow{\sim} H^2(X, \mathbb{Z})$.

Let $\mathbb{P}_*(T \otimes \mathbb{C})$ denote the projective space of 1-dimensional subspaces of $T \otimes \mathbb{C}$. We put

$$\Omega_T := \{\, \mathbb{C}\omega \in \mathbb{P}_*(T \otimes \mathbb{C}) \mid \langle \omega, \omega \rangle = 0,\ \langle \omega, \bar{\omega} \rangle > 0 \,\}.$$

A non-zero vector $\omega = u + \sqrt{-1}v \in T \otimes \mathbb{C}$ with $u, v \in V := T \otimes \mathbb{R}$ satisfies $\mathbb{C}\omega \in \Omega_T$ if and only if (u, v) belongs to

$$Z := \{\, (x, y) \in V \times V \mid \langle x, x \rangle = \langle y, y \rangle > 0, \langle x, y \rangle = 0 \,\}.$$

The image Z_1 of the first projection $\mathrm{pr}\colon Z \to V$ is connected and $\pi_1(Z_1) \cong \mathbb{Z}$. Since $t_+ = 2$, the orthogonal complement of a vector $u \in Z_1$ in V has signature $(1, t_-)$, and hence $\mathrm{pr}^{-1}(u) = \{y \in V \mid \langle y, y \rangle = \langle u, u \rangle, \langle u, y \rangle = 0\}$ has two connected components. We can easily see that $\pi_1(Z_1)$ acts on the set of these connected components trivially. Therefore Ω_T has exactly two connected components, and they are complex conjugate to each other. For $\mathbb{C}\omega \in \Omega_T$, the ordered pair of vectors $\mathrm{Re}\,\omega$ and $\mathrm{Im}\,\omega$ in $T \otimes \mathbb{R}$ defines an oriented positive definite 2-dimensional subspace of $T \otimes \mathbb{R}$. By this correspondence, the set of connected components of Ω_T can be identified with the set of positive sign structures of T. Let $\Omega_{(T,\theta)}$ be the connected component corresponding to θ.

By the theory of the refined period map of marked $K3$ surfaces, which is stated in Barth, Hulek, Peters and Van de Ven [3, Chapter VIII], we see that there exists a universal family

$$\mathcal{F}_{(M,T,\alpha,\theta)}\colon \mathcal{X} \to \Omega^0_{(T,\theta)}$$

of H-marked $K3$ surfaces (X_t, μ_t) with parameter space $\Omega^0_{(T,\theta)}$ being an open dense subset of $\Omega_{(T,\theta)}$ such that, for each $t \in \Omega^0_{(T,\theta)}$, the H-marking $\mu_t\colon H \xrightarrow{\sim} H^2(X_t, \mathbb{Z})$ satisfies the following:

- $\mu_t \otimes \mathbb{C}$ maps $\mathbb{C}\omega_t$ to $H^{2,0}(X_t)$, where $\mathbb{C}\omega_t$ is the 1-dimensional subspace of $T \otimes \mathbb{C} \subset H \otimes \mathbb{C}$ corresponding to the point t, and hence the Néron-Severi lattice $\mathrm{NS}(X_t)$ of X_t contains the primitive sublattice $\mu_t(U \oplus M)$ of $H^2(X_t, \mathbb{Z})$, and
- μ_t maps the vectors $v_{\rm fib} \in U$, $v_{\rm zero} \in U$, and each $r \in \Phi \subset H$ to the classes of certain irreducible curves F, Z, and C_r on X_t, respectively.

Then the complete linear system $|F|$ defines an elliptic fibration $f_t\colon X_t \to \mathbb{P}^1$ and Z provides us with a section s_t of f_t. Moreover, the set $\{C_r \mid r \in \Phi\}$ is the set of smooth rational curves on X_t contracted by f_t and disjoint from s_t, and hence $\Phi_t := \Phi_{f_t}$ is equal to $\{[C_r] \mid r \in \Phi\}$. We have $M(\Phi_t)/L(\Phi_t) \cong M/L(\Phi) \cong A$. Therefore (X_t, f_t, s_t) is of type (Φ, A), and the H-marking μ_t yields a marking $\phi_t\colon \Phi \xrightarrow{\sim} \Phi_t$.

Thus, each element (M, T, α, θ) of $\mathcal{Q}(\Phi, A)$ gives a connected family $\mathcal{F}_{(M,T,\alpha,\theta)}$ of marked elliptic $K3$ surfaces of type (Φ, A). By the existence of H-markings, the monodromy of the family $\{\Phi_t \mid t \in \Omega^0_{(T,\theta)}\}$ of ADE-configurations is trivial. Any marked elliptic $K3$ surface (X, f, s, ϕ) of type (Φ, A) is isomorphic to a member of the family $\mathcal{F}_{\zeta(X,f,s,\phi)}$. Hence the surjectivity of $\bar\zeta$ follows. It follows from the universality of the family $\mathcal{F}_{\zeta(X,f,s,\phi)}$ that, if $(M, T, \alpha, \theta) \sim_G (M', T', \alpha', \theta')$, then each member of $\mathcal{F}_{\zeta(X,f,s,\phi)}$ is G-isomorphic to a member of $\mathcal{F}_{(M',T',\alpha',\theta')}$. Hence $\bar\zeta$ is injective. □

3.3. **Computation of the set $\mathcal{Q}(\Phi, A)/{\sim_G}$.** Thus our problem of computing the set $\mathfrak{C}(\Phi, A, G)$ is reduced to the calculation of the set $\mathcal{Q}(\Phi, A)/{\sim_G}$.

Recall that G acts on the set $\mathcal{E}(\Phi, A)$ from the right. We have a projection

$$\mathrm{pr}_1\colon \mathcal{Q}(\Phi, A)/{\sim_G} \to \mathcal{E}(\Phi, A)/G$$

given by $(M, T, \alpha, \theta) \mapsto M$. The set of even overlattices of $L(\Phi)$ and the action of G on it can be easily calculated by Proposition 2.7. From each G-orbit, we choose an even overlattice M, calculate $M/L(\Phi)$ and $\mathrm{Roots}(M)$, and determine whether $\mathcal{G}(M)$ is empty or not by the criterion of Theorem 2.12. In this way, we can compute the set $\mathcal{E}(\Phi, A)/G$.

Remark 3.6. For the calculation of $\mathrm{Roots}(M)$, the technique of the lattice reduction bases due to Lenstra, Lenstra, and Lovász [12] is very useful. See [5, Chapter 2].

The notion of algebraic equivalence of connected components defined in Definitions 1.3 and 1.4 is now succinctly defined as follows.

Definition 3.7. We say that two connected components $\mathcal{C}_1, \mathcal{C}_2 \in \mathfrak{C}(\Phi, A, G)$ are *algebraically equivalent* if $\mathrm{pr}_1(\bar\zeta(\mathcal{C}_1)) = \mathrm{pr}_1(\bar\zeta(\mathcal{C}_2))$ holds.

For a positive sign structure θ of T, let $-\theta$ denote the other positive sign structure. The mapping $(M, T, \alpha, \theta) \mapsto (M, T, \alpha, -\theta)$ defines an involution

$$c\colon \mathcal{Q}(\Phi, A)/{\sim_G} \;\to\; \mathcal{Q}(\Phi, A)/{\sim_G}.$$

It is obvious that, via the bijection $\bar{\zeta}$, this involution c corresponds to the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathfrak{C}(\Phi, A, G)$.

Definition 3.8. Let $\mathcal{C}_1, \mathcal{C}_2 \in \mathfrak{C}(\Phi, A, G)$ be two connected components. We say that $\mathcal{C}_1$ is *complex conjugate to* $\mathcal{C}_2$ if $c(\bar{\zeta}(\mathcal{C}_1)) = \bar{\zeta}(\mathcal{C}_2)$ holds. We say that $\mathcal{C}_1$ is *real* if $c(\bar{\zeta}(\mathcal{C}_1)) = \bar{\zeta}(\mathcal{C}_1)$ holds.

We fix an even overlattice $M \in \mathcal{E}(\Phi, A)$. Our next task is to calculate the fiber of pr_1 over the G-orbit $[M]$ containing M. Let $\mathrm{Stab}(M) \subset G$ denote the stabilizer subgroup of M for the action of G on $\mathcal{E}(\Phi, A)$. Then we have a natural homomorphism

$$\mathrm{Stab}(M) \to \mathrm{O}(M).$$

To ease the notation in the next section, we put

$$\mathcal{G} := \mathcal{G}(M), \quad (D_{\mathcal{G}}, q_{\mathcal{G}}) := (D_M, -q_M).$$

We further denote by

$$\bar{G} \subset \mathrm{O}(D_{\mathcal{G}}, q_{\mathcal{G}})$$

the image of $\mathrm{Stab}(M)$ by the natural homomorphism $\mathrm{O}(M) \to \mathrm{O}(D_{\mathcal{G}}, q_{\mathcal{G}})$.

Definition 3.9. Let $\mathcal{T}_{\mathcal{G}}$ be the set of triples (T, α, θ), where T is an even $\mathbb{Z}$-lattice belonging to $\mathcal{G}$, α is an isomorphism $(D_{\mathcal{G}}, q_{\mathcal{G}}) \xrightarrow{\sim} (D_T, q_T)$ of finite quadratic forms, and θ is a positive sign structure of T.

We define an equivalence relation $\sim_{\bar{G}}$ on $\mathcal{T}_{\mathcal{G}}$ by the following.

Definition 3.10. Let (T, α, θ) and (T', α', θ') be triples belonging to $\mathcal{T}_{\mathcal{G}}$. We put $(T, \alpha, \theta) \sim_{\bar{G}} (T', \alpha', \theta')$ if there exist an element $g \in \bar{G}$ and an isometry $\phi \colon T \xrightarrow{\sim} T'$ that satisfy the following:

- The diagram
$$\begin{array}{ccc} (D_{\mathcal{G}}, q_{\mathcal{G}}) & \xrightarrow{g} & (D_{\mathcal{G}}, q_{\mathcal{G}}) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ (D_T, q_T) & \xrightarrow[q_\phi]{} & (D_{T'}, q_{T'}) \end{array}$$
commutes.
- The isometry ϕ maps θ to θ'.

Then it is easy to see that the fiber of pr_1 over $[M] \in \mathcal{E}(\Phi, A)/G$ is canonically identified with $\mathcal{T}_{\mathcal{G}}/\!\sim_{\bar{G}}$. In the next section, we present an algorithm to calculate the set $\mathcal{T}_{\mathcal{G}}/\!\sim_{\bar{G}}$.

4. Miranda-Morrison theory

This section and the next section are devoted to purely lattice-theoretic investigations, and are completely independent of the geometry of $K3$ surfaces.

Let $\mathcal{G}$ be a non-empty genus of even $\mathbb{Z}$-lattices determined by a signature (t_+, t_-) with $t_+ = 2$ and a non-degenerate finite quadratic form $(D_{\mathcal{G}}, q_{\mathcal{G}})$. Let $\bar{G}$ be a subgroup of $\mathrm{O}(D_{\mathcal{G}}, q_{\mathcal{G}})$. We give an algorithm to calculate the set $\mathcal{T}_{\mathcal{G}}/\!\sim_{\bar{G}}$ defined in Definitions 3.9 and 3.10. We put

$$\mathrm{Sign} := \{1, -1\}.$$

For a vector v of an even lattice, we put

$$Q(v) := \frac{\langle v, v\rangle}{2}.$$

4.1. **Spinor norm.** Let R be $\mathbb{Z}$, $\mathbb{Z}_p$, or $\mathbb{R}$, let k denote the quotient field of R, and let L be an even R-lattice. Let v be a vector of $L \otimes k$ such that $Q(v) \neq 0$. Then we have the *reflection* $\tau(v) \in \mathrm{O}(L \otimes k)$ defined by

$$\tau(v)\colon x \mapsto x - \frac{\langle x, v\rangle}{Q(v)} v.$$

The classical theorem of Cartan (see [4, Chapter 1]) says that $\mathrm{O}(L\otimes k)$ is generated by reflections. Suppose that an isometry $g \in \mathrm{O}(L)$ is decomposed into a product $\tau(v_1)\cdots\tau(v_m)$ of reflections in $\mathrm{O}(L \otimes k)$. We define the *spinor norm* $\mathrm{spin}(g)$ of g by

$$\mathrm{spin}(g) := Q(v_1)\cdots Q(v_m) \bmod (k^\times)^2.$$

It is known that $\mathrm{spin}(g) \in k^\times/(k^\times)^2$ does not depend on the choice of the decomposition $g = \tau(v_1)\cdots\tau(v_m)$, and hence the map $\mathrm{spin}\colon \mathrm{O}(L) \to k^\times/(k^\times)^2$ is a group homomorphism. (See [4, Chapter 10].)

Remark 4.1. We use the definition of $\mathrm{spin}(g)$ of $g = \tau(v_1)\cdots\tau(v_m)$ given in [14], which differs from the one given in [4] by the multiplicative factor $2^m \in k^\times/(k^\times)^2$.

The following is due to [13]. See also [15].

Proposition 4.2. *Suppose that L is an $\mathbb{R}$-lattice, so that the spinor norm takes values in* Sign. *The action of an isometry $g \in \mathrm{O}(L)$ on the set of positive sign structures of L is trivial if and only if* $\det(g)\cdot\mathrm{spin}(g) > 0$ *holds.*

4.2. **The case of positive definite genus.** Suppose that $t_- = 0$, so that $\mathcal{G}$ is a genus of even positive definite $\mathbb{Z}$-lattices of rank 2. By an algorithm that goes back to Gauss (see, for example, [6, Chapter 15]), we can make the complete set of isomorphism classes of even positive definite $\mathbb{Z}$-lattices of rank 2 with discriminant $|D_{\mathcal{G}}|$. From this list, we sort out those lattices whose discriminant forms are isomorphic to $(D_{\mathcal{G}}, q_{\mathcal{G}})$, and calculate a complete set

$$\{T_1, \dots, T_k\}$$

of representatives of the genus $\mathcal{G}$. For each T in this list, we calculate the finite groups $\mathrm{O}(T)$ and $\mathrm{O}(D_T, q_T)$, an isomorphism $\alpha_0\colon (D_{\mathcal{G}}, q_{\mathcal{G}}) \xrightarrow{\sim} (D_T, q_T)$, and the natural homomorphism $\mathrm{O}(T) \to \mathrm{O}(D_T, q_T)$. Then the set of isomorphisms from $(D_{\mathcal{G}}, q_{\mathcal{G}})$ to (D_T, q_T) is equal to

$$\{\ \alpha_0 \cdot h \ \mid\ h \in \mathrm{O}(D_T, q_T)\ \} = \{\ h' \cdot \alpha_0 \ \mid\ h' \in \mathrm{O}(D_{\mathcal{G}}, q_{\mathcal{G}})\ \}.$$

Let $\alpha_{0*}\colon \mathrm{O}(D_{\mathcal{G}}, q_{\mathcal{G}}) \xrightarrow{\sim} \mathrm{O}(D_T, q_T)$ be the isomorphism induced by α_0. Since T is positive definite, an isometry $\tilde{h}$ of T preserves the positive sign structures of T if and only if $\det(\tilde{h}) = 1$. We make $\bar{G} \subset \mathrm{O}(D_{\mathcal{G}}, q_{\mathcal{G}})$ act on $\mathrm{O}(D_T, q_T) \times \mathrm{Sign}$ from the left by

$$(g, (\gamma, \theta)) \mapsto (\ \alpha_{0*}(g)\cdot\gamma,\ \theta\ ),\ \ \text{where } g \in \bar{G} \text{ and } (\gamma,\theta) \in \mathrm{O}(D_T, q_T) \times \mathrm{Sign}.$$

We also make $\mathrm{O}(T)$ act on $\mathrm{O}(D_T, q_T) \times \mathrm{Sign}$ from the right by

$$((\gamma,\theta),\ \tilde{h}) \mapsto (\ \gamma\cdot h,\ \det(\tilde{h})\cdot\theta\ ),\ \ \text{where } h \in \mathrm{O}(D_T, q_T) \text{ is induced by } \tilde{h} \in \mathrm{O}(T).$$

We consider the set of orbits

$$\mathrm{Orb}(T)\ :=\ \bar{G}\backslash(\mathrm{O}(D_T, q_T) \times \mathrm{Sign})/\,\mathrm{O}(T)$$

under these actions. Then the set of all $(\ T,\ \alpha_0\cdot\gamma,\ \theta\ ) \in \mathcal{T}_{\mathcal{G}}$, where T runs through the set $\{T_1, \dots, T_k\}$, and for each T, (γ, θ) runs through the set of representatives of $\mathrm{Orb}(T)$, is a complete set of representatives of $\mathcal{T}_{\mathcal{G}}/{\sim_{\bar{G}}}$. By this algorithm, we compute Table I.

4.3. **Miranda-Morrison theory.** From now on to the end of this section, we assume that $t_- > 0$. Hence $\mathcal{G}$ is a genus of even indefinite $\mathbb{Z}$-lattices of rank ≥ 3. We formulate a refinement of Miranda-Morrison theory [14] on the structure of a genus of this kind.

We first review the original version of Miranda-Morrison theory, which calculates the set $\mathcal{T}'_{\mathcal{G}}/{\sim}$ defined as follows. Let $\mathcal{T}'_{\mathcal{G}}$ be the set of pairs (T, α), where T is an even $\mathbb{Z}$-lattice belonging to $\mathcal{G}$, and α is an isomorphism $(D_{\mathcal{G}}, q_{\mathcal{G}}) \xrightarrow{\sim} (D_T, q_T)$. For elements (T, α) and (T', α') of $\mathcal{T}'_{\mathcal{G}}$, we put $(T, \alpha) \sim (T', \alpha')$ if there exists an isometry $\phi\colon T \xrightarrow{\sim} T'$ such that the diagram

$$\begin{array}{ccc} (D_{\mathcal{G}}, q_{\mathcal{G}}) & = & (D_{\mathcal{G}}, q_{\mathcal{G}}) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ (D_T, q_T) & \underset{q_\phi}{\longrightarrow} & (D_{T'}, q_{T'}) \end{array}$$

commutes.

We fix an element (L, λ) of $\mathcal{T}'_{\mathcal{G}}$, and put

$$\begin{aligned} \mathrm{O}_{\mathbb{A},0}(L) &:= \prod_p \mathrm{O}(L \otimes \mathbb{Z}_p), \\ \mathrm{O}_{\mathbb{A}}(L) &:= \{\, (\sigma_p) \in \prod_p \mathrm{O}(L \otimes \mathbb{Q}_p) \;\mid\; \sigma_p \in \mathrm{O}(L \otimes \mathbb{Z}_p) \text{ for almost all } p \,\}. \end{aligned}$$

Note that we have a natural homomorphism $\mathrm{O}(L \otimes \mathbb{Q}) \to \mathrm{O}_{\mathbb{A}}(L)$. Let $\boldsymbol{\sigma} = (\sigma_p)$ be an element of $\mathrm{O}_{\mathbb{A}}(L)$. Then there exists a unique $\mathbb{Z}$-submodule $L^{\boldsymbol{\sigma}}$ of $L \otimes \mathbb{Q}$ such that $L^{\boldsymbol{\sigma}} \otimes \mathbb{Z}_p = (L \otimes \mathbb{Z}_p)^{\sigma_p}$ holds in $L \otimes \mathbb{Q}_p$ for all p, where $(L \otimes \mathbb{Z}_p)^{\sigma_p}$ is the image of $L \otimes \mathbb{Z}_p \subset L \otimes \mathbb{Q}_p$ by $\sigma_p \in \mathrm{O}(L \otimes \mathbb{Q}_p)$. (See Theorem 4.1 in [14, Chapter VI].) We restrict the symmetric bilinear form of $L \otimes \mathbb{Q}$ to $L^{\boldsymbol{\sigma}}$. Since the $\mathbb{Z}_p$-lattices $L^{\boldsymbol{\sigma}} \otimes \mathbb{Z}_p$ and $L \otimes \mathbb{Z}_p$ are isomorphic for all p, we see that $L^{\boldsymbol{\sigma}}$ is an even $\mathbb{Z}$-lattice belonging to $\mathcal{G}$. Note that we have $L^{\boldsymbol{\sigma}} = L$ if and only if $\boldsymbol{\sigma} \in \mathrm{O}_{\mathbb{A},0}(L)$. Let $\boldsymbol{\tau} = (\tau_p)$ be an element of $\mathrm{O}_{\mathbb{A}}(L)$. Then each component τ_p of $\boldsymbol{\tau}$ induces an isometry $L^{\boldsymbol{\sigma}} \otimes \mathbb{Z}_p \xrightarrow{\sim} L^{\boldsymbol{\sigma\tau}} \otimes \mathbb{Z}_p$, and hence induces an isomorphism

$$q_{\tau_p}\colon (D_{L^{\boldsymbol{\sigma}} \otimes \mathbb{Z}_p}, q_{L^{\boldsymbol{\sigma}} \otimes \mathbb{Z}_p}) \xrightarrow{\sim} (D_{L^{\boldsymbol{\sigma\tau}} \otimes \mathbb{Z}_p}, q_{L^{\boldsymbol{\sigma\tau}} \otimes \mathbb{Z}_p}).$$

Their product over the primes p dividing $|\mathrm{disc}(L)| = |D_{\mathcal{G}}|$ gives rise to an isomorphism

$$q_{\boldsymbol{\tau}}|_{L^{\boldsymbol{\sigma}}}\colon (D_{L^{\boldsymbol{\sigma}}}, q_{L^{\boldsymbol{\sigma}}}) \xrightarrow{\sim} (D_{L^{\boldsymbol{\sigma\tau}}}, q_{L^{\boldsymbol{\sigma\tau}}}).$$

If $\boldsymbol{\tau} \in \mathrm{O}_{\mathbb{A},0}(L)$, then $q_{\boldsymbol{\tau}}|_L \in \mathrm{O}(D_L, q_L)$. As a corollary of Proposition 2.6, we obtain the following:

Proposition 4.3. *The homomorphism* $\mathrm{O}_{\mathbb{A},0}(L) \to \mathrm{O}(D_L, q_L)$ *given by* $\boldsymbol{\tau} \mapsto q_{\boldsymbol{\tau}}|_L$ *is surjective.*

For $\boldsymbol{\sigma} \in \mathrm{O}_{\mathbb{A}}(L)$, we put

$$\lambda^{\boldsymbol{\sigma}} := \lambda \cdot q_{\boldsymbol{\sigma}}|_L\colon (D_{\mathcal{G}}, q_{\mathcal{G}}) \xrightarrow{\sim} (D_{L^{\boldsymbol{\sigma}}}, q_{L^{\boldsymbol{\sigma}}}).$$

Thus we obtain a map $\mathrm{O}_{\mathbb{A}}(L) \to \mathcal{T}'_{\mathcal{G}}$ given by $\boldsymbol{\sigma} \mapsto (L^{\boldsymbol{\sigma}}, \lambda^{\boldsymbol{\sigma}})$. We show that this map is surjective. Let (T, α) be an arbitrary element of $\mathcal{T}'_{\mathcal{G}}$. By Proposition 2.14, we can assume that T is embedded into $L \otimes \mathbb{Q}$ isometrically, and hence we have $L \otimes \mathbb{Q} = T \otimes \mathbb{Q}$. Then the equality $L \otimes \mathbb{Z}_p = T \otimes \mathbb{Z}_p$ holds in $L \otimes \mathbb{Q}_p$ for almost all p. For each p, we have an isometry $\sigma_p\colon L\otimes\mathbb{Z}_p \xrightarrow{\sim} T\otimes\mathbb{Z}_p$, which we regard as an element of $\mathrm{O}(L\otimes\mathbb{Q}_p)$. We put $\boldsymbol{\sigma} := (\sigma_p)$. Since $L\otimes\mathbb{Z}_p = T\otimes\mathbb{Z}_p$ for almost all p, we see that $\boldsymbol{\sigma}$ belongs to $\mathrm{O}_{\mathbb{A}}(L)$, and we have $T = L^{\boldsymbol{\sigma}}$. We also obtain an isomorphism $q_{\boldsymbol{\sigma}}|_L\colon (D_L, q_L) \xrightarrow{\sim} (D_T, q_T)$. Consider the diagram

$$\begin{tikzcd} & (D_{\mathcal{G}}, q_{\mathcal{G}}) \arrow[dl, "\alpha"'] \arrow[dr, "\lambda"] & \\ (D_T, q_T) & & (D_L, q_L). \arrow[ll, "q_{\boldsymbol{\sigma}}|L"] \end{tikzcd}$$

We see that $\lambda^{-1}\cdot\alpha\cdot(q_{\boldsymbol{\sigma}}|_L)^{-1}$ belongs to $\mathrm{O}(D_L, q_L)$. By Proposition 4.3, there exists an element $\boldsymbol{\rho}\in\mathrm{O}_{\mathbb{A},0}(L)$ such that $q_{\boldsymbol{\rho}}|_L = \lambda^{-1}\cdot\alpha\cdot(q_{\boldsymbol{\sigma}}|_L)^{-1}$. Then we have $(T,\alpha) = (L^{\boldsymbol{\rho\sigma}}, \lambda^{\boldsymbol{\rho\sigma}})$. Therefore the mapping $\boldsymbol{\sigma}\mapsto (L^{\boldsymbol{\sigma}}, \lambda^{\boldsymbol{\sigma}})$ is surjective, and we obtain

$$\mathrm{O}_{\mathbb{A}}(L) \twoheadrightarrow \mathcal{T}'_{\mathcal{G}} \twoheadrightarrow \mathcal{T}'_{\mathcal{G}}/\!\sim.$$

Let U_p denote the image of the natural homomorphism $\mathbb{Z}_p^\times \hookrightarrow \mathbb{Q}_p^\times \to \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$. Recall that $\mathrm{Det} = \{1,-1\}$. We put

$$\Gamma_{p,0} := \mathrm{Det}\times U_p \quad\subset\quad \Gamma_p := \mathrm{Det}\ \times\ \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2.$$

Note that Γ_p is an elementary 2-group of rank 4 if $p=2$ and of rank 3 if $p>2$, and that $\Gamma_{p,0}$ is of index 2 in Γ_p. We consider the homomorphism

$$(\det, \mathrm{spin})\colon \mathrm{O}(L\otimes\mathbb{Z}_p)\to\Gamma_p.$$

Definition 4.4. Let $\mathrm{O}^\sharp(L\otimes\mathbb{Z}_p)$ denote the kernel of the natural homomorphism $\mathrm{O}(L\otimes\mathbb{Z}_p)\to\mathrm{O}(D_{L\otimes\mathbb{Z}_p}, q_{L\otimes\mathbb{Z}_p})$, and let $\Sigma^\sharp(L\otimes\mathbb{Z}_p)$ denote the image of $\mathrm{O}^\sharp(L\otimes\mathbb{Z}_p)$ by $(\det, \mathrm{spin})$.

The abelian group $\Sigma^\sharp(L\otimes\mathbb{Z}_p)$ is completely calculated in [16] and [14, Chapter VII]. In particular, we have the following proposition. (See Theorems 12.1-12.4 and Corollary 12.11 in [14, Chapter VII].) Recall that we have assumed that L is of rank ≥ 3.

Proposition 4.5. (1) *We have* $\Sigma^\sharp(L\otimes\mathbb{Z}_p)\subset\Gamma_{p,0}$.
(2) *If* $L\otimes\mathbb{Z}_p$ *is unimodular, then* $\Sigma^\sharp(L\otimes\mathbb{Z}_p) = \Gamma_{p,0}$.

We put

$$\Gamma_{\mathbb{A},0} := \prod_p \Gamma_{p,0} \quad\subset\quad \Gamma_{\mathbb{A}} := \{\ (\gamma_p)\in\prod_p\Gamma_p\ \mid\ \gamma_p\in\Gamma_{p,0}\ \text{ for almost all }\ p\ \}.$$

If $L\otimes\mathbb{Z}_p$ is unimodular, we have $\mathrm{O}^\sharp(L\otimes\mathbb{Z}_p) = \mathrm{O}(L\otimes\mathbb{Z}_p)$, and hence Proposition 4.5 implies that the image of $\mathrm{O}(L\otimes\mathbb{Z}_p)$ by $(\det,\mathrm{spin})$ is $\Gamma_{p,0}$. Since $L\otimes\mathbb{Z}_p$ is unimodular for almost all p, we obtain a homomorphism

$$(\det,\mathrm{spin})\colon \mathrm{O}_{\mathbb{A}}(L)\to\Gamma_{\mathbb{A}}.$$

We put

$$\Sigma^\sharp_{\mathbb{A}}(L) := \prod_p \Sigma^\sharp(L\otimes\mathbb{Z}_p) \quad\subset\quad \Gamma_{\mathbb{A},0}.$$

Finally, we put

$$\Gamma_{\mathbb{Q}} := \mathrm{Det}\ \times\ \mathbb{Q}^\times/(\mathbb{Q}^\times)^2,$$

and embed $\Gamma_{\mathbb{Q}}$ into $\Gamma_{\mathbb{A}}$ naturally. We have the following proposition. See Proposition 6.1 in [14, Chapter V].

Proposition 4.6. *If V is an indefinite $\mathbb{Q}$-lattice of rank ≥ 3, then the homomorphism* $(\det, \mathrm{spin})\colon \mathrm{O}(V) \to \Gamma_{\mathbb{Q}}$ *is surjective.*

One of the principal results of Miranda-Morrison theory is as follows (see Theorem 3.1 in [14, Chapter VIII]).

Theorem 4.7. *Let $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ be in $\mathrm{O}_{\mathbb{A}}(L)$. Then we have $(L^{\boldsymbol{\sigma}}, \lambda^{\boldsymbol{\sigma}}) \sim (L^{\boldsymbol{\tau}}, \lambda^{\boldsymbol{\tau}})$ if and only if*

$$(\det(\boldsymbol{\sigma}), \mathrm{spin}(\boldsymbol{\sigma})) \equiv (\det(\boldsymbol{\tau}), \mathrm{spin}(\boldsymbol{\tau})) \mod \Gamma_{\mathbb{Q}} \cdot \Sigma^{\sharp}_{\mathbb{A}}(L)$$

holds in $\Gamma_{\mathbb{A}}$. In particular, we can endow the set $\mathcal{T}'_{\mathcal{G}}/\!\sim$ with a structure of abelian group.

The main ingredient of the proof of Theorem 4.7 is the following corollary (Theorem 2.2 in [14, Chapter VIII]) of the strong approximation theorem (Theorem 7.1 in [4, Chapter 10]) for the *spin group*

$$\Theta_{\mathbb{A}}(L) := \mathrm{Ker}((\det, \mathrm{spin})\colon \mathrm{O}_{\mathbb{A}}(L) \to \Gamma_{\mathbb{A}}).$$

Recall that L is indefinite of rank ≥ 3.

Theorem 4.8. *Let $\boldsymbol{\sigma}$ be an element of $\mathrm{O}_{\mathbb{A}}(L)$. For any element $\boldsymbol{\psi}'$ of $\Theta_{\mathbb{A}}(L)$, there exists an isometry $\psi \in \mathrm{O}(L \otimes \mathbb{Q})$ such that $(\det(\psi), \mathrm{spin}(\psi)) = (1, 1)$, that $L^{\boldsymbol{\sigma}\psi} = L^{\boldsymbol{\sigma}\boldsymbol{\psi}'}$, and that $q_{\psi}|_{L^{\boldsymbol{\sigma}}}\colon (D_{L^{\boldsymbol{\sigma}}}, q_{L^{\boldsymbol{\sigma}}}) \xrightarrow{\sim} (D_{L^{\boldsymbol{\sigma}\psi}}, q_{L^{\boldsymbol{\sigma}\psi}})$ is equal to $q_{\boldsymbol{\psi}'}|_{L^{\boldsymbol{\sigma}}}$.*

Indeed, the set of all $\boldsymbol{\tau} \in \Theta_{\mathbb{A}}(L)$ that satisfy $L^{\boldsymbol{\sigma}\boldsymbol{\tau}} = L^{\boldsymbol{\sigma}\boldsymbol{\psi}'}$ and $q_{\boldsymbol{\tau}}|_{L^{\boldsymbol{\sigma}}} = q_{\boldsymbol{\psi}'}|_{L^{\boldsymbol{\sigma}}}$ is a non-empty open subset of $\Theta_{\mathbb{A}}(L)$ whose p-component coincides with

$$\Theta(L \otimes \mathbb{Z}_p) := \mathrm{Ker}((\det, \mathrm{spin})\colon \mathrm{O}(L \otimes \mathbb{Z}_p) \to \Gamma_p)$$

for almost all p.

Remark 4.9. Even though the definition of the spinor norm in [14] and in this paper differs from the one given in [4], the definition of the spin group is not affected, because, for any element $g = \tau(v_1) \cdots \tau(v_m)$ of a spin group, the condition $\det(g) = 1$ implies $m \equiv 0 \bmod 2$. See Remark 4.1.

4.4. **A refinement of Miranda-Morrison theory.** We refine Theorem 4.7 in order to incorporate the positive sign structures and the action of $\bar{G}$.

As in the previous section, we fix an element (L, λ, θ) of $\mathcal{T}_{\mathcal{G}}$. For each $\boldsymbol{\sigma} \in \mathrm{O}_{\mathbb{A}}(L)$, we have $L^{\boldsymbol{\sigma}} \otimes \mathbb{R} = L \otimes \mathbb{R}$, and hence the fixed positive sign structure θ of L induces a positive sign structure on $L^{\boldsymbol{\sigma}}$, which is denoted by the same symbol θ, and let $-\theta$ denote the other positive sign structure of $L^{\boldsymbol{\sigma}}$. Recall that $\mathrm{Sign} = \{\pm 1\}$. The surjectivity of the map $\mathrm{O}_{\mathbb{A}}(L) \to \mathcal{T}'_{\mathcal{G}}/\!\sim$ defined in the previous section implies that the mapping

$$(\boldsymbol{\sigma}, \varepsilon) \mapsto (L^{\boldsymbol{\sigma}}, \lambda^{\boldsymbol{\sigma}}, \varepsilon\theta)$$

induces a surjective map

$$\mathrm{O}_{\mathbb{A}}(L) \times \mathrm{Sign} \to \mathcal{T}_{\mathcal{G}}/\!\sim_{\bar{G}}.$$

Definition 4.10. We define a homomorphism

$$\Psi_p\colon \mathrm{O}(D_{L\otimes\mathbb{Z}_p}, q_{L\otimes\mathbb{Z}_p}) \to \Gamma_p/\Sigma^{\sharp}(L \otimes \mathbb{Z}_p)$$

by $g \mapsto (\det(\tilde{g}), \mathrm{spin}(\tilde{g})) \bmod \Sigma^{\sharp}(L \otimes \mathbb{Z}_p)$, where $\tilde{g} \in \mathrm{O}(L \otimes \mathbb{Z}_p)$ is an isometry that induces g on $(D_{L\otimes\mathbb{Z}_p}, q_{L\otimes\mathbb{Z}_p})$. (Since the natural homomorphism $\mathrm{O}(L\otimes\mathbb{Z}_p) \to \mathrm{O}(D_{L\otimes\mathbb{Z}_p}, q_{L\otimes\mathbb{Z}_p})$ is surjective (see Proposition 2.6), we can always find a lift $\tilde{g}$ of

g, and by the definition of $\Sigma^\sharp(L \otimes \mathbb{Z}_p)$, we see that $\Psi_p(g)$ does not depend on the choice of the lift $\tilde{g}$.)

Since $\mathrm{O}(D_L, q_L)$ is a product of $\mathrm{O}(D_{L\otimes\mathbb{Z}_p}, q_{L\otimes\mathbb{Z}_p})$, all of which is trivial except for p dividing $|\mathrm{disc}(L)| = |D_\mathcal{G}|$, we obtain a homomorphism

$$\Psi_\mathbb{A} \colon \mathrm{O}(D_L, q_L) \to \Gamma_\mathbb{A}/\Sigma^\sharp_\mathbb{A}(L).$$

Definition 4.11. Let $\bar{G}^\lambda$ denote the subgroup of $\mathrm{O}(D_L, q_L)$ corresponding to the given subgroup $\bar{G} \subset \mathrm{O}(D_\mathcal{G}, q_\mathcal{G})$ by the fixed isomorphism $\lambda \colon (D_\mathcal{G}, q_\mathcal{G}) \xrightarrow{\sim} (D_L, q_L)$:

$$\bar{G}^\lambda := \{\ \lambda^{-1} \cdot g \cdot \lambda \in \mathrm{O}(D_L, q_L) \ \mid \ g \in \bar{G}\ \}.$$

We then define $\Sigma(L, \bar{G}^\lambda)$ to be the subgroup of $\Gamma_\mathbb{A}$ containing $\Sigma^\sharp_\mathbb{A}(L)$ such that

$$\Psi_\mathbb{A}(\bar{G}^\lambda) = \Sigma(L, \bar{G}^\lambda)/\Sigma^\sharp_\mathbb{A}(L) \tag{4.1}$$

holds; that is,

$$\Sigma(L, \bar{G}^\lambda) := \left\{ \ \boldsymbol{\gamma} \in \Gamma_\mathbb{A} \ \left| \ \begin{array}{l} \text{there exists an element } \boldsymbol{\sigma} \in \mathrm{O}_{\mathbb{A},0}(L) \text{ such that} \\ q_{\boldsymbol{\sigma}}|_L \in \bar{G}^\lambda \text{ and that } (\det(\boldsymbol{\sigma}), \mathrm{spin}(\boldsymbol{\sigma})) = \boldsymbol{\gamma} \end{array} \right. \right\}.$$

We have a natural homomorphism $\Gamma_\mathbb{Q} \to \mathrm{Sign}$ that maps (d, s) to the sign of ds, where $d \in \mathrm{Det}$ and $s \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$. We then define an embedding

$$\Gamma_\mathbb{Q} \hookrightarrow \Gamma_\mathbb{A} \times \mathrm{Sign}$$

by the product of the natural embedding $\Gamma_\mathbb{Q} \hookrightarrow \Gamma_\mathbb{A}$ and the above homomorphism $\Gamma_\mathbb{Q} \to \mathrm{Sign}$, and denote by $\Gamma^\sim_\mathbb{Q}$ the image of $\Gamma_\mathbb{Q}$ in $\Gamma_\mathbb{A} \times \mathrm{Sign}$.

The main result of this subsection is as follows:

Theorem 4.12. *Let* $(\boldsymbol{\sigma}, \varepsilon)$ *and* $(\boldsymbol{\tau}, \eta)$ *be elements of* $\mathrm{O}_\mathbb{A}(L) \times \mathrm{Sign}$. *Then we have* $(L^{\boldsymbol{\sigma}}, \lambda^{\boldsymbol{\sigma}}, \varepsilon) \sim_{\bar{G}} (L^{\boldsymbol{\tau}}, \lambda^{\boldsymbol{\tau}}, \eta)$ *if and only if*

$$(\det(\boldsymbol{\sigma}), \mathrm{spin}(\boldsymbol{\sigma}), \varepsilon) \equiv (\det(\boldsymbol{\tau}), \mathrm{spin}(\boldsymbol{\tau}), \eta) \mod \Gamma^\sim_\mathbb{Q} \cdot (\Sigma(L, \bar{G}^\lambda) \times \{1\}) \tag{4.2}$$

holds in $\Gamma_\mathbb{A} \times \mathrm{Sign}$.

Proof. The proof is parallel to that of Theorem 4.7 in [14, Chapter VIII]. Suppose that $(L^{\boldsymbol{\sigma}}, \lambda^{\boldsymbol{\sigma}}, \varepsilon) \sim_{\bar{G}} (L^{\boldsymbol{\tau}}, \lambda^{\boldsymbol{\tau}}, \eta)$ holds. Then there exist an element $g \in \bar{G}^\lambda$ and an isometry $\phi \colon L^{\boldsymbol{\sigma}} \xrightarrow{\sim} L^{\boldsymbol{\tau}}$ of even $\mathbb{Z}$-lattices such that the diagram

$$\begin{array}{ccc} (D_L, q_L) & \xrightarrow{g} & (D_L, q_L) \\ {\scriptstyle q_{\boldsymbol{\sigma}}|_L} \downarrow & & \downarrow {\scriptstyle q_{\boldsymbol{\tau}}|_L} \\ (D_{L^{\boldsymbol{\sigma}}}, q_{L^{\boldsymbol{\sigma}}}) & \xrightarrow[q_\phi]{} & (D_{L^{\boldsymbol{\tau}}}, q_{L^{\boldsymbol{\tau}}}) \end{array} \tag{4.3}$$

commutes, and that

$$\varepsilon \cdot \det(\phi \otimes \mathbb{R}) \cdot \mathrm{spin}(\phi \otimes \mathbb{R}) = \eta \tag{4.4}$$

holds by Proposition 4.2. (Note that we have $\mathrm{spin}(\phi \otimes \mathbb{R}) \in \mathbb{R}^\times/(\mathbb{R}^\times)^2 = \{\pm 1\}$.) We have an element $\tilde{\boldsymbol{g}} \in \mathrm{O}_{\mathbb{A},0}(L)$ that induces g on (D_L, q_L) by the surjectivity of $\mathrm{O}_{\mathbb{A},0}(L) \to \mathrm{O}(D_L, q_L)$ (see Proposition 4.3). Then the product $\tilde{\boldsymbol{g}} \cdot \boldsymbol{\tau} \cdot \phi^{-1} \cdot \boldsymbol{\sigma}^{-1}$ belongs to $\mathrm{O}_{\mathbb{A},0}(L)$, and it induces an identity on (D_L, q_L) by the commutativity of (4.3). Hence we have

$$(\det(\boldsymbol{\sigma} \cdot \phi \cdot \boldsymbol{\tau}^{-1}),\ \mathrm{spin}(\boldsymbol{\sigma} \cdot \phi \cdot \boldsymbol{\tau}^{-1})) \bmod \Sigma^\sharp_\mathbb{A}(L) \ \ = \ \ \Psi_\mathbb{A}(g).$$

In particular, we have

$$(\det(\boldsymbol{\sigma}), \mathrm{spin}(\boldsymbol{\sigma})) \cdot (\det(\boldsymbol{\tau}), \mathrm{spin}(\boldsymbol{\tau}))^{-1} \cdot (\det(\phi), \mathrm{spin}(\phi)) \in \Sigma(L, \bar{G}^\lambda).$$

Note that $(\det(\phi), \mathrm{spin}(\phi)) \in \Gamma_{\mathbb{Q}}$, and that (4.4) implies that $(\det(\phi), \mathrm{spin}(\phi), \varepsilon^{-1}\eta)$ belongs to $\Gamma_{\mathbb{Q}}^{\sim}$. Hence (4.2) holds.

Conversely, suppose that (4.2) holds. By the definition of $\Sigma(L, \bar{G}^{\lambda})$ and the surjectivity of $\mathrm{O}(L \otimes \mathbb{Q}) \to \Gamma_{\mathbb{Q}}$ (see Proposition 4.6), we obtain $\tilde{\boldsymbol{g}} \in \mathrm{O}_{\mathbb{A},0}(L)$ and $\xi \in \mathrm{O}(L \otimes \mathbb{Q})$ such that

- (i) the automorphism g of (D_L, q_L) induced by $\tilde{\boldsymbol{g}}$ belongs to $\bar{G}^{\lambda}$,
- (ii) $(\det(\tilde{\boldsymbol{g}}), \mathrm{spin}(\tilde{\boldsymbol{g}})) \cdot (\det(\boldsymbol{\tau}), \mathrm{spin}(\boldsymbol{\tau})) = (\det(\boldsymbol{\sigma}), \mathrm{spin}(\boldsymbol{\sigma})) \cdot (\det(\xi), \mathrm{spin}(\xi))$ holds in $\Gamma_{\mathbb{A}}$, and
- (iii) $\varepsilon \cdot \det(\xi \otimes \mathbb{R}) \cdot \mathrm{spin}(\xi \otimes \mathbb{R}) = \eta$.

We put

$$\boldsymbol{\psi}' := \xi^{-1} \cdot \boldsymbol{\sigma}^{-1} \cdot \tilde{\boldsymbol{g}} \cdot \boldsymbol{\tau} \in \mathrm{O}_{\mathbb{A}}(L).$$

Note that we have $L^{\boldsymbol{\sigma}\xi\boldsymbol{\psi}'} = L^{\boldsymbol{\tau}}$. Since we have $(\det(\boldsymbol{\psi}'), \mathrm{spin}(\boldsymbol{\psi}')) = (1,1)$ by property (ii) above, Theorem 4.8 implies that there exists an element $\psi \in \mathrm{O}(L \otimes \mathbb{Q})$ such that $(\det(\psi), \mathrm{spin}(\psi)) = (1,1)$, that $L^{\boldsymbol{\sigma}\xi\psi} = L^{\boldsymbol{\tau}}$, and that the isomorphism from $(D_{L^{\boldsymbol{\sigma}\xi}}, q_{L^{\boldsymbol{\sigma}\xi}})$ to $(D_{L^{\boldsymbol{\tau}}}, q_{L^{\boldsymbol{\tau}}})$ induced by ψ is equal to

$$q_{\boldsymbol{\psi}'}|_{L^{\boldsymbol{\sigma}\xi}} = (q_{\xi}|_{L^{\boldsymbol{\sigma}}})^{-1} \cdot (q_{\boldsymbol{\sigma}}|_{L})^{-1} \cdot g \cdot (q_{\boldsymbol{\tau}}|_{L}).$$

We put

$$\phi := \xi \cdot \psi,$$

which belongs to $\mathrm{O}(L \otimes \mathbb{Q})$. Then the diagram (4.3) commutes. Moreover, since $(\det(\psi), \mathrm{spin}(\psi)) = (1,1)$, we have $\varepsilon \cdot \det(\phi \otimes \mathbb{R}) \cdot \mathrm{spin}(\phi \otimes \mathbb{R}) = \eta$ by property (iii) above. Therefore we obtain $(L^{\boldsymbol{\sigma}}, \lambda^{\boldsymbol{\sigma}}, \varepsilon) \sim_{\bar{G}} (L^{\boldsymbol{\tau}}, \lambda^{\boldsymbol{\tau}}, \eta)$. □

Therefore the set $\mathcal{T}_{\mathcal{G}}/\!\sim_{\bar{G}}$ can be equipped with a structure of abelian group:

$$\mathcal{T}_{\mathcal{G}}/\!\sim_{\bar{G}} \;=\; (\Gamma_{\mathbb{A}} \times \mathrm{Sign}) \,/\, (\Gamma_{\mathbb{Q}}^{\sim} \cdot (\Sigma(L, \bar{G}^{\lambda}) \times \{1\})). \tag{4.5}$$

Let $P(d) = \{p_1, \dots, p_m\}$ denote the set of primes that divide

$$d := |D_{\mathcal{G}}| = |D_L| = |\mathrm{disc}(L)|.$$

We put

$$\Gamma_d := \prod_{p \in P(d)} \Gamma_p, \quad \Gamma_{\mathbb{A},d} := \left(\prod_{p \notin P(d)} \Gamma_{p,0} \right) \times \Gamma_d.$$

We show that the abelian group $\mathcal{T}_{\mathcal{G}}/\!\sim_{\bar{G}}$ is isomorphic to a quotient of $\Gamma_d \times \mathrm{Sign}$, and present a set of generators of the kernel K of the quotient homomorphism $\Gamma_d \times \mathrm{Sign} \to \mathcal{T}_{\mathcal{G}}/\!\sim_{\bar{G}}$. Note that the finite abelian group $\Gamma_d \times \mathrm{Sign}$ is 2-elementary, and hence the computation below can be carried out by linear algebra over $\mathbb{F}_2$.

Lemma 4.13. *We have* $\Gamma_{\mathbb{A}} \times \mathrm{Sign} = (\Gamma_{\mathbb{A},d} \times \mathrm{Sign}) \cdot \Gamma_{\mathbb{Q}}^{\sim}$.

Proof. This follows from $\Gamma_{\mathbb{A},0} \subset \Gamma_{\mathbb{A},d}$ and $\Gamma_{\mathbb{A}} = \Gamma_{\mathbb{A},0} \cdot \Gamma_{\mathbb{Q}}$ (see Lemma 4.1 in [14, Chapter VIII]). □

By this lemma, we have an exact sequence

$$0 \;\to\; (\Gamma_{\mathbb{A},d} \times \mathrm{Sign}) \cap \Gamma_{\mathbb{Q}}^{\sim} \;\to\; \Gamma_{\mathbb{A},d} \times \mathrm{Sign} \;\to\; (\Gamma_{\mathbb{A}} \times \mathrm{Sign})/\Gamma_{\mathbb{Q}}^{\sim} \;\to\; 0.$$

By the definition (4.1) of $\Sigma(L, \bar{G}^{\lambda})$ and Proposition 4.5, we see that $\Sigma(L, \bar{G}^{\lambda})$ is contained in $\Gamma_{\mathbb{A},d}$. Hence the finite abelian group $\mathcal{T}_{\mathcal{G}}/\!\sim_{\bar{G}}$ is isomorphic to the cokernel of

$$(\Gamma_{\mathbb{A},d} \times \mathrm{Sign}) \cap \Gamma_{\mathbb{Q}}^{\sim} \;\hookrightarrow\; \Gamma_{\mathbb{A},d} \times \mathrm{Sign} \;\twoheadrightarrow\; (\,\Gamma_{\mathbb{A},d} \,/\, \Sigma(L, \bar{G}^{\lambda})\,) \times \mathrm{Sign}.$$

Recall that $\bar{G}^\lambda$ is a subgroup of

$$\mathrm{O}(D_L, q_L) = \prod_{p \in P(d)} \mathrm{O}(D_{L\otimes\mathbb{Z}_p}, q_{L\otimes\mathbb{Z}_p}). \tag{4.6}$$

Suppose that $\bar{G}^\lambda$ is generated by $g_1, \dots, g_k$. Let p be a prime in $P(d)$. We denote by $g_i[p]$ the p-component of $g_i \in \bar{G}^\lambda$ under the direct-sum decomposition (4.6). We then choose an isometry

$$g_i[p]^\sim \ \in \ \mathrm{O}(L \otimes \mathbb{Z}_p)$$

that induces $g_i[p]$ on $(D_{L\otimes\mathbb{Z}_p}, q_{L\otimes\mathbb{Z}_p})$. Then $\Psi_p(g_i) \in \Gamma_p/\Sigma^\sharp(L\otimes\mathbb{Z}_p)$ is represented by $(\det(g_i[p]^\sim), \mathrm{spin}(g_i[p]^\sim)) \in \Gamma_p$. We put

$$\gamma(g_i) \ := \ \big(\ (\det(g_i[p]^\sim),\ \mathrm{spin}(g_i[p]^\sim)) \ \mid\ p \in P(d)\ \big) \ \in \ \Gamma_d.$$

Remark that $\gamma(g_i)$ *does* depend on the choice of the lifts $g_i[p]^\sim$ of $g_i[p]$, but that $\gamma(g_i)$ modulo $\prod_{p\in P(d)} \Sigma^\sharp(L\otimes\mathbb{Z}_p)$ is uniquely determined by g_i. By Proposition 4.5, the projection $\Gamma_{\mathbb{A},d} \to \Gamma_d$ induces an isomorphism from $\Gamma_{\mathbb{A},d} \,/\, \Sigma(L, \bar{G}^\lambda)$ to the group

$$\Gamma_d \ / \ \langle\ \Sigma^\sharp(L\otimes\mathbb{Z}_{p_1}), \dots, \Sigma^\sharp(L\otimes\mathbb{Z}_{p_m}),\ \gamma(g_1), \dots, \gamma(g_k)\ \rangle.$$

On the other hand, the group $(\Gamma_{\mathbb{A},d} \times \mathrm{Sign}) \cap \Gamma_{\mathbb{Q}}^\sim$ is generated by the following $2 + |P(d)|$ elements of $\Gamma_{\mathbb{Q}}^\sim \subset \mathrm{Det} \times (\mathbb{Q}^\times/(\mathbb{Q}^\times)^2) \times \mathrm{Sign}$:

$$(-1, 1, -1),\ \ (1, -1, -1),\ \ \text{and}\ \ (1, p_j, 1)\ \ \text{for}\ p_j \in P(d).$$

We put $S(d) := \{p_1^{\nu_1} \cdots p_m^{\nu_m} \mid \nu_j = 0 \text{ or } 1 \text{ for } j = 1, \dots, m\}$. The image of $(\varepsilon, \eta s, \varepsilon\eta) \in (\Gamma_{\mathbb{A},d} \times \mathrm{Sign}) \cap \Gamma_{\mathbb{Q}}^\sim$, where $\varepsilon \in \mathrm{Det}$, $\eta \in \mathrm{Sign}$, and $s \in S(d)$, by the projection $\Gamma_{\mathbb{A},d} \times \mathrm{Sign} \to \Gamma_d \times \mathrm{Sign}$ is

$$\beta(\varepsilon, \eta s, \varepsilon\eta) := \big(\ (\,\varepsilon,\ \eta s \bmod (\mathbb{Q}_{p_j}^\times)^2\,) \mid p_j \in P(d)),\ \ \varepsilon\eta\ \big).$$

Hence we obtain the following:

Proposition 4.14. *The finite abelian group $\mathcal{T}_{\mathcal{G}}/{\sim_{\bar{G}}}$ is isomorphic to $(\Gamma_d \times \mathrm{Sign})/K$, where K is generated by the following subgroups and elements of $\Gamma_d \times \mathrm{Sign}$:*

(i) $\Sigma^\sharp(L\otimes\mathbb{Z}_{p_j}) \times \{1\}$ *for* $p_j \in P(d)$,
(ii) $(\gamma(g_i), 1)$, *where $\bar{G}^\lambda$ is generated by $g_1, \dots, g_k$, and*
(iii) $\beta(-1, 1, -1)$, $\beta(1, -1, -1)$, *and* $\beta(1, p_j, 1)$ *for* $p_j \in P(d)$.

The groups $\Sigma^\sharp(L\otimes\mathbb{Z}_p)$ for $p \in P(d)$ have been calculated in terms of $\mathrm{rank}(L\otimes\mathbb{Z}_p)$, $\mathrm{disc}(L\otimes\mathbb{Z}_p)$, and $(D_{L\otimes\mathbb{Z}_p}, q_{L\otimes\mathbb{Z}_p})$ in [16] and [14, Chapter VII]. The computation of $\beta(\varepsilon, \eta s, \varepsilon\eta)$ can be carried out by an elementary number theory. Therefore, in order to make an algorithm to calculate $\mathcal{T}_{\mathcal{G}}/{\sim_{\bar{G}}}$, it is enough to write a sub-algorithm to calculate $\gamma(g)$ for an arbitrary element $g \in \mathrm{O}(D_L, q_L)$. For $p \in P(d)$, the p-part $g[p] \in \mathrm{O}(D_{L\otimes\mathbb{Z}_p}, q_{L\otimes\mathbb{Z}_p})$ of g is easily calculated, because $D_{L\otimes\mathbb{Z}_p}$ is the p-part of the finite abelian group D_L. An algorithm to find a lift $g[p]^\sim \in \mathrm{O}(L\otimes\mathbb{Z}_p)$ and to calculate its value by $(\det, \mathrm{spin})$ is presented in the next section.

Remark 4.15. Let K' denote the subgroup of $\Gamma_d \times \mathrm{Sign}$ generated by the subgroups in (i) and the elements in (iii) of Proposition 4.14. If $\dim_{\mathbb{F}_2}(\Gamma_d \times \mathrm{Sign})/K' = 0$, then $\mathcal{T}_{\mathcal{G}}/{\sim_{\bar{G}}}$ is obviously trivial, and we do not have to calculate $\Psi_p(g[p])$. We have $\dim_{\mathbb{F}_2}(\Gamma_d \times \mathrm{Sign})/K' > 0$ for 319 algebraic equivalence classes of connected components. See Section 6.2 for some cases where $K \neq K'$.

Remark 4.16. The cokernel of the natural $\mathbb{F}_2$-linear homomorphism

$$K \quad \hookrightarrow \quad \Gamma_d \times \mathrm{Sign} \xrightarrow{\mathrm{pr}_1} \Gamma_d$$

calculates the set of connected components modulo complex conjugation. By this method, we can show that the two connected components of the moduli of each type (Φ, A) in Corollary 1.5 are complex conjugate to each other.

5. Computation of the homomorphism Ψ_p

Throughout this section, we fix a prime p, a non-degenerate p-adic finite quadratic form (D, q), and an automorphism $g \in \mathrm{O}(D, q)$. We assume the following:

Assumption 5.1. The finite quadratic form (D, q) is isomorphic to the discriminant form of an even $\mathbb{Z}_p$-lattice L of rank r and discriminant d. (By Proposition 2.5, this even $\mathbb{Z}_p$-lattice L is unique up to isomorphism.)

Our goal is to construct an algorithm to calculate an element of Γ_p that represents $\Psi_p(g) \in \Gamma_p/\Sigma^\sharp(L)$; that is, an algorithm that finds an isometry $\tilde{g} \in \mathrm{O}(L)$ inducing g on (D, q), and then calculates $(\det(\tilde{g}), \mathrm{spin}(\tilde{g})) \in \Gamma_p$. Let $b\colon D \times D \to \mathbb{Q}/\mathbb{Z}$ denote the bilinear form associated with (D, q). We put

$$\ell := \mathrm{leng}(D),$$

and suppose that

$$D \quad \cong \quad \mathbb{Z}/p^{\nu_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{\nu_\ell}\mathbb{Z}.$$

We fix generators $\varepsilon_1, \dots, \varepsilon_\ell$ of D such that ε_j generates the jth factor $\mathbb{Z}/p^{\nu_j}\mathbb{Z}$ of D. We denote by $\mathrm{M}_\ell(R)$ the R-module of $\ell \times \ell$ matrices with components in R, where R is $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{F}_p$, or the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at the prime ideal (p). A matrix in $\mathrm{M}_\ell(R)$ is said to be *even symmetric* if it is symmetric and its diagonal components are in $2R$. We denote by $\Delta(R)$ the submodule of $\mathrm{M}_\ell(R)$ consisting of even symmetric matrices. Note that, if $A \in \Delta(R)$ and $B \in \mathrm{M}_\ell(R)$, then we have $B \cdot A \cdot {}^tB \in \Delta(R)$, where ${}^tB$ is the transpose of B.

Then the quadratic form q on D is expressed by

$$F_q \bmod \Delta(\mathbb{Z}),$$

where F_q is a symmetric matrix in $\mathrm{M}_\ell(\mathbb{Q})$ whose (i, j)-component represents

$$q(\varepsilon_i) \in \mathbb{Q}/2\mathbb{Z} \quad \text{if } i = j, \qquad b(\varepsilon_i, \varepsilon_j) \in \mathbb{Q}/\mathbb{Z} \quad \text{if } i \neq j.$$

Since q is non-degenerate, we have $\det F_q \neq 0$.

We denote by $N_D(R)$ the submodule of $\mathrm{M}_\ell(R)$ consisting of matrices whose components in the jth column are divisible by p^{ν_j}. The given automorphism $v \mapsto v^g$ of the finite abelian group D is expressed by

$$T_0 \bmod N_D(\mathbb{Z}),$$

where T_0 is an element of $\mathrm{M}_\ell(\mathbb{Z})$ whose (i, j)-components t_{ij} satisfy

$$\varepsilon_i^g = \sum_{j=1}^{\ell} t_{ij}\varepsilon_j.$$

Since g preserves q, we have

$$T_0 \cdot F_q \cdot {}^tT_0 \;\equiv\; F_q \mod \Delta(\mathbb{Z}). \tag{5.1}$$

Since $g^{-1} \in \mathrm{O}(D, q)$ exists, there exists a matrix $T_0^{(-1)} \in \mathrm{M}_\ell(\mathbb{Z})$ such that

$$T_0 \cdot T_0^{(-1)} \;\equiv\; I_\ell \mod N_D(\mathbb{Z}).$$

In particular, the matrix $T_0 \bmod p \in \mathrm{M}_\ell(\mathbb{F}_p)$ is invertible.

Therefore, the algorithm we are going to construct is specified as follows:

Input (1) A sequence $[p^{\nu_1}, \dots, p^{\nu_\ell}]$ that describes the order of each element in a minimal set of generators $\varepsilon_1, \dots, \varepsilon_\ell$ of D.

(2) A symmetric matrix $F_q \in \mathrm{M}_\ell(\mathbb{Q})$ that represents q with respect to $\varepsilon_1, \dots, \varepsilon_\ell$.

(3) A matrix $T_0 \in \mathrm{M}_\ell(\mathbb{Z})$ that represents the automorphism $g \in \mathrm{O}(D, q)$ with respect to $\varepsilon_1, \dots, \varepsilon_\ell$.

Output An element $(\det(\tilde{g}), \mathrm{spin}(\tilde{g}))$ of Γ_p that represents $\Psi_p(g)$.

5.1. **Step 1.** By the normal form theorem (Proposition 2.3) of non-degenerate p-adic finite quadratic forms, there exists an algorithm to calculate an automorphism $v \mapsto v^h$ of D represented by $H \bmod N_D(\mathbb{Z})$ such that $H \cdot F_q \cdot {}^t H$ is equivalent modulo $\Delta(\mathbb{Z})$ to a matrix $F' \in \mathrm{M}_\ell(\mathbb{Q})$ in normal form; that is, F' is a block-diagonal matrix with diagonal components being matrices that appear in Table 2.1. We replace the basis $\varepsilon_1, \dots, \varepsilon_\ell$ of D with the new basis, and assume that F_q is in normal form. Accordingly, we replace the matrix T_0 representing $g \in \mathrm{O}(D, q)$ by $H\, T_0\, H^{(-1)}$, where $H^{(-1)} \in \mathrm{M}_\ell(\mathbb{Z})$ is a matrix such that $H^{(-1)} \bmod N_D(\mathbb{Z})$ represents h^{-1}.

5.2. **Step 2.** We put

$$M := F_q^{-1} \in \mathrm{M}_\ell(\mathbb{Q}).$$

Looking at Table 2.1, we see that

$$M \in \Delta(\mathbb{Z}_{(p)}) \quad \text{and} \quad M \equiv O \bmod p, \tag{5.2}$$

where O is the zero matrix. Let Λ be an even $\mathbb{Z}_p$-lattice of rank ℓ with a fixed basis $e_1, \dots, e_\ell$ whose Gram matrix is M. Then the discriminant form of Λ is isomorphic to (D, q). Recall from Assumption 5.1 that L is an even $\mathbb{Z}_p$-lattice of rank r, discriminant d, and with $(D_L, q_L) \cong (D, q)$. By the normal form theorem of even $\mathbb{Z}_p$-lattices (Proposition 2.4), we obtain the following.

- Suppose that $r > \ell$. Then there exists an even *unimodular* $\mathbb{Z}_p$-lattice Λ_0 such that L is isomorphic to the orthogonal direct-sum $\Lambda_0 \oplus \Lambda$. (In particular, if $p = 2$, we have $r \equiv \ell \bmod 2$.)
- Suppose that $r = \ell$ and p is odd. Then L is isomorphic to Λ.
- Suppose that $r = \ell$ and $p = 2$. Suppose that L is not isomorphic to Λ. Then at least one of the matrices on the diagonal of F_q is of the form $[\varepsilon/2]$, where $\varepsilon \in \{1, 3\}$. We replace one of such components with $[5\varepsilon/2]$, and re-calculate $M := F_q^{-1}$. (This change does not affect the class of F_q modulo $\Delta(\mathbb{Z})$ and preserves the property (5.2).) Then L is isomorphic to Λ.

Thus we obtain an even $\mathbb{Z}_p$-lattice Λ of rank ℓ with the following properties.

(i) The $\mathbb{Z}_p$-lattice Λ is an orthogonal direct summand of L.

(ii) The Gram matrix M of Λ with respect to a basis $e_1, \dots, e_\ell$ satisfies the property (5.2), and $M^{-1} \bmod \Delta(\mathbb{Z})$ expresses (D, q). More precisely, let $e_1^\vee, \dots, e_\ell^\vee$ denote the basis of $\Lambda^\vee$ dual to $e_1, \dots, e_\ell$. Then the homomorphism $\Lambda^\vee \to D$ given by $e_i^\vee \mapsto \varepsilon_i$ induces an isomorphism $(D_\Lambda, q_\Lambda) \xrightarrow{\sim} (D, q)$.

We have a surjective homomorphism $\mathrm{O}(\Lambda) \to \mathrm{O}(D, q)$ by Proposition 2.6. By property (i) of Λ, every isometry $\tilde{h}_\Lambda$ of Λ can be extended to an isometry $\tilde{h}_L$ of L by letting it act on the orthogonal complement $\Lambda^\perp \subset L$ trivially. Note that $\tilde{h}_\Lambda$ and $\tilde{h}_L$ induce the same automorphism on (D, q), and their (det, spin)-values are equal. Therefore it suffices to find an element $\tilde{g}_\Lambda \in \mathrm{O}(\Lambda)$ that induces the given automorphism g of (D, q), and then to calculate $(\det(\tilde{g}_\Lambda), \mathrm{spin}(\tilde{g}_\Lambda))$.

5.3. **Step 3.** Our next task is to find a sequence $\tilde{T}_\nu$ ($\nu = 0, 1, \dots$) of matrices in $\mathrm{M}_\ell(\mathbb{Z}_{(p)})$ converging to a matrix $\tilde{T} \in \mathrm{M}_\ell(\mathbb{Z}_p)$ that represents with respect to the basis $e_1, \dots, e_\ell$ of Λ an isometry $\tilde{g} \in \mathrm{O}(\Lambda)$ inducing $g \in \mathrm{O}(D, q)$.

Lemma 5.2. *We have*

$$N_D(\mathbb{Z}_p) = \{\, YM \mid Y \in \mathrm{M}_\ell(\mathbb{Z}_p) \,\}.$$

Proof. Let $v = (p^{\nu_1} a_1, \dots, p^{\nu_\ell} a_\ell)$ be a row vector of a matrix belonging to $N_D(\mathbb{Z}_p)$. Since $p^{\nu_i} \varepsilon_i = 0$ holds for $i = 1, \dots, \ell$, the mapping $x \mapsto vF_q\, {}^t x$ expresses the homomorphism $D \to \mathbb{Q}_p/\mathbb{Z}_p$ given by $x \mapsto b(0, x)$. Therefore $vF_q = vM^{-1}$ has components in $\mathbb{Z}_p$. Conversely, note that the i-th row vector $(m_{i1}, \dots, m_{i\ell})$ of M is a vector representation of $e_i \in \Lambda$ with respect to the dual basis $e_1^\vee, \dots, e_\ell^\vee$ of $\Lambda^\vee$. Therefore $m_{i1}\varepsilon_1 + \dots + m_{i\ell}\varepsilon_\ell = 0$ holds in D, and hence $m_{ij}\varepsilon_j = 0$ holds for $j = 1, \dots, \ell$. In particular, we have $M \in N_D(\mathbb{Z}_p)$. □

The Gram matrix of $\Lambda^\vee$ with respect to the basis $e_1^\vee, \dots, e_\ell^\vee$ is $M^{-1} = F_q$. Recall that $T_0 \in \mathrm{M}_\ell(\mathbb{Z})$ is a matrix such that $T_0 \bmod N_D(\mathbb{Z})$ represents $g \in \mathrm{O}(D, q)$.

Lemma 5.3. *For $\tilde{T} \in \mathrm{M}_\ell(\mathbb{Q}_p)$, the following conditions are equivalent.*

(i) *The matrix $\tilde{T}$ represents with respect to the basis $e_1, \dots, e_\ell$ of Λ an isometry $\tilde{g} \in \mathrm{O}(\Lambda)$ that induces $g \in \mathrm{O}(D, q)$.*

(ii) *We put*
$$T := M^{-1}\tilde{T}M.$$
Then $(T - T_0)M^{-1} \in \mathrm{M}_\ell(\mathbb{Z}_p)$ and $T \cdot M^{-1} \cdot {}^tT = M^{-1}$ hold.

Proof. Suppose that $\tilde{T}$ satisfies condition (i). The isometry $\tilde{g}$ induces an isometry of $\Lambda^\vee$, and $T = M^{-1}\tilde{T}M$ is the matrix representation of this isometry with respect to $e_1^\vee, \dots, e_\ell^\vee$. Hence we have $T \cdot M^{-1} \cdot {}^tT = M^{-1}$. Since T induces g on D and the identification $D = \Lambda^\vee/\Lambda$ is given by $e_i^\vee \mapsto \varepsilon_i$, we have

$$T \equiv T_0 \bmod N_D(\mathbb{Z}_p).$$

By Lemma 5.2, we have $(T - T_0)M^{-1} \in \mathrm{M}_\ell(\mathbb{Z}_p)$.

Conversely, suppose that $\tilde{T}$ satisfies condition (ii). Then $\tilde{T} = MTM^{-1}$ satisfies $\tilde{T} \cdot M \cdot {}^t\tilde{T} = M$. We show that $\tilde{T}$ has components in $\mathbb{Z}_p$. As was seen above, $T_0 \bmod p \in \mathrm{M}_\ell(\mathbb{F}_p)$ is invertible, and hence we have ${}^tT_0^{-1} \in \mathrm{M}_\ell(\mathbb{Z}_p)$. Since g preserves q and M^{-1} is equal to F_q, we see from (5.1) that

$$E_0 := T_0 \cdot M^{-1} \cdot {}^tT_0 - M^{-1} \;\in\; \Delta(\mathbb{Z}_{(p)}) \subset \Delta(\mathbb{Z}_p). \tag{5.3}$$

In particular, we have

$$MT_0M^{-1} = (I_\ell + ME_0)\, {}^tT_0^{-1} \in \mathrm{M}_\ell(\mathbb{Z}_p).$$

By the assumption, we have $T = T_0 + YM$ for some $Y \in \mathrm{M}_\ell(\mathbb{Z}_p)$. Therefore $\tilde{T} = MT_0M^{-1} + MY$ has components in $\mathbb{Z}_p$. Hence $\tilde{T}$ is a matrix representation of an isometry of Λ. Since $T = T_0 + YM$, this isometry induces g on $D = \Lambda^\vee/\Lambda$. □

We denote by $\mathrm{M}_{\ell,p}$ the set of square matrices of size ℓ whose components are in $\{0,\dots,p-1\}\subset\mathbb{Z}$. By the surjectivity of $\mathrm{O}(\Lambda)\to\mathrm{O}(D,q)$ and Lemma 5.3, there exists a sequence $Z_0, Z_1, Z_2, \dots$ of elements of $\mathrm{M}_{\ell,p}$ such that the matrix

$$T := T_0 + YM, \quad \text{where } Y := Z_0 + pZ_1 + p^2 Z_2 + \cdots \in \mathrm{M}_\ell(\mathbb{Z}_p),$$

satisfies

$$T\cdot M^{-1}\cdot {}^tT = M^{-1}.$$

Let $Z_0, Z_1, Z_2, \dots$ be such a sequence. For $\nu > 0$, we put

$$\begin{aligned} Y_{\nu-1} &:= Z_0 + pZ_1 + \cdots + p^{\nu-1}Z_{\nu-1}, \\ T_\nu &:= T_0 + Y_{\nu-1}M. \end{aligned}$$

Since $M\equiv O \bmod p$, we have $T\equiv T_\nu\equiv T_0 \bmod p$. Then, for $\nu\geq 0$, we have

$$T_\nu\cdot M^{-1}\cdot {}^tT_\nu = M^{-1} + p^\nu E_\nu \quad \text{for some } E_\nu\in\Delta(\mathbb{Z}_{(p)}). \tag{5.4}$$

Indeed, since $T_\nu = T + p^\nu WM$ for some $W\in\mathrm{M}_\ell(\mathbb{Z}_p)$, we have

$$T_\nu\cdot M^{-1}\cdot {}^tT_\nu - M^{-1} = p^\nu(W\,{}^tT_\nu + T_\nu\,{}^tW + p^\nu\, W\cdot M\cdot {}^tW).$$

By (5.2), we see that

$$E_\nu := W\,{}^tT_\nu + T_\nu\,{}^tW + p^\nu\, W\cdot M\cdot {}^tW \;\;\in\;\; \Delta(\mathbb{Z}_p).$$

Since $E_\nu = p^{-\nu}(T_\nu\cdot M^{-1}\cdot {}^tT_\nu - M^{-1})$ has components in $\mathbb{Q}$, we have $E_\nu\in\Delta(\mathbb{Z}_{(p)})$.

We calculate such a sequence Z_ν ($\nu = 0,1,\cdots$) inductively on ν. Suppose that we have obtained $T_\nu\in\mathrm{M}_\ell(\mathbb{Z}_{(p)})$ satisfying (5.4) and

$$T_\nu\equiv T_0 \bmod p. \tag{5.5}$$

(By (5.3), we can use the input data T_0 for $\nu = 0$.) Our task is to search for $Z_\nu\in\mathrm{M}_{\ell,p}$ such that $T_{\nu+1} := T_\nu + p^\nu Z_\nu M$ satisfies (5.4) with ν replaced by $\nu+1$. Since

$$T_{\nu+1}\cdot M^{-1}\cdot {}^tT_{\nu+1} - M^{-1} = p^\nu(E_\nu + Z_\nu\,{}^tT_\nu + T_\nu\,{}^tZ_\nu + p^\nu\, Z_\nu\cdot M\cdot {}^tZ_\nu),$$

it is enough to find a matrix $X\in\mathrm{M}_{\ell,p}$ that satisfies

$$\frac{1}{p}(E_\nu + X\,{}^tT_\nu + T_\nu\,{}^tX) + p^{\nu-1}\,X\cdot M\cdot {}^tX \;\in\; \Delta(\mathbb{Z}_{(p)}). \tag{5.6}$$

5.3.1. *Suppose that* $p>2$. Then every symmetric matrix in $\mathrm{M}_\ell(\mathbb{Z}_{(p)})$ is even symmetric. Since $M\equiv O \bmod p$, we see that $p^{\nu-1}\,X\cdot M\cdot {}^tX$ is a symmetric matrix in $\mathrm{M}_\ell(\mathbb{Z}_{(p)})$ for any $X\in\mathrm{M}_{\ell,p}$ even when $\nu = 0$. It is obvious that $E_\nu + X\,{}^tT_\nu + T_\nu\,{}^tX$ is symmetric for any $X\in\mathrm{M}_{\ell,p}$. Therefore, combining this with (5.5), we see that the condition (5.6) is equivalent to the affine linear equation

$$E_\nu + X\,{}^tT_0 + T_0\,{}^tX \;\equiv\; O \mod p \tag{5.7}$$

over $\mathbb{F}_p$. We solve (5.7) and lift a solution in $\mathrm{M}_\ell(\mathbb{F}_p)$ to $Z_\nu\in\mathrm{M}_{\ell,p}$.

5.3.2. *Suppose that $p=2$.* We put

$$\begin{aligned} h_{ii} &:= \frac{1}{2}(\text{ the } (i,i)\text{-component of } E_\nu \text{ }) \mod 2, \\ f_{ii}(X) &:= \text{ the } (i,i)\text{-component of } X\,{}^tT_0 \mod 2, \end{aligned}$$

for $i=1,\dots,\ell$. Note that, since $E_\nu \in \Delta(\mathbb{Z}_{(2)})$, the definition of $h_{ii} \in \mathbb{F}_2$ above makes sense.

Suppose that $\nu > 0$. Then we have $2^{\nu-1}\cdot X\cdot M\cdot {}^tX \in \Delta(\mathbb{Z}_{(2)})$ for any $X \in \mathrm{M}_{\ell,2}$. Therefore, by (5.5), we see that the condition (5.6) is equivalent to the affine linear equation

$$\begin{cases} E_\nu + X\,{}^tT_0 + T_0\,{}^tX \;\equiv\; O \mod 2, \\ h_{ii} + f_{ii}(X) \;\equiv\; 0 \mod 2 \quad (i=1,\dots,\ell) \end{cases} \tag{5.8}$$

over $\mathbb{F}_2$. We solve (5.8) and lift a solution in $\mathrm{M}_\ell(\mathbb{F}_2)$ to $Z_\nu \in \mathrm{M}_{\ell,2}$.

Suppose that $\nu = 0$. Then $2^{-1}\,X\cdot M\cdot {}^tX$ is symmetric with components in $\mathbb{Z}_{(2)}$, but some of its diagonal components may fail to be even. Hence we put

$$g_{ii}(X) := \frac{1}{2}(\text{ the } (i,i)\text{-component of } X\cdot M\cdot {}^tX \text{ }) \mod 2,$$

which is a homogeneous quadratic polynomial over $\mathbb{F}_2$ of the components of $X = (x_{ij})$. Note that, since $M \equiv O \bmod 2$, the definition of $g_{ii}(X)$ makes sense. Recall that $M = F_q^{-1}$ is block-diagonal with diagonal components

$$W_{\mu,\varepsilon} := \left[\frac{2^\mu}{\varepsilon}\right] \quad (\varepsilon \in \{1,3,5,7\}), \quad U_\mu := 2^\mu \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ or } \; V_\mu := \frac{2^\mu}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

where $\mu > 0$. Note that the quadratic forms

$$[x,y]\;U_\mu \begin{bmatrix} x \\ y \end{bmatrix} = 2^{\mu+1}xy, \quad [x,y]\;V_\mu \begin{bmatrix} x \\ y \end{bmatrix} = \frac{2^{\mu+1}}{3}(x^2 - xy + y^2)$$

are always divisible by 4 in $\mathbb{Z}_{(2)}$. We put

$$J := \{\, j \mid \text{the } (j,j)\text{-component of } M \text{ is } 2/\varepsilon_j \,\}.$$

Since $\varepsilon_j \equiv 1 \bmod 2$, the quadratic polynomial $g_{ii}(X)$ is of the form

$$\sum_{j\in J} x_{ij}^2/\varepsilon_j = \sum_{j\in J} x_{ij}^2.$$

Since $x^2 = x$ in $\mathbb{F}_2$, the equation $g_{ii}(X) = b$ over $\mathbb{F}_2$ with $b \in \mathbb{F}_2$ is equivalent to the affine linear equation $\bar{g}_{ii}(X) = b$, where

$$\bar{g}_{ii}(X) := \sum_{j\in J} x_{ij}.$$

Therefore, by (5.5), we see that the condition (5.6) is equivalent to the affine linear equation

$$\begin{cases} E_\nu + X\,{}^tT_\nu + T_\nu\,{}^tX \;\equiv\; O \mod 2, \\ h_{ii} + f_{ii}(X) + \bar{g}_{ii}(X) \;\equiv\; 0 \mod 2 \quad (i=1,\dots,\ell) \end{cases} \tag{5.9}$$

over $\mathbb{F}_2$. We solve (5.9) and lift a solution in $\mathrm{M}_\ell(\mathbb{F}_2)$ to $Z_\nu \in \mathrm{M}_{\ell,2}$.

Remark 5.4. The fact that the equations (5.7) and (5.8) always have solutions in $\mathrm{M}_\ell(\mathbb{F}_p)$ is easily proved from $\det T_0 \not\equiv 0 \bmod p$. For example, when $p > 2$, the image of the linear map $\mathrm{M}_\ell(\mathbb{F}_p) \to \mathrm{M}_\ell(\mathbb{F}_p)$ given by $X \mapsto X\,{}^tT_0 + T_0\,{}^tX$ is equal to $\Delta(\mathbb{F}_p)$. The fact that the equation (5.9) is always soluble in $\mathrm{M}_\ell(\mathbb{F}_2)$ is non-trivial; it is a consequence of the surjectivity of $\mathrm{O}(\Lambda) \to \mathrm{O}(D_\Lambda, q_\Lambda)$.

For an element $a \in \mathbb{Q}_p^\times$, let $\mathrm{ord}_p(a)$ denote the maximal integer n such that $p^{-n}a \in \mathbb{Z}_p$. We put $\mathrm{ord}_p(0) := \infty$. For a matrix $M = (m_{ij})$ with components in $\mathbb{Z}_p$, we put

$$\mathrm{minord}_p(M) := \text{the minimum of } \mathrm{ord}_p(m_{ij}).$$

We define $\mathrm{minord}_p(v)$ for a vector v with components in $\mathbb{Z}_p$ in the same way. By the argument above, we have proved the following:

Proposition 5.5. *For an arbitrarily large integer ν, we can calculate a matrix $T_\nu \in \mathrm{M}_\ell(\mathbb{Z}_{(p)})$ such that there exists a matrix $T \in \mathrm{M}_\ell(\mathbb{Z}_p)$ with the following properties:*

(i) $\mathrm{minord}_p(T - T_\nu) \geq \nu$,
(ii) *MTM^{-1} represents an isometry $\tilde{g}$ of Λ with respect to $e_1, \dots, e_\ell$, and*
(iii) *$\tilde{g}$ induces the given automorphism g on (D, q).*

5.4. **Step 4.** Let Λ and $\tilde{g} \in \mathrm{O}(\Lambda)$ be as in Step 3. Let ν be a sufficiently large integer. We put

$$V := \Lambda \otimes \mathbb{Q}_p = \Lambda^\vee \otimes \mathbb{Q}_p.$$

In Step 3, we have calculated a matrix

$${}^{\mathrm{a}}T := T_\nu \in \mathrm{M}_\ell(\mathbb{Z}_{(p)})$$

that is approximate to the matrix $T \in \mathrm{M}_\ell(\mathbb{Z}_p)$ representing $\tilde{g} \in \mathrm{O}(V)$ with respect to the basis $e_1^\vee, \dots, e_\ell^\vee$ of V. The approximate accuracy $\mathrm{minord}_p(T - {}^{\mathrm{a}}T)$ of ${}^{\mathrm{a}}T$ satisfies

$$\mathrm{minord}_p(T - {}^{\mathrm{a}}T) \geq \nu.$$

In order to calculate $(\det(\tilde{g}), \mathrm{spin}(\tilde{g}))$, we present an algorithm to decompose $\tilde{g}$ into a product of reflections in $\mathrm{O}(V)$ using only the computed matrix ${}^{\mathrm{a}}T$. This algorithm works when ν is sufficiently large.

Remark 5.6. It is possible to state explicitly how large ν should be for the algorithm to work. However, the result would be complicated, and, for most practical applications, the theoretical bound seems to be unnecessarily large. Therefore we present an algorithm of the style that, if it fails to continue at some point because ν is not large enough, it quits, goes back to Step 3, re-calculate an approximate matrix ${}^{\mathrm{a}}T$ with higher accuracy ν, and re-start the algorithm from the beginning. If this algorithm reaches the end, the result $(\det(\tilde{g}), \mathrm{spin}(\tilde{g}))$ is correct.

Note that the Gram matrix M^{-1} of V with respect to $e_1^\vee, \dots, e_\ell^\vee$ has components in $\mathbb{Q}$. Hence we can find an orthogonal basis $f_1, \dots, f_\ell$ of V by the Gram-Schmidt orthogonalization *in* $\mathbb{Q}$; that is, we can calculate an invertible matrix $S \in \mathrm{M}_\ell(\mathbb{Q})$ of basis transformation such that the new Gram matrix

$$M_V := S \cdot M^{-1} \cdot {}^tS$$

with respect to the new basis $f_1, \dots, f_\ell$ is diagonal. We replace T and ${}^{\mathrm{a}}T$ by $S\,T\,S^{-1}$ and $S\,{}^{\mathrm{a}}T\,S^{-1}$, respectively, so that T represents $\tilde{g}$ with respect to $f_1, \dots, f_\ell$. The lower bound ν of the approximate accuracy $\operatorname{minord}_p(T - {}^{\mathrm{a}}T)$ is replaced by

$$\nu + \min(0, \operatorname{minord}_p(S)) + \min(0, \operatorname{minord}_p(S^{-1})).$$

(See Lemma 5.7 below.)

To simplify the notation, we fix this orthogonal basis $f_1, \dots, f_\ell$ of V in the rest of this section. We identify vectors in V with row vectors in $\mathbb{Q}_p^\ell$, and linear transformations of V with matrices in $\mathrm{M}_\ell(\mathbb{Q}_p)$. In particular, if $A \in \mathrm{M}_\ell(\mathbb{Q}_p)$ represents $a \in \mathrm{O}(V)$, we write vA instead of v^a for $v \in V$. For $v \in \mathbb{Q}_p^\ell$ and $A \in \mathrm{M}_\ell(\mathbb{Q}_p)$, we use the notation ${}^{\mathrm{a}}v \in \mathbb{Q}^\ell$ and ${}^{\mathrm{a}}A \in \mathrm{M}_\ell(\mathbb{Q})$ to denote computed objects that we intend to be approximate values of v and A, respectively.

For k with $0 \le k \le \ell$, let $\langle f_1, \dots, f_k \rangle$ denote the subspace of V generated by $f_1, \dots, f_k$. Then the orthogonal complement of $\langle f_1, \dots, f_k \rangle$ in V is $\langle f_{k+1}, \dots, f_\ell \rangle$. We put

$$\gamma_i := \operatorname{ord}_p(\langle f_i, f_i \rangle), \quad \gamma := \operatorname{minord}_p(M_V) = \min(\gamma_1, \dots, \gamma_\ell).$$

The following lemma is easy to prove, and will be used frequently.

Lemma 5.7. (1) *Let A, B be matrices in $\mathrm{M}_\ell(\mathbb{Q}_p)$, and suppose that ${}^{\mathrm{a}}A, {}^{\mathrm{a}}B \in \mathrm{M}_\ell(\mathbb{Q})$ satisfy*

$$\operatorname{minord}_p(A - {}^{\mathrm{a}}A) \ge \alpha, \;\; \operatorname{minord}_p(B - {}^{\mathrm{a}}B) \ge \beta.$$

Then we have

$$\operatorname{minord}_p(AB - {}^{\mathrm{a}}A\,{}^{\mathrm{a}}B) \ge \min(\, \operatorname{minord}_p({}^{\mathrm{a}}A) + \beta, \; \alpha + \operatorname{minord}_p({}^{\mathrm{a}}B), \; \alpha + \beta \,).$$

(2) *For any $u, v \in \mathbb{Q}_p^\ell$, we have*

$$\operatorname{ord}_p(\langle u, v \rangle) \;\ge\; \gamma + \operatorname{minord}_p(u) + \operatorname{minord}_p(v).$$

We also put

$$\delta := \begin{cases} 1 & \text{if } p = 2, \\ 0 & \text{if } p > 2. \end{cases}$$

The following lemma is also easy to prove.

Lemma 5.8. *Let a and b be elements of the multiplicative group $\mathbb{Q}_p^\times$.*
(1) *If $\operatorname{ord}_p(a) + \delta < \operatorname{ord}_p(b - a)$, then we have $\operatorname{ord}_p(a + b) = \operatorname{ord}_p(a) + \delta$.*
(2) *We have $a \equiv b \bmod (\mathbb{Q}_p^\times)^2$ if $\operatorname{ord}_p(1 - a/b) \ge 1 + 2\delta$.*

Our algorithm proceeds as follows. We start from

$$T^{(0)} := T, \quad {}^{\mathrm{a}}T^{(0)} := {}^{\mathrm{a}}T, \quad \nu_0 := \nu, \quad i(0) := 0.$$

By the induction on k up to $k = \ell$, we compute a matrix ${}^{\mathrm{a}}T^{(k)} \in \mathrm{M}_\ell(\mathbb{Q})$, an integer ν_k, and a sequence ${}^{\mathrm{a}}r_j$ of vectors in $\mathbb{Q}^\ell$ for $j = i(k-1)+1, \dots, i(k)$ with the following properties:

(P1) For j with $i(k-1) < j \le i(k)$, we have $\langle {}^{\mathrm{a}}r_j, {}^{\mathrm{a}}r_j \rangle \ne 0$. In particular, we have the reflection $\tau({}^{\mathrm{a}}r_j) \in \mathrm{O}(V)$.

(P2) The vector ${}^{\mathrm{a}}r_j$ is approximate to a vector $r_j \in \langle f_1, \dots, f_k \rangle \subset \mathbb{Q}_p^\ell$ with accuracy high enough to ensure that $\langle r_j, r_j \rangle \ne 0$ and $\langle r_j, r_j \rangle \equiv \langle {}^{\mathrm{a}}r_j, {}^{\mathrm{a}}r_j \rangle \bmod (\mathbb{Q}_p^\times)^2$ holds in the multiplicative group $\mathbb{Q}_p^\times$. In particular, we have the reflection $\tau(r_j) \in \mathrm{O}(V)$.

(P3) The isometry

$$T^{(k)} := T^{(0)}\tau(r_1)\cdots\tau(r_{i(k)}) = T^{(k-1)}\tau(r_{i(k-1)+1})\cdots\tau(r_{i(k)})$$

preserves the subspace $\langle f_1,\dots,f_k\rangle$ of V, and acts trivially on $\langle f_1,\dots,f_k\rangle$.

(P4) The matrix

$${}^{\mathrm{a}}T^{(k)} := {}^{\mathrm{a}}T^{(0)}\tau({}^{\mathrm{a}}r_1)\cdots\tau({}^{\mathrm{a}}r_{i(k)}) = {}^{\mathrm{a}}T^{(k-1)}\tau({}^{\mathrm{a}}r_{i(k-1)+1})\cdots\tau({}^{\mathrm{a}}r_{i(k)}).$$

is approximate to $T^{(k)}$, and we have $\mathrm{minord}_p(T^{(k)} - {}^{\mathrm{a}}T^{(k)}) \ge \nu_k$.

Suppose that we reach $k=\ell$. Then $T^{(\ell)}$ is the identity matrix by property (3), and hence we have

$$T = \tau(r_{i(\ell)})\cdots\tau(r_1).$$

Therefore we have $\det(T) = (-1)^{i(\ell)}$ and

$$\mathrm{spin}(T) = Q(r_{i(\ell)})\cdots Q(r_1) \bmod (\mathbb{Q}_p^\times)^2 = Q({}^{\mathrm{a}}r_{i(\ell)})\cdots Q({}^{\mathrm{a}}r_1) \bmod (\mathbb{Q}_p^\times)^2,$$

where the second equality follows from property (2). Since ${}^{\mathrm{a}}r_1,\dots,{}^{\mathrm{a}}r_{i(\ell)}$ are computed, $(\det(\tilde g), \mathrm{spin}(\tilde g))$ is also computed.

Suppose that we have calculated ${}^{\mathrm{a}}T^{(k-1)}$, ν_{k-1} and ${}^{\mathrm{a}}r_1,\dots,{}^{\mathrm{a}}r_{i(k-1)}$. Recall that f_k is the vector $(0,\dots,1,\dots,0)\in\mathbb{Q}^\ell$, where 1 is at the kth position. We put

$$g_k := f_k\,T^{(k-1)}, \quad {}^{\mathrm{a}}g_k := f_k\,{}^{\mathrm{a}}T^{(k-1)}.$$

By the induction hypothesis, the isometry $T^{(k-1)}$ of V preserves the subspace $\langle f_1,\dots,f_{k-1}\rangle$, and hence preserves $\langle f_1,\dots,f_{k-1}\rangle^\perp = \langle f_k,\dots,f_\ell\rangle$. Therefore we have $g_k\in\langle f_k,\dots,f_\ell\rangle$. Since $T^{(k-1)}$ is an isometry, we have

$$\langle g_k, g_k\rangle = \langle f_k, f_k\rangle, \tag{5.10}$$

and hence we have $\mathrm{ord}_p(\langle g_k,g_k\rangle) = \mathrm{ord}_p(\langle f_k,f_k\rangle) = \gamma_k$. We estimate the approximation error $\langle g_k,g_k\rangle - \langle {}^{\mathrm{a}}g_k, {}^{\mathrm{a}}g_k\rangle$. For this purpose, we put

$$\lambda_k := \mathrm{minord}_p({}^{\mathrm{a}}g_k), \quad \rho := \min(\ \delta+\nu_{k-1}+\gamma+\lambda_k,\ 2\nu_{k-1}+\gamma\ ).$$

By property (4) for ν_{k-1}, we have a matrix $A\in\mathrm{M}_\ell(\mathbb{Z}_p)$ such that $T^{(k-1)} = {}^{\mathrm{a}}T^{(k-1)} + p^{\nu_{k-1}}A$. Hence we have a vector $v\in\mathbb{Z}_p^\ell$ such that

$$g_k = {}^{\mathrm{a}}g_k + p^{\nu_{k-1}}v.$$

Therefore we have

$$\langle g_k,g_k\rangle - \langle {}^{\mathrm{a}}g_k,{}^{\mathrm{a}}g_k\rangle = 2\langle {}^{\mathrm{a}}g_k, p^{\nu_{k-1}}v\rangle + \langle\, p^{\nu_{k-1}}v,\ p^{\nu_{k-1}}v\rangle.$$

From $\mathrm{ord}_p(\langle {}^{\mathrm{a}}g_k, p^{\nu_{k-1}}v\rangle) \ge \lambda_k+\gamma+\nu_{k-1}$ and $\mathrm{ord}_p(\langle\, p^{\nu_{k-1}}v,\ p^{\nu_{k-1}}v\rangle) \ge 2\nu_{k-1}+\gamma$, we obtain

$$\mathrm{ord}_p(\ \langle g_k,g_k\rangle - \langle {}^{\mathrm{a}}g_k,{}^{\mathrm{a}}g_k\rangle\ ) \ge \rho.$$

If $\rho\le\gamma_k+\delta$, then we quit and go to the re-calculation process (Remark 5.6). Suppose that $\rho>\gamma_k+\delta$. Then, by Lemma 5.8, we have

$$\mathrm{ord}_p(\ \langle g_k,g_k\rangle + \langle {}^{\mathrm{a}}g_k,{}^{\mathrm{a}}g_k\rangle\ ) = \gamma_k+\delta. \tag{5.11}$$

We put

$$b^+ := f_k+g_k,\ \ {}^{\mathrm{a}}b^+ := f_k+{}^{\mathrm{a}}g_k,\ \ b^- := f_k-g_k,\ \ {}^{\mathrm{a}}b^- := f_k-{}^{\mathrm{a}}g_k.$$

We have

$$\langle {}^{\mathrm{a}}b^+, {}^{\mathrm{a}}b^+\rangle + \langle {}^{\mathrm{a}}b^-, {}^{\mathrm{a}}b^-\rangle = 2(\langle f_k,f_k\rangle + \langle {}^{\mathrm{a}}g_k,{}^{\mathrm{a}}g_k\rangle).$$

By (5.10) and (5.11), we see that the ord_p of at least one of $\langle {}^{\mathrm{a}}b^+, {}^{\mathrm{a}}b^+\rangle$ or $\langle {}^{\mathrm{a}}b^-, {}^{\mathrm{a}}b^-\rangle$ is $\le \gamma_k + 2\delta$. If $\mathrm{ord}_p(\langle {}^{\mathrm{a}}b^-, {}^{\mathrm{a}}b^-\rangle) \le \gamma_k + 2\delta$, we put $b := b^-$ and ${}^{\mathrm{a}}b := {}^{\mathrm{a}}b^-$; otherwise, we put $b := b^+$ and ${}^{\mathrm{a}}b := {}^{\mathrm{a}}b^+$. Note that we have $b \in \langle f_k, \dots, f_\ell\rangle$. Then we have

$$b = {}^{\mathrm{a}}b + p^{\nu_{k-1}} w,$$

where $w = \pm v \in \mathbb{Z}_p^\ell$. In order to estimate $\langle b, b\rangle - \langle {}^{\mathrm{a}}b, {}^{\mathrm{a}}b\rangle$, we put

$$\sigma := \min(\, \delta + \nu_{k-1} + \gamma,\ \ \delta + \nu_{k-1} + \gamma + \lambda_k,\ \ 2\nu_{k-1} + \gamma\,).$$

Since

$$\langle b, b\rangle - \langle {}^{\mathrm{a}}b, {}^{\mathrm{a}}b\rangle = 2\langle f_k, p^{\nu_{k-1}} w\rangle \pm 2\langle {}^{\mathrm{a}}g_k, p^{\nu_{k-1}} w\rangle + \langle\, p^{\nu_{k-1}} w,\, p^{\nu_{k-1}} w\rangle,$$

we have

$$\mathrm{ord}_p(\, \langle b, b\rangle - \langle {}^{\mathrm{a}}b, {}^{\mathrm{a}}b\rangle\,) \ge \sigma.$$

We then put

$$\kappa := \sigma - (\gamma_k + 2\delta).$$

Since $\mathrm{ord}_p(\langle {}^{\mathrm{a}}b, {}^{\mathrm{a}}b\rangle) \le \gamma_k + 2\delta$, we see that

$$\langle b, b\rangle = \langle {}^{\mathrm{a}}b, {}^{\mathrm{a}}b\rangle(1 + p^\kappa c) \ \text{ for some } \ c \in \mathbb{Z}_p.$$

If $\kappa < 1 + 2\delta$, then we quit and go to the re-calculation process (Remark 5.6). Suppose that $\kappa \ge 1 + 2\delta$. Then, by Lemma 5.8, we have

$$\langle b, b\rangle \equiv \langle {}^{\mathrm{a}}b, {}^{\mathrm{a}}b\rangle \bmod (\mathbb{Q}_p^\times)^2.$$

When $b = b^-$, we put

$$i(k) := i(k-1) + 1,\ \ r_{i(k-1)+1} := b,\ \ {}^{\mathrm{a}}r_{i(k-1)+1} := {}^{\mathrm{a}}b,$$

so that

$$T^{(k)} = T^{(k-1)}\tau(b),\ \ {}^{\mathrm{a}}T^{(k)} = {}^{\mathrm{a}}T^{(k-1)}\tau({}^{\mathrm{a}}b).$$

When $b = b^+$, we put

$$i(k) := i(k-1)+2,\ \ r_{i(k-1)+1} := b,\ \ {}^{\mathrm{a}}r_{i(k-1)+1} := {}^{\mathrm{a}}b,\ \ r_{i(k-1)+2} := {}^{\mathrm{a}}r_{i(k-1)+2} := f_k,$$

so that

$$T^{(k)} = T^{(k-1)}\tau(b)\tau(f_k),\ \ {}^{\mathrm{a}}T^{(k)} = {}^{\mathrm{a}}T^{(k-1)}\tau({}^{\mathrm{a}}b)\tau(f_k).$$

By construction, we have $f_k\, T^{(k)} = f_k$. Using the induction hypothesis on $T^{(k-1)}$, we can easily verify that $T^{(k)}$ preserves $\langle f_1, \dots, f_k\rangle$ and acts on $\langle f_1, \dots, f_k\rangle$ trivially. Thus the constructed data satisfies the properties (P1), (P2), (P3).

It remains to give a lower bound ν_k of $\mathrm{minord}_p(T^{(k)} - {}^{\mathrm{a}}T^{(k)})$. First we calculate $\mathrm{minord}_p(\tau(b) - \tau({}^{\mathrm{a}}b))$. For $x \in \mathbb{Q}_p^\ell$, we have

$$x \cdot (\tau(b) - \tau({}^{\mathrm{a}}b)) = \frac{2}{\langle b, b\rangle}\phi(x),$$

where

$$\begin{aligned} \phi(x) &= (1 + p^\kappa c)\,\langle {}^{\mathrm{a}}b, x\rangle\, {}^{\mathrm{a}}b - \langle {}^{\mathrm{a}}b + p^{\nu_{k-1}} w, x\rangle\,({}^{\mathrm{a}}b + p^{\nu_{k-1}} w) \\ &= p^\kappa c\,\langle {}^{\mathrm{a}}b, x\rangle\, {}^{\mathrm{a}}b - \langle\, p^{\nu_{k-1}} w, x\rangle\, {}^{\mathrm{a}}b - \langle {}^{\mathrm{a}}b, x\rangle\, p^{\nu_{k-1}} w - \langle\, p^{\nu_{k-1}} w, x\rangle\, p^{\nu_{k-1}} w. \end{aligned}$$

Since $\mathrm{minord}_p(f_k) = 0$, we have $\mathrm{minord}_p({}^{\mathrm{a}}b) \ge \bar\lambda_k := \min(0, \lambda_k)$. Hence, whenever $\mathrm{minord}_p(x) \ge 0$, we have

$$\mathrm{minord}_p(\phi(x)) \ge \theta := \min(\, \kappa + 2\bar\lambda_k + \gamma,\ \nu_{k-1} + \gamma + \bar\lambda_k,\ 2\nu_{k-1} + \gamma\,).$$

Combining this with $\mathrm{ord}_p(\langle b, b\rangle) \le \gamma_k + 2\delta$, we see that

$$\mathrm{minord}_p(\tau(b) - \tau({}^{\mathrm{a}}b)) \ge \delta + \theta - (\gamma_k + 2\delta) = \theta - \gamma_k - \delta. \tag{5.12}$$

We put

$$\lambda := \mathrm{minord}_p({}^{\mathrm{a}}T^{(k-1)}), \quad \alpha := \mathrm{minord}_p(\tau({}^{\mathrm{a}}b)), \quad \beta := \mathrm{minord}_p(\tau(f_k)),$$

and

$$\nu' := \min(\, \nu_{k-1} + \alpha,\ \lambda + \theta - \gamma_k - \delta,\ \nu_{k-1} + \theta - \gamma_k - \delta\,).$$

By Lemma 5.7 and (5.12), we see that

$$\mathrm{minord}_p(\, T^{(k-1)}\tau(b) - {}^{\mathrm{a}}T^{(k-1)}\tau({}^{\mathrm{a}}b)\,) \geq \nu'.$$

Therefore, in the case where $b = b^-$, we put $\nu_k := \nu'$. In the case where $b = b^+$, we have

$$\mathrm{minord}_p(\, T^{(k-1)}\tau(b)\tau(f_k) - {}^{\mathrm{a}}T^{(k-1)}\tau({}^{\mathrm{a}}b)\tau(f_k)\,) \geq \nu' + \beta,$$

and hence we put $\nu_k := \nu' + \beta$.

The values of γ_k, γ and β do not depend on the initial approximate accuracy $\nu_0 = \nu$. The values of λ_k, λ and α stabilize to constants when ν goes to infinity. Suppose that ν_{k-1}/ν converges to 1 when ν goes to infinity. By definitions, we see that σ/ν_{k-1}, κ/ν_{k-1} and θ/ν_{k-1} also converge to 1, and hence ν_k/ν converges to 1. Therefore, if ν is large enough, this algorithm reaches $k = \ell$.

6. Examples

6.1. **Algebraically distinguished connected components.** Let (X, f, s) be an elliptic $K3$ surface. We use the notation A_f, $U_f, L(\Phi_f)$, and $M(\Phi_f)$ defined in Introduction. Since we can perturb (X, f, s) to an elliptic $K3$ surface (X', f', s') in such a way that $A_{f'} \cong A_f$ and $S_{X'} \cong U_f \oplus M(\Phi_f)$, we see that, for each torsion section $\tau \in A_f$, the class $[\tau] \in H^2(X, \mathbb{Z})$ of the curve $\tau(\mathbb{P}^1)$ is contained in $U_f \oplus M(\Phi_f)$. In this subsection, we present a method to calculate these classes $[\tau]$.

We denote by $v_f \in U_f$ the class of a fiber of f. Let $P \in \mathbb{P}^1$ be a point such that $f^{-1}(P)$ is reducible. Suppose that the reduced part of $f^{-1}(P)$ consists of $\rho+1$ smooth rational curves. A smooth rational curve Θ in $f^{-1}(P)$ is said to be *a simple component of* $f^{-1}(P)$ if the divisor $f^{-1}(P)$ of X is reduced at a general point of Θ. If a section of f intersects $f^{-1}(P)$ at a point of Θ, then Θ is a simple component. Let Θ_0 be a simple component of $f^{-1}(P)$, and let $\Theta_1, \dots, \Theta_\rho$ be the other smooth rational curves in $f^{-1}(P)$. Let θ_ν be the class of Θ_ν for $\nu = 0, \dots, \rho$. Then $\theta_1, \dots, \theta_\rho$ span a root sublattice $L(P)$ in $U_f \oplus L(\Phi_f)$, and $\theta_1, \dots, \theta_\rho$ form a fundamental root system of $L(P)$. Moreover, v_f is orthogonal to $L(P)$, and $\theta_0 \in \mathbb{Z}v_f \oplus L(P)$.

Proposition 6.1. *Let $\{P_1, \dots, P_N\}$ be the set of points $P_i \in \mathbb{P}^1$ such that $f^{-1}(P_i)$ is reducible. A vector $u \in U_f \oplus M(\Phi_f)$ is the class $[\tau]$ of a torsion section $\tau \in A_f$ if and only if u satisfies the following:*

(i) *$\langle u, u \rangle = -2$ and $\langle u, v_f \rangle = 1$.*

(ii) *For each $i = 1, \dots, N$, there exists a simple component $\Theta_0^{(i)}$ of $f^{-1}(P_i)$ such that $\langle u, \Theta_0^{(i)} \rangle = 1$, and that $\langle u, \Theta_\nu^{(i)} \rangle = 0$ holds for all smooth rational curves $\Theta_\nu^{(i)}$ in $f^{-1}(P_i)$ other than $\Theta_0^{(i)}$.*

For the proof, we need a preparation. Let $P \in \mathbb{P}^1$, $L(P)$, $\Theta_0, \dots, \Theta_\rho$, and $\theta_0, \dots, \theta_\rho$ be as above. We have

$$\theta_0 = v_f - \sum_{i=1}^{\rho} m_\nu \theta_\nu,$$

where $m_\nu \in \mathbb{Z}_{>0}$ is the multiplicity of Θ_ν in the divisor $f^{-1}(P)$. The values of m_ν are classically known for all types of singular fibers of elliptic surfaces. (See [11]. See also [8, Figure 1.8].) The following lemma can be confirmed by explicit computation.

Lemma 6.2. *Let $\theta_1^\vee, \dots, \theta_\rho^\vee$ be the basis of $L(P)^\vee$ dual to the fundamental root system $\theta_1, \dots, \theta_\rho$ of $L(P)$. Then there exists no index $\mu > 0$ such that $m_\mu = 1$ and $\theta_\mu^\vee \in L(P)$.*

Proof of Proposition 6.1. The necessity of conditions (i) and (ii) is obvious. Suppose that u satisfies (i) and (ii). By condition (i), we see that u is the class of an effective divisor

$$H + \sum_{i=0}^{N} \Gamma_i$$

on X, where H is a reduced curve mapped isomorphically to $\mathbb{P}^1$ by f, and Γ_i is an effective divisor whose support is contained in the support of $f^{-1}(P_i)$. It is enough to show that $\Gamma_i = 0$ for each i. Indeed, if u is the class of a section H, then H must be a torsion section because $u \in U_f \oplus M(\Phi_f)$.

Let $\Theta_1^{(i)}, \dots, \Theta_{\rho(i)}^{(i)}$ be the smooth rational curves in $f^{-1}(P_i)$ other than the simple component $\Theta_0^{(i)}$ given in condition (ii). Since H is a section, there exists an index $\mu(i)$ with $0 \le \mu(i) \le \rho(i)$ such that

$$\langle H, \Theta_\nu^{(i)} \rangle = \begin{cases} 1 & \text{if } \nu = \mu(i), \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to show that $\mu(i) = 0$. Indeed, suppose that $\mu(i) = 0$. Then we have $\langle u, \Theta_\nu^{(i)} \rangle = \langle H, \Theta_\nu^{(i)} \rangle = 0$ for all $\nu > 0$, and hence $\langle \Gamma_i, \Theta_\nu^{(i)} \rangle = 0$ for all $\nu > 0$. Since the root lattice $L(P_i)$ spanned by the classes of $\Theta_1^{(i)}, \dots, \Theta_{\rho(i)}^{(i)}$ is non-degenerate, and $[\Gamma_i] \in \mathbb{Z}v_f \oplus L(P_i)$, we see that Γ_i is a multiple of the divisor $f^{-1}(P_i)$. We put $\Gamma_i = k_i\, f^{-1}(P_i)$ with $k_i \in \mathbb{Z}_{\ge 0}$. Then we have $u = [H] + k\, v_f$, where $k = \sum_{i=1}^N k_i$. From $\langle u, u \rangle = \langle H, H \rangle = -2$ and $\langle H, v_f \rangle = 1$, we obtain $k = 0$.

Now we prove $\mu(i) = 0$. Let $\theta_1^\vee, \dots, \theta_{\rho(i)}^\vee$ be the basis of $L(P_i)^\vee$ dual to the basis $[\Theta_1^{(i)}], \dots, [\Theta_{\rho(i)}^{(i)}]$ of $L(P_i)$. Suppose that $\mu(i) > 0$. By $\langle u, \Theta_0^{(i)} \rangle = 1$ and $\langle H, \Theta_0^{(i)} \rangle = 0$, we have

$$\langle \Gamma_i, \Theta_0^{(i)} \rangle = 1. \tag{6.1}$$

By $\langle u, \Theta_{\mu(i)}^{(i)} \rangle = 0$ and $\langle H, \Theta_{\mu(i)}^{(i)} \rangle = 1$, we have

$$\langle \Gamma_i, \Theta_{\mu(i)}^{(i)} \rangle = -1. \tag{6.2}$$

If $\nu \ne 0$ and $\nu \ne \mu(i)$, then we have $\langle u, \Theta_\nu^{(i)} \rangle = \langle H, \Theta_\nu^{(i)} \rangle = 0$, and hence we have

$$\langle \Gamma_i, \Theta_\nu^{(i)} \rangle = 0. \tag{6.3}$$

Let $z \in L(P_i)$ be the image of $[\Gamma_i] \in \mathbb{Z}v_f \oplus L(P_i)$ by the projection $\mathbb{Z}v_f \oplus L(P_i) \to L(P_i)$. Then (6.2) and (6.3) imply that z is equal to $-\theta_{\mu(i)}^\vee$. In particular, $\theta_{\mu(i)}^\vee$ is in $L(P_i)$. On the other hand, (6.1) implies that the coefficient $m_{\mu(i)}$ of $[\Theta_0^{(i)}] = v_f - \sum_\nu m_\nu [\Theta_\nu^{(i)}]$ is 1, which contradicts Lemma 6.2. $\square$

Let $v_s \in U_f$ denote the class of the zero section s. It is easy to make the complete list of vectors u_L of $U_f \oplus L(\Phi_f)^\vee$ that satisfies condition (ii) in Proposition 6.1. If $u_L \in L(\Phi_f)^\vee$ satisfies condition (ii) in Proposition 6.1 and belongs to $U_f \oplus M(\Phi_f)$, then

$$-\frac{\langle u_L, u_L\rangle}{2} v_f + v_s + u_L$$

is the class of a torsion section. The classes of all torsion sections are obtained in this way. Thus we can calculate the set $\{[\tau] \mid \tau \in A_f\}$, and see how the torsion sections intersect irreducible components of reducible fibers.

We say that a torsion section $\tau \in A_f$ is *narrow at* $P \in \mathbb{P}^1$ if τ and s intersect the same irreducible component of $f^{-1}(P)$.

Example 6.3. We consider the extremal elliptic $K3$ surfaces (X, f, s) of type

$$(\, A_9 + A_5 + A_3 + A_1,\ \mathbb{Z}/2\mathbb{Z}\,),$$

which have two algebraically distinguished connected components that cannot be distinguished by the transcendental lattices. (See no. 64 of Table I.) Let $P(A_l) \in \mathbb{P}^1$ denote the point such that $f^{-1}(P_i)$ is of type A_l. The non-trivial torsion section of an elliptic $K3$ surface in one connected components is not narrow at $P(A_9), P(A_3), P(A_1)$, and narrow at $P(A_5)$, whereas the non-trivial torsion section of an elliptic $K3$ surface in the other connected components is not narrow at $P(A_9), P(A_5)$, and narrow at $P(A_3), P(A_1)$.

Example 6.4. We consider the non-extremal elliptic $K3$ surfaces of type

$$(\, A_5 + A_3 + 6A_1,\ \mathbb{Z}/2\mathbb{Z}\,),$$

which have three algebraically distinguished connected components. (See no. 91 of Table II.) These connected components can be distinguished by the narrowness of the non-trivial torsion section as follows:

A_5	A_3	$A_1, A_1, A_1, A_1, A_1, A_1$
narrow	not narrow	not narrow at all 6 points
not narrow	narrow	narrow at only one point
not narrow	not narrow	narrow at exacly 3 points.

6.2. **Connected components when G is trivial.**

Example 6.5. Consider the combinatorial type

$$(\Phi, A) = (7A_2, \mathbb{Z}/3\mathbb{Z}).$$

We have $|\mathfrak{C}(\Phi, A, \mathrm{Aut}(\Phi))| = 1$. The discriminant form of $L(\Phi)$ is isomorphic to $(\mathbb{F}_3, [4/3])^7$, and we have

$$\mathrm{Aut}(\Phi) = (\mathbb{Z}/2\mathbb{Z})^7 \rtimes \mathfrak{S}_7.$$

The set $\mathcal{E}(\Phi, A)/\mathrm{Aut}(\Phi)$ consists of only one element $[M]$, where M corresponds to the totally isotropic subspace of dimension 1 over $\mathbb{F}_3$ generated by $(0, 1, 1, 1, 1, 1, 1)$. Hence we have

$$\mathrm{Stab}(M) = (\mathbb{Z}/2\mathbb{Z}) \times ((\mathbb{Z}/2\mathbb{Z})^6 \rtimes \mathfrak{S}_6).$$

The genus $\mathcal{G}$ determined by the signature $(2, 4)$ and the discriminant form $(D_M, -q_M)$ consists of only one isomorphism class. We have $|\mathfrak{C}(\Phi, A, \{\mathrm{id}\})| = 2$, and the two connected components in $\mathfrak{C}(\Phi, A, \{\mathrm{id}\})$ are complex conjugate to each other.

Example 6.6. For the combinatorial type

$$(\Phi, A) = (4A_4, \{0\}),$$

we have $|\mathfrak{C}(\Phi, A, \mathrm{Aut}(\Phi))| = 1$, whereas $|\mathfrak{C}(\Phi, A, \{\mathrm{id}\})| = 2$, and the two connected components in $\mathfrak{C}(\Phi, A, \{\mathrm{id}\})$ are real.

Example 6.7. For the combinatorial type $(\Phi, A) = (2D_4 + 4A_2, \{0\})$, we have $|\mathfrak{C}(\Phi, A, \mathrm{Aut}(\Phi))| = 1$, whereas $|\mathfrak{C}(\Phi, A, \{\mathrm{id}\})| = 4$, and the four connected components in $\mathfrak{C}(\Phi, A, \{\mathrm{id}\})$ are divided into two complex conjugate pairs.

7. Tables

See Introduction for the explanation of the entries of the tables below.

7.1. Table I: Non-connected moduli of extremal elliptic $K3$ surfaces.

no.	Φ	A	T	$[r, c]$
1	$E_8 + A_9 + A_1$	[1]	$[2, 0, 10]$	$[2, 0]$
2	$E_8 + A_6 + A_3 + A_1$	[1]	$[6, 2, 10]$	$[0, 2]$
3	$E_8 + 2A_5$	[1]	$[6, 0, 6]$	$[0, 2]$
4	$E_7 + E_6 + A_5$	[1]	$[6, 0, 6]$	$[0, 2]$
5	$E_7 + D_5 + A_6$	[1]	$[6, 2, 10]$	$[0, 2]$
6	$E_7 + A_{11}$	[1]	$[4, 0, 6]$	$[0, 2]$
7	$E_7 + A_{10} + A_1$	[1]	$[2, 0, 22]$	$[1, 0]$
			$[6, 2, 8]$	$[0, 2]$
8	$E_7 + A_8 + A_2 + A_1$	[1]	$[6, 0, 18]$	$[1, 2]$
9	$E_7 + A_7 + A_4$	[1]	$[6, 2, 14]$	$[0, 2]$
10	$E_7 + A_7 + A_3 + A_1$	[2]	$[4, 0, 8]$	$[0, 2]$
11	$E_7 + A_6 + A_5$	[1]	$[4, 2, 22]$	$[0, 2]$
12	$E_7 + A_6 + A_4 + A_1$	[1]	$[2, 0, 70]$	$[1, 0]$
			$[8, 2, 18]$	$[0, 2]$
13	$E_7 + A_5 + A_4 + A_2$	[1]	$[6, 0, 30]$	$[2, 0]$
14	$E_6 + D_5 + A_7$	[1]	$[8, 0, 12]$	$[0, 2]$
15	$E_6 + A_{12}$	[1]	$[4, 1, 10]$	$[0, 2]$
16	$E_6 + A_{11} + A_1$	[1]	$[6, 0, 12]$	$[0, 2]$
17	$E_6 + A_9 + A_2 + A_1$	[1]	$[12, 6, 18]$	$[0, 2]$
18	$E_6 + A_8 + A_4$	[1]	$[12, 3, 12]$	$[1, 2]$
19	$E_6 + A_8 + A_3 + A_1$	[1]	$[12, 0, 18]$	$[1, 2]$
20	$E_6 + A_7 + A_5$	[1]	$[6, 0, 24]$	$[0, 2]$
21	$E_6 + A_6 + A_5 + A_1$	[1]	$[6, 0, 42]$	$[0, 2]$
22	$E_6 + A_6 + A_3 + A_2 + A_1$	[1]	$[6, 0, 84]$	$[1, 0]$
			$[12, 0, 42]$	$[1, 0]$
23	$E_6 + A_5 + A_4 + A_3$	[1]	$[12, 0, 30]$	$[2, 0]$
24	$D_{11} + A_6 + A_1$	[1]	$[6, 2, 10]$	$[0, 2]$
25	$D_9 + D_5 + A_4$	[1]	$[4, 0, 20]$	$[2, 0]$
26	$D_7 + A_6 + A_3 + A_2$	[1]	$[8, 4, 44]$	$[0, 2]$
27	$D_6 + A_9 + A_2 + A_1$	[2]	$[4, 2, 16]$	$[1, 0]$
			$[6, 0, 10]$	$[1, 0]$
28	$D_6 + A_7 + A_4 + A_1$	[2]	$[6, 2, 14]$	$[0, 2]$
29	$D_6 + 2A_6$	[1]	$[14, 0, 14]$	$[0, 2]$
30	$D_5 + A_{13}$	[1]	$[6, 2, 10]$	$[0, 2]$
31	$D_5 + A_{12} + A_1$	[1]	$[2, 0, 52]$	$[1, 0]$
			$[6, 2, 18]$	$[0, 2]$
32	$D_5 + A_{10} + A_2 + A_1$	[1]	$[14, 4, 20]$	$[0, 2]$
33	$D_5 + A_9 + A_4$	[1]	$[10, 0, 20]$	$[1, 2]$
34	$D_5 + A_9 + A_3 + A_1$	[2]	$[8, 4, 12]$	$[0, 2]$

(continues)

(continued)

no.	Φ	A	T	$[r,c]$
35	$D_5+A_8+A_5$	[1]	$[12,0,18]$	$[1,2]$
36	$D_5+A_8+A_4+A_1$	[1]	$[2,0,180]$	$[1,0]$
			$[18,0,20]$	$[1,0]$
37	$D_5+2A_6+A_1$	[1]	$[14,0,28]$	$[0,2]$
38	$D_5+A_6+A_5+A_2$	[1]	$[6,0,84]$	$[1,0]$
			$[12,0,42]$	$[1,0]$
39	$A_{17}+A_1$	[1]	$[4,2,10]$	$[0,2]$
40	$A_{16}+2A_1$	[1]	$[2,0,34]$	$[1,0]$
			$[4,2,18]$	$[1,0]$
41	$A_{15}+A_2+A_1$	[1]	$[10,2,10]$	$[0,2]$
42	$A_{14}+A_4$	[1]	$[10,5,10]$	$[0,2]$
43	$A_{14}+A_3+A_1$	[1]	$[10,0,12]$	$[0,2]$
44	$A_{14}+A_2+2A_1$	[1]	$[12,6,18]$	$[0,2]$
45	$A_{13}+A_5$	[1]	$[4,2,22]$	$[0,2]$
46	$A_{13}+A_4+A_1$	[1]	$[2,0,70]$	$[1,0]$
			$[8,2,18]$	$[0,2]$
47	$A_{13}+A_3+2A_1$	[2]	$[6,2,10]$	$[0,2]$
48	$A_{12}+A_5+A_1$	[1]	$[10,2,16]$	$[0,2]$
49	$A_{12}+A_4+2A_1$	[1]	$[2,0,130]$	$[1,0]$
			$[18,8,18]$	$[1,0]$
50	$A_{11}+A_6+A_1$	[1]	$[4,0,42]$	$[0,2]$
51	$A_{11}+A_4+A_2+A_1$	[1]	$[12,0,30]$	$[0,4]$
52	$A_{11}+A_3+A_2+2A_1$	[2]	$[12,0,12]$	$[0,2]$
53	$A_{10}+A_7+A_1$	[1]	$[2,0,88]$	$[1,0]$
			$[10,2,18]$	$[0,2]$
54	$A_{10}+A_6+A_2$	[1]	$[4,1,58]$	$[0,2]$
			$[16,5,16]$	$[1,0]$
55	$A_{10}+A_6+2A_1$	[1]	$[12,2,26]$	$[0,2]$
56	$A_{10}+A_5+A_3$	[1]	$[4,0,66]$	$[1,0]$
			$[12,0,22]$	$[1,0]$
57	$A_{10}+A_5+A_2+A_1$	[1]	$[6,0,66]$	$[1,0]$
			$[18,6,24]$	$[0,2]$
58	$A_{10}+A_4+A_3+A_1$	[1]	$[12,4,38]$	$[0,2]$
			$[20,0,22]$	$[1,0]$
59	$A_{10}+A_4+2A_2$	[1]	$[6,3,84]$	$[1,0]$
			$[24,9,24]$	$[1,0]$
60	$2A_9$	[1]	$[10,0,10]$	$[2,0]$
61	$A_9+A_8+A_1$	[1]	$[10,0,18]$	$[2,0]$
62	$A_9+A_6+A_2+A_1$	[1]	$[10,0,42]$	$[2,0]$
63	$A_9+A_5+A_4$	[1]	$[10,0,30]$	$[1,2]$
64	$A_9+A_5+A_3+A_1$	[2]	$[10,0,12]$	$[1,0]$
			$[10,0,12]$	$[1,0]$
65	$A_9+2A_4+A_1$	[5]	$[2,0,10]$	$[2,0]$
66	$A_9+A_4+A_3+2A_1$	[2]	$[10,0,20]$	$[1,2]$
67	$2A_8+2A_1$	[1]	$[18,0,18]$	$[1,2]$
68	$A_8+A_7+A_2+A_1$	[1]	$[18,0,24]$	$[1,2]$
69	$A_8+A_6+A_3+A_1$	[1]	$[10,4,52]$	$[0,2]$
70	$A_8+A_6+A_2+2A_1$	[1]	$[18,0,42]$	$[1,2]$
71	$A_8+A_5+A_4+A_1$	[1]	$[18,0,30]$	$[1,2]$
72	$A_8+A_5+2A_2+A_1$	[3]	$[6,0,18]$	$[1,2]$
73	$A_8+A_4+A_3+A_2+A_1$	[1]	$[6,0,180]$	$[1,2]$
74	$2A_7+2A_2$	[1]	$[24,0,24]$	$[0,2]$
75	$A_7+A_6+A_5$	[1]	$[16,4,22]$	$[0,2]$

(continues)

(continued)

no.	Φ	A	T	$[r, c]$
76	$A_7 + A_6 + A_4 + A_1$	$[1]$	$[2, 0, 280]$	$[1, 0]$
			$[18, 4, 32]$	$[0, 2]$
77	$A_7 + A_6 + A_3 + A_2$	$[1]$	$[4, 0, 168]$	$[0, 2]$
78	$A_7 + A_6 + A_3 + 2A_1$	$[2]$	$[12, 4, 20]$	$[0, 2]$
79	$A_7 + 2A_5 + A_1$	$[2]$	$[6, 0, 24]$	$[0, 2]$
80	$A_7 + A_5 + A_4 + A_2$	$[1]$	$[6, 0, 120]$	$[1, 0]$
			$[24, 0, 30]$	$[1, 0]$
81	$A_7 + A_5 + A_3 + A_2 + A_1$	$[2]$	$[12, 0, 24]$	$[2, 0]$
82	$A_7 + A_4 + A_3 + 2A_2$	$[1]$	$[12, 0, 120]$	$[2, 0]$
83	$2A_6 + A_4 + A_2$	$[1]$	$[28, 7, 28]$	$[2, 0]$
84	$2A_6 + 2A_3$	$[1]$	$[28, 0, 28]$	$[0, 2]$
85	$2A_6 + 2A_2 + 2A_1$	$[1]$	$[42, 0, 42]$	$[2, 0]$
86	$A_6 + A_5 + A_4 + A_2 + A_1$	$[1]$	$[18, 6, 72]$	$[0, 2]$
			$[30, 0, 42]$	$[1, 0]$
87	$A_6 + 2A_4 + A_3 + A_1$	$[1]$	$[10, 0, 140]$	$[1, 0]$
			$[20, 0, 70]$	$[1, 0]$
88	$2A_5 + 2A_4$	$[1]$	$[30, 0, 30]$	$[2, 0]$
89	$2A_5 + 4A_2$	$[3, 3]$	$[6, 0, 6]$	$[0, 2]$

7.2. **Table II: Non-connected moduli of non-extremal elliptic $K3$ surfaces.**

no.	r	Φ	A	$[c_1, \dots, c_k]$
1	17	$E_7 + D_6 + A_3 + A_1$	$[2]$	$[1, 1]$
2	17	$E_7 + 2A_5$	$[1]$	$[2]$
3	17	$E_7 + A_5 + A_3 + 2A_1$	$[2]$	$[1, 1]$
4	17	$E_6 + A_{11}$	$[1]$	$[2]$
5	17	$E_6 + A_6 + A_5$	$[1]$	$[2]$
6	17	$E_6 + 2A_5 + A_1$	$[1]$	$[2]$
7	17	$D_{12} + A_3 + 2A_1$	$[2]$	$[1, 1]$
8	17	$D_{10} + D_6 + A_1$	$[2]$	$[1, 1]$
9	17	$D_8 + A_7 + 2A_1$	$[2]$	$[1, 1]$
10	17	$D_8 + A_5 + A_3 + A_1$	$[2]$	$[1, 1]$
11	17	$2D_6 + A_3 + 2A_1$	$[2, 2]$	$[1, 1]$
12	17	$D_6 + D_5 + A_5 + A_1$	$[2]$	$[1, 1]$
13	17	$D_6 + A_9 + 2A_1$	$[2]$	$[1, 1]$
14	17	$D_6 + A_7 + A_3 + A_1$	$[2]$	$[1, 1]$
15	17	$D_6 + A_7 + A_2 + 2A_1$	$[2]$	$[1, 1]$
16	17	$D_6 + A_5 + A_3 + A_2 + A_1$	$[2]$	$[1, 1]$
17	17	$D_6 + A_5 + A_3 + 3A_1$	$[2, 2]$	$[1, 1]$
18	17	$D_5 + 2A_6$	$[1]$	$[2]$
19	17	$D_4 + 2A_6 + A_1$	$[1]$	$[2]$
20	17	$A_{11} + A_5 + A_1$	$[1]$	$[2]$
21	17	$A_9 + A_5 + 3A_1$	$[2]$	$[1, 1]$
22	17	$A_9 + A_3 + A_2 + 3A_1$	$[2]$	$[1, 1]$
23	17	$A_7 + 2A_5$	$[1]$	$[2]$
24	17	$A_7 + A_5 + A_3 + 2A_1$	$[2]$	$[1, 1]$
25	17	$2A_6 + A_3 + 2A_1$	$[1]$	$[2]$
26	17	$A_6 + 2A_5 + A_1$	$[1]$	$[2]$
27	17	$2A_5 + 2A_3 + A_1$	$[2]$	$[1, 1]$
28	17	$2A_5 + A_3 + A_2 + 2A_1$	$[2]$	$[1, 1]$
29	16	$E_7 + D_6 + 3A_1$	$[2]$	$[1, 1]$
30	16	$E_7 + 2A_3 + 3A_1$	$[2]$	$[1, 1]$

(continues)

(continued)

no.	r	Φ	A	$[c_1, \ldots, c_k]$
31	16	$E_6 + 2A_5$	[1]	[2]
32	16	$D_{10} + A_3 + 3A_1$	[2]	$[1, 1]$
33	16	$D_8 + D_6 + 2A_1$	[2]	$[1, 1]$
34	16	$D_8 + A_5 + 3A_1$	[2]	$[1, 1]$
35	16	$D_8 + 2A_3 + 2A_1$	[2]	$[1, 1, 1]$
36	16	$2D_6 + A_3 + A_1$	[2]	$[1, 1]$
37	16	$2D_6 + 4A_1$	$[2, 2]$	$[1, 1]$
38	16	$D_6 + D_5 + A_3 + 2A_1$	[2]	$[1, 1]$
39	16	$D_6 + D_4 + A_5 + A_1$	[2]	$[1, 1]$
40	16	$D_6 + A_9 + A_1$	[2]	$[1, 1]$
41	16	$D_6 + A_7 + 3A_1$	[2]	$[1, 1]$
42	16	$D_6 + A_5 + A_3 + 2A_1$	[2]	$[1, 1, 1]$
43	16	$D_6 + A_5 + A_2 + 3A_1$	[2]	$[1, 1]$
44	16	$D_6 + 3A_3 + A_1$	[2]	$[1, 1]$
45	16	$D_6 + 2A_3 + A_2 + 2A_1$	[2]	$[1, 1]$
46	16	$D_6 + 2A_3 + 4A_1$	$[2, 2]$	$[1, 1]$
47	16	$D_5 + A_5 + A_3 + 3A_1$	[2]	$[1, 1]$
48	16	$D_4 + A_7 + A_3 + 2A_1$	[2]	$[1, 1]$
49	16	$A_{11} + A_3 + 2A_1$	[2]	$[1, 1]$
50	16	$A_9 + A_3 + 4A_1$	[2]	$[1, 1]$
51	16	$A_7 + A_5 + 4A_1$	[2]	$[1, 1]$
52	16	$A_7 + A_3 + A_2 + 4A_1$	[2]	$[1, 1]$
53	16	$3A_5 + A_1$	[1]	[2]
54	16	$2A_5 + A_3 + 3A_1$	[2]	$[1, 1, 1]$
55	16	$A_5 + 3A_3 + 2A_1$	[2]	$[1, 1]$
56	16	$A_5 + 2A_3 + A_2 + 3A_1$	[2]	$[1, 1]$
57	16	$A_5 + 2A_3 + 5A_1$	$[2, 2]$	$[1, 1]$
58	15	$E_7 + A_3 + 5A_1$	[2]	$[1, 1]$
59	15	$D_8 + A_3 + 4A_1$	[2]	$[1, 1, 1]$
60	15	$2D_6 + 3A_1$	[2]	$[1, 1]$
61	15	$D_6 + D_5 + 4A_1$	[2]	$[1, 1]$
62	15	$D_6 + D_4 + A_3 + 2A_1$	[2]	$[1, 1]$
63	15	$D_6 + A_7 + 2A_1$	[2]	$[1, 1]$
64	15	$D_6 + A_5 + A_3 + A_1$	[2]	$[1, 1]$
65	15	$D_6 + A_5 + 4A_1$	[2]	$[1, 1]$
66	15	$D_6 + 2A_3 + 3A_1$	[2]	$[1, 1, 1]$
67	15	$D_6 + A_3 + A_2 + 4A_1$	[2]	$[1, 1]$
68	15	$D_6 + A_3 + 6A_1$	$[2, 2]$	$[1, 1]$
69	15	$D_5 + 2A_3 + 4A_1$	[2]	$[1, 1]$
70	15	$D_4 + A_5 + A_3 + 3A_1$	[2]	$[1, 1]$
71	15	$D_4 + 3A_3 + 2A_1$	[2]	$[1, 1]$
72	15	$A_9 + A_3 + 3A_1$	[2]	$[1, 1]$
73	15	$A_7 + 2A_3 + 2A_1$	[2]	$[1, 1]$
74	15	$A_7 + A_3 + 5A_1$	[2]	$[1, 1]$
75	15	$2A_5 + A_3 + 2A_1$	[2]	$[1, 1]$
76	15	$2A_5 + 5A_1$	[2]	$[1, 1]$
77	15	$A_5 + 2A_3 + 4A_1$	[2]	$[1, 1, 1]$
78	15	$A_5 + A_3 + A_2 + 5A_1$	[2]	$[1, 1]$
79	15	$3A_3 + A_2 + 4A_1$	[2]	$[1, 1]$
80	15	$3A_3 + 6A_1$	$[2, 2]$	$[1, 1]$

(continues)

(continued)

no.	r	Φ	A	$[c_1, \dots, c_k]$
81	14	$D_8 + 6A_1$	$[2]$	$[1, 1]$
82	14	$D_6 + D_4 + 4A_1$	$[2]$	$[1, 1]$
83	14	$D_6 + A_5 + 3A_1$	$[2]$	$[1, 1]$
84	14	$D_6 + 2A_3 + 2A_1$	$[2]$	$[1, 1]$
85	14	$D_6 + A_3 + 5A_1$	$[2]$	$[1, 1, 1]$
86	14	$D_6 + A_2 + 6A_1$	$[2]$	$[1, 1]$
87	14	$D_5 + A_3 + 6A_1$	$[2]$	$[1, 1]$
88	14	$D_4 + 2A_3 + 4A_1$	$[2]$	$[1, 1]$
89	14	$A_7 + A_3 + 4A_1$	$[2]$	$[1, 1]$
90	14	$A_5 + 2A_3 + 3A_1$	$[2]$	$[1, 1]$
91	14	$A_5 + A_3 + 6A_1$	$[2]$	$[1, 1, 1]$
92	14	$4A_3 + 2A_1$	$[2]$	$[1, 1]$
93	14	$3A_3 + 5A_1$	$[2]$	$[1, 1]$
94	14	$2A_3 + A_2 + 6A_1$	$[2]$	$[1, 1]$
95	14	$2A_3 + 8A_1$	$[2, 2]$	$[1, 1]$
96	13	$D_6 + A_3 + 4A_1$	$[2]$	$[1, 1]$
97	13	$D_6 + 7A_1$	$[2]$	$[1, 1]$
98	13	$D_4 + A_3 + 6A_1$	$[2]$	$[1, 1]$
99	13	$A_5 + A_3 + 5A_1$	$[2]$	$[1, 1]$
100	13	$A_5 + 8A_1$	$[2]$	$[1, 1]$
101	13	$3A_3 + 4A_1$	$[2]$	$[1, 1]$
102	13	$2A_3 + 7A_1$	$[2]$	$[1, 1]$
103	13	$A_3 + A_2 + 8A_1$	$[2]$	$[1, 1]$
104	12	$D_6 + 6A_1$	$[2]$	$[1, 1]$
105	12	$2A_3 + 6A_1$	$[2]$	$[1, 1]$
106	12	$A_3 + 9A_1$	$[2]$	$[1, 1]$
107	11	$A_3 + 8A_1$	$[2]$	$[1, 1]$

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