A NOTE ON MIRANDA-MORRISON THEORY

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Let (X, f, s) be a complex elliptic $K3$ surface; that is, X is a complex $K3$ surface, $f: X \to \mathbb{P}^1$ is a fibration whose general fiber is a curve of genus 1, and $s: \mathbb{P}^1 \to X$ a section of f. As in [2], we consider the following objects:

- S_X is the Néron-Severi lattice of X, embedded primitively into the even unimodular lattice $H^2(X, \mathbb{Z})$ with the cup product.
- U_f is the sublattice of S_X generated by the class of a fiber of f and the class of the zero section $s(\mathbb{P}^1)$. Thus U_f is a hyperbolic plane.
- Φ_f is the set of classes of smooth rational curves on X that are mapped by f to a point and disjoint from $s(\mathbb{P}^1)$. It is well-known that Φ_f is a fundamental system of roots of type ADE.
- $L(\Phi_f)$ is the sublattice of S_X generated by Φ_f.
- M_f is the primitive closure of $L(\Phi_f)$ in S_X. It is obvious that $L(\Phi_f)$ is orthogonal to U_f in S_X, and hence the orthogonal direct sum $U_f \oplus M_f$ is embedded primitively into S_X.
- A_f is the finite abelian group $M_f/L(\Phi_f)$. It is well-known that A_f is isomorphic to the torsion part of the Mordell-Weil group of (X, f, s).
- T_f is the orthogonal complement of $U_f \oplus M_f$ in $H^2(X, \mathbb{Z})$. Then we have an isomorphism $q_{T_f} \cong -q_{M_f}$.
- $\mathrm{O}(T_f) \to \mathrm{O}(q_{T_f})$ is the natural homomorphism from the orthogonal group of T_f to the automorphism group of the discriminant form q_{T_f} of T_f.
- $\mathcal{G}_{T_f}$ is the genus of lattices containing the isomorphism class of T_f.

Suppose that $\operatorname{rank} T_f \geq 3$; that is, $\operatorname{rank} L(\Phi_f) \leq 17$. Miranda-Morrison theory [1] enables us to put a structure of the abelian group on $\mathcal{G}_{T_f}$, and to calculate a group $\mathcal{M}_{T_f}$ that fits in the exact sequence

$$0 \;\to\; \mathrm{Coker}(\mathrm{O}(T_f) \to \mathrm{O}(q_{T_f})) \;\to\; \mathcal{M}_{T_f} \;\to\; \mathcal{G}_{T_f} \;\to\; 0.$$

We have calculated this group $\mathcal{M}_{T_f}$ for all elliptic $K3$ surface (X, f, s) by means of the computational tools developed in [2].

Theorem 1. *Suppose that* $\operatorname{rank} T_f \geq 3$. *Then the group* $\mathcal{G}_{T_f}$ *is trivial.*

Theorem 2. *The list in the following page is the list of all the cases where* $\operatorname{rank} T_f \geq 3$ *and* $\mathcal{M}_{T_f}$ *is non-trivial.*

Remark 3. In order to calculate the connected components of the moduli of elliptic $K3$ surfaces in [2], we have to take into account the positive sign structures of T_f, and the action of automorphisms of q_{T_f} coming from the automorphisms of the diagram Φ_f via the isomorphism $q_{T_f} \cong -q_{M_f}$. For this purpose, we presented a refinement of the Miranda-Morrison theory in Section 4.4 of [2]. The purpose of this note is to present a simple part of the calculation, for which we need not use the refinement of Miranda-Morrison theory.

The contents of the following table are

$$\operatorname{rank} L(\Phi_f) \qquad \text{the } ADE \text{ type of } \Phi_f \qquad A_f \qquad |\mathcal{M}_{T_f}|.$$

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