ENRIQUES INVOLUTIONS ON SINGULAR K3 SURFACES OF SMALL DISCRIMINANTS: COMPUTATIONAL DATA

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1. Introduction

This note is an explanation of the computational data obtained in the second half of the paper [2] (joint work with Davide Cesare Veniani). The data are available from the author’s webpage:

http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3andEnriques.html

The data consists of 11 files, each of which contains a record of GAP [1] that describes a data of the singular K3 surface $X$ with the transcendental lattice $T_X$:

\[
\begin{align*}
Xd3a2b1c2 & : T_X = [2,1,2] \\
Xd4a2b0c2 & : T_X = [2,0,2] \\
Xd7a2b1c4 & : T_X = [2,1,4] \\
Xd8a2b0c4 & : T_X = [2,0,4] \\
Xd12a2b0c6 & : T_X = [2,0,6] \\
Xd12a4b2c4 & : T_X = [4,2,4] \\
Xd15a2b1c8 & : T_X = [2,1,8] \\
Xd16a4b0c4 & : T_X = [4,0,4] \\
Xd20a4b2c6 & : T_X = [4,2,6] \\
Xd24a2b0c12 & : T_X = [2,0,12] \\
Xd36a6b0c6 & : T_X = [6,0,6]
\end{align*}
\]

Each of these records is written in the txt file with the same name.

In the file NKrecs.txt, we give a list of 7 records NKrec, each of which describes the nef-chamber of an Enriques surface with finite automorphism group.

Every lattice $L$ is equipped with a basis, which is fixed during the whole calculation. The dual lattice $L^\vee$ is regarded as a submodule of $L \otimes \mathbb{Q}$, and hence its elements are given by a vector with rational components.

The discriminant form $q(L)$ of an even lattice $L$ is described by a record, whose components are explained in Section 3. In particular, the natural projection $L^\vee / L$ and the automorphism group $O(q(L))$ are given in this record, and the natural homomorphism $O(L) \to O(q(L))$ is calculated by means of this record.

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Let \( \mathbf{Xrec} \) be one of the records in (1.1). Then \( \mathbf{Xrec} \) has the following components:

- \( \mathbf{Xrec.GramTX} \): The Gram matrix of \( T_X \).
- \( \mathbf{Xrec.GramSX} \): The Gram matrix of \( S_X \).
- \( \mathbf{Xrec.GramHX} \): The Gram matrix of \( H^2(X,Z) \).
- \( \mathbf{Xrec.GramL} \): The Gram matrix of \( L_{26} \).
- \( \mathbf{Xrec.GramR} \): The Gram matrix of the orthogonal complement \( R := [\iota_X] \perp \) of the image of \( \iota_X: S_X \rightarrow L_{26} \).
- \( \mathbf{Xrec.m2R} \): The list of \((-2)\)-vectors of \( R \).
- \( \mathbf{Xrec.embTXHX} \): The embedding of \( T_X \) into \( H^2(X,Z) \).
- \( \mathbf{Xrec.embSXHX} \): The embedding of \( S_X \) into \( H^2(X,Z) \).
- \( \mathbf{Xrec.embSXL} \): The embedding of \( S_X \) into \( L_{26} \).
- \( \mathbf{Xrec.embRL} \): The embedding of \( R \) into \( L_{26} \).
- \( \mathbf{Xrec.projHXTX} \): The orthogonal projection \( H^2(X,Q) \rightarrow T_X \otimes Q \).
- \( \mathbf{Xrec.projHSX} \): The orthogonal projection \( H^2(X,Q) \rightarrow S_X \otimes Q \).
- \( \mathbf{Xrec.projLSX} \): The orthogonal projection \( L_{26} \otimes Q \rightarrow S_X \otimes Q \).
- \( \mathbf{Xrec.projLR} \): The orthogonal projection \( L_{26} \otimes Q \rightarrow R \otimes Q \).
- \( \mathbf{Xrec.OTX} \): The elements of the group \( O(T_X) \).
- \( \mathbf{Xrec.OR} \): The elements of the group \( O(R) \).
- \( \mathbf{Xrec.qTX} \): The record of \( q(T_X) \) (see Section 3).
- \( \mathbf{Xrec.qSX} \): The record of \( q(S_X) \) (see Section 3).
- \( \mathbf{Xrec.qR} \): The record of \( q(R) \) (see Section 3).
- \( \mathbf{Xrec.kerOTXOqTX} \): The elements of the kernel of the natural homomorphism \( O(T_X) \rightarrow O(q(T_X)) \).
- \( \mathbf{Xrec.imageOTXOqTX} \): The elements of the image of the natural homomorphism \( O(T_X) \rightarrow O(q(T_X)) \).
- \( \mathbf{Xrec.imageOROqR} \): The elements of the image of the natural homomorphism \( O(R) \rightarrow O(q(R)) \).
- \( \mathbf{Xrec.isomqSXqTX} \): The isomorphism \( q(S_X) \rightarrow -q(T_X) \) induced by \( H^2(X,Z) \).
- \( \mathbf{Xrec.isomqTXqSX} \): The inverse of \( \mathbf{Xrec.isomqSXqTX} \).
- \( \mathbf{Xrec.isomqSXqR} \): The isomorphism \( q(S_X) \rightarrow -q(R) \) induced by \( L_{26} \).
- \( \mathbf{Xrec.isomqRqSX} \): The inverse of \( \mathbf{Xrec.isomqSXqTR} \).
- \( \mathbf{Xrec.OTXomega} \): The elements of \( O(T_X,\omega) \).
- \( \mathbf{Xrec.OqTXomega} \): The elements of \( O(q(T_X),\omega) \).
- \( \mathbf{Xrec.OqSXomega} \): The elements of \( O(q(S_X),\omega) \).
- \( \mathbf{Xrec.OqRomega} \): The image of \( O(q(S_X),\omega) \) by the isomorphism \( O(q(S_X)) \rightarrow O(q(R)) \) induced by \( L_{26} \).
- \( \mathbf{Xrec.weyl} \): The Weyl vector \( \mathbf{w} \) for the Conway chamber inducing \( D_0 \).
- \( \mathbf{Xrec.weylS} \): The image of the Weyl vector \( \mathbf{w} \) by the orthogonal projection \( L_{26} \rightarrow S_X \).
• **Xrec.autXD0**: The elements of the group \(\text{aut}(X, D_0)\).

• **Xrec.D0wallorbitrecs**: The list of records \(\text{orbrec}\) that describe orbits of the walls of \(D_0\) under the action of \(\text{aut}(X, D_0)\). Each \(\text{orbrec}\) has the following components. Let \(w\) be a representative of the orbit \(o\) described by \(\text{orbrec}\).
  
  – \(\text{orbrec.}\text{definingv}\): The primitive vector \(v\) of \(S_X^\vee\) that defines the representative \(w\) of \(o\).
  
  – \(\text{orbrec.}\text{definingvs}\): The list of primitive vectors of \(S_X^\vee\) that define the walls in \(o\).
  
  – \(\text{orbrec.}\text{n}\): \(n = \langle v, v \rangle\).
  
  – \(\text{orbrec.}\text{a}\): \(a = \langle w, v \rangle\).
  
  – \(\text{orbrec.}\text{size}\): The size \(|o|\) of \(\text{orbrec.}\text{definingvs}\).
  
  – \(\text{orbrec.}\text{innout}\): If \(w\) is inner, then "inner", whereas if \(w\) is outer, then "outer".

If \(w\) is inner, then \(\text{orbrec}\) has the following components. If \(w\) is outer, then the components below are the string "not defined".

  – \(\text{orbrec.}\text{type}\): The type (no.) of the orbit of inner-walls given in Tables of \([2]\).
  
  – \(\text{orbrec.}\text{roots}\): The list of \((-2)\)-vectors \(r\) of \(L_{26}\) such that \(\langle w, r \rangle = 1\) and that \((r)^\perp\) passes through \(\iota_X(w) \subset \mathcal{P}_{26}\).
  
  – \(\text{orbrec.}\text{adetype}\): The ADE-type of a fundamental root system \(\text{orbrec.}\text{roots}\), which is a list of strings "A1", "A2", \ldots, "E8".
  
  – \(\text{orbrec.}\text{adjweyl}\): The Weyl vector \(w'\) corresponding to an Conway chamber \(C'\) that induces the \(\iota_X R_{26}^\perp\)-chamber \(D'\) adjacent to \(D_0\) across the wall \(w\).
  
  – \(\text{orbrec.}\text{adjweylS}\): The projection of \(w' \in L_{26}\) to \(S_X^\vee\), which is the image of \(\text{Xrec.weylS}\) by an extra automorphism \(g_S\) associated with \(w\).
  
  – \(\text{orbrec.}\text{gl}\): An isometry \(g_L \in O^+(L_{26})\) that maps \(C_0\) to \(C'\), preserves \(S_X\), and such that \(g_L|S_X\) is an element of \(\text{aut}(X)\) that maps \(D_0\) to \(D'\).
  
  – \(\text{orbrec.}\text{gS}\): An isometry \(g_S \in \text{aut}(X)\) that is an extra automorphism associated with \(w\).
  
  – \(\text{orbrec.}\text{dg}\): The degree \(\langle w^{g_S}, w \rangle\).

• **Xrec.D0wallrecs**: The list of records \(\text{wallrec}\) that describe the walls of \(D_0\). Each record \(\text{wallrec}\) has the following components.

  – \(\text{wallrec.}\text{no}\): The number (index) of the wall, which will be used in \(\text{facerec}\) below.
  
  – \(\text{wallrec.}\text{definingv}\): The primitive vector \(v\) of \(S_X^\vee\) that defines the wall \(w\).
  
  – \(\text{wallrec.}\text{innout}\): If \(w\) is inner, then "inner", whereas if \(w\) is outer, then "outer".
– wallrec.type: If \( w \) is inner, then the type given in Tables of [2]. If \( w \) is outer, then "not defined".

– wallrec.extraaut: If \( w \) is inner, then an extra automorphism \( g \in \text{aut}(X) \) associated with \( w \). If \( w \) is outer, then "not defined".

• Xrec.facerecs: The list of records that describe the \( N_X \)-inner faces of \( D_0 \). An \( \text{aut}(X) \)-equivalence class of \( N_X \)-inner faces of \( D_0 \) is described by a record facerec whose components are as follows. Let \( f \) be a representative of the \( \text{aut}(X) \)-equivalence class.

– facerec.no: The number (index) of this \( \text{aut}(X) \)-equivalence class, which will be used in enrrec below.

– facerec.dim: The dimension of \( f \).

– facerec.basis: A basis of the vector subspace \( \langle f \rangle \) of \( S_X \otimes \mathbb{Q} \).

– facerec.passingwalls: The list of wallrec.no of walls of \( D_0 \) containing \( f \).

– facerec.Gammaf: A list \( \Gamma(f) \) of elements of \( g \in \text{aut}(X) \) such that \( \{D_0^g | g \in \Gamma(f)\} \) is equal to \( \mathcal{D}(f) \) and that, if \( g \neq g' \in \Gamma(f) \), then \( D_0^g \neq D_0^{g'} \).

– facerec.autXf: The list of elements of the group \( \text{aut}(X, f) \),

– facerec.orbrerecs: The list of \( \text{aut}(X, D_0) \)-orbits contained in this \( \text{aut}(X) \)-equivalence class of \( N_X \)-inner faces of \( D_0 \). Let \( o \) be an \( \text{aut}(X, D_0) \)-orbit in this \( \text{aut}(X) \)-equivalence class. Then \( o \) is described by the record orbrecc with the following components. Let \( f' \) be a representative of \( o \).

* orbrecc.size: The size of \( o \).

* orbrecc.g: An element \( g \in \text{aut}(X) \) such that \( f^g = f' \).

* orbrecc.passingwalls: The list of wallrec.no of walls of \( D_0 \) containing \( f' \).

* orbrecc.passingwalltypes: The list of wallrec.type of walls containing \( f' \).

• Xrec.Yrecs: The list of records that describe the Enriques involutions on \( X \) (see Section 4).

3. DISCREC

The discriminant form \( q(L) \) of an even lattice \( L \) of rank \( r \) with a fixed basis is described by a record discrec. We put

\[
A(L) := L'/L \cong \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_l\mathbb{Z}
\]

and fix, one and for all, a set of generators \( \vec{v}_1, \ldots, \vec{v}_l \) of \( A(L) \), where \( \vec{v}_i \) is a generator of the \( i \)th component \( \mathbb{Z}/a_i\mathbb{Z} \). Let \( b(L): A(L) \times A(L) \to \mathbb{Q}/\mathbb{Z} \) be the bilinear form associated with the quadratic form \( q(L): A(L) \to \mathbb{Q}/2\mathbb{Z} \). The record discrec has the following components.
ENRIQUES SURFACES

• discrec.discg: The list \([a_1, \ldots, a_l]\).
• discrec.discf: The Gram matrix of \(q(L)\), whose \((i, j)\)-component is \(q(L)(\bar{v}_i) \in \mathbb{Q}/2\mathbb{Z}\) if \(i = j\) and \(b(L)(\bar{v}_i, \bar{v}_j) \in \mathbb{Q}/\mathbb{Z}\) if \(i \neq j\).
• discrec.lifts: The list of vectors \(v_1, \ldots, v_l \in L^\vee\), where \(\bar{v}_i = v_i \mod L\).
• discrec.proj: The \(r \times l\) matrix \(M\) such that \(v \mapsto vM\) gives the natural projection \(L^\vee \rightarrow A(L)\), where \(v \in L^\vee\) is written as a vector of length \(r\) with rational components with respect to the basis of \(L\).
• discrec.Oq: The list of elements of \(O(q(L))\).

Note that, by this record, we can calculate the action \(q(g) \in O(q(L))\) of a given isometry \(g\) of \(L\) on \(q(L)\).

4. ENRREC

An Enriques involution \(\varepsilon \in \text{aut}(X)\) on \(X\) is is described by a record \textit{enrrec} whose components are as follows.

• \textit{enrrec.no}: The number of \(\varepsilon\) in Table (Enriques involutions) of [2].
• \textit{enrrec.fenr}: The number \textit{facerec.no} of the \(\text{aut}(X)\)-equivalence class of \(N_X\)-inner faces containing \(f_\varepsilon\).
• \textit{enrrec.invol}: The matrix representation of \(\varepsilon \in O^+(S_X)\).
• \textit{enrrec.GramSY}: The Gram matrix of \(S_Y\).
• \textit{enrrec.GramP}: The Gram matrix of the orthogonal complement \(P\) of the image of \(\pi^*: S_Y(2) \hookrightarrow S_X\).
• \textit{enrrec.embSY2SX}: The embedding \(\pi^*: S_Y(2) \hookrightarrow S_X\).
• \textit{enrrec.embPSX}: The embedding \(P \hookrightarrow S_X\).
• \textit{enrrec.projSXSY2}: The orthogonal projection \(S_X \otimes \mathbb{Q} \rightarrow S_Y(2) \otimes \mathbb{Q}\).
• \textit{enrrec.projSXP}: The orthogonal projection \(S_X \otimes \mathbb{Q} \rightarrow P \otimes \mathbb{Q}\).
• \textit{enrrec.m4P}: The list of \((-4)\)-vectors of \(P\).
• \textit{enrrec.autXenrfenr}: The list of elements of \(\text{aut}(X, \varepsilon, f_\varepsilon)\).
• \textit{enrrec.kerres}: The list of elements of the kernel of the restriction homomorphism \(\text{aut}(X, \varepsilon, f_\varepsilon) \rightarrow \text{aut}(Y, E_0)\). The quotient of \textit{enrrec.kerres} by \(\langle \varepsilon \rangle\) is identified with the kernel of \(\rho_Y: \text{Aut}(Y) \rightarrow \text{aut}(Y)\).
• \textit{enrrec.autYE0}: The list of elements of \(\text{aut}(Y, E_0)\).
• \textit{enrrec.E0NKtype}: The Nikulin-Kondo type of the walls of \(E_0\). For the Enriques involution No. 24 in Table (Enriques involutions) of [2], this data is "not defined".
• \textit{enrrec.E0walls}: The list of walls of defining \((-2)\)-vectors of the walls of \(E_0\).
• \textit{enrrec.E0wallrecs}: The list of records \textit{E0wallrec}, each of which describes a wall \(w\) of \(E_0\). The components of the record \textit{E0wallrec} are as follows.
– \texttt{E0wallrec.no}: The number of the wall. Except for the Enriques involution \texttt{No. 24}, the walls are numbered according to the corresponding Nikulin-Kondo configuration recorded in \texttt{NKrecs} (see Section 5), and hence this numbering coincides with the figures of Nikulin-Kondo configurations in [2].

– \texttt{E0wallrec.definingv}: The \((-2)\)-vector \(r \in \mathcal{R}_Y\) of \(S_Y\) that defines the wall \(w\) of \(E_0\).

– \texttt{wallrec.innout}: If \(w\) is inner, then "inner", whereas if \(w\) is outer, then "outer".

– \texttt{E0wallrec.embwSX}: A basis of the minimal linear subspace of \(S_X \otimes \mathbb{Q}\) containing \(\pi^*(w) \subset \mathcal{P}_X\).

– \texttt{E0wallrec.passingwalls}: The list of \texttt{wallrec.no} of walls of \(D_0\) passing through \(\pi^*(w) \subset \mathcal{P}_X\).

– \texttt{E0wallrec.fenrw}: If \(w\) is inner, then the \texttt{facerc.no} of the \(\text{aut}(X)\)-equivalence class of \(N_X\)-inner faces containing \(f_e(w)\). If \(w\) is outer, then "not defined".

– \texttt{E0wallrec.extraaut}: If \(w\) is inner, then an extra automorphism \(g \in \text{aut}(X, \varepsilon)\) associated with \(w\). If \(w\) is outer, then "not defined".

– \texttt{E0wallrec.extraautY}: If \(w\) is inner, then the automorphism \(g|_{S_Y} \in \text{aut}(Y)\), where \(g \in \text{aut}(X, \varepsilon)\) is \texttt{E0wallrec.extraaut}. If \(w\) is outer, then "not defined".

• \texttt{enrrec.autXenrgenerators}: A generating set of \(\text{aut}(X, \varepsilon)\).

• \texttt{enrrec.autYgenerators}: A generating set of \(\text{aut}(Y)\).

5. \texttt{NKrec}

In the file \texttt{NKrecs.txt}, we give a list of 7 records \texttt{NKrec}, each of which describes the nef-chamber of an Enriques surface \(Y\) with finite automorphism group. The components of \texttt{NKrec} are as follows.

• \texttt{NKrec.type}: The Nikulin-Kondo type of \(Y\) (a number between 1 and 7).

• \texttt{NKrec.configmat}: The adjacency matrix of the dual graph of the smooth rational curves on \(Y\).

In the figures of Nikulin-Kondo configurations in [2], the nodes are numbered according to this record.

References


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