ENRIQUES INVOLUTIONS ON SINGULAR K3 SURFACES
OF SMALL DISCRIMINANTS

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Abstract. We classify Enriques involutions on a K3 surface, up to conjugation in the automorphism group, in terms of lattice theory. We enumerate such involutions on singular K3 surfaces with transcendental lattice of discriminant smaller than or equal to $36$. For 11 of these K3 surfaces, we apply Borcherds method to compute the automorphism group of the Enriques surfaces covered by them. In particular, we investigate the structure of the two most algebraic Enriques surfaces.

1. Introduction

1.1. Background. Let $X$ be a complex K3 surface. We denote by $S_X = H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ the lattice of numerical equivalence classes of divisors on $X$, and by $T_X$ the orthogonal complement of $S_X$ in $H^2(X,\mathbb{Z})$, which we call the transcendental lattice of $X$. Suppose that $X$ is singular, that is, the Picard number $\text{rank} S_X$ attains the possible maximum $h^{1,1}(X) = 20$. The discriminant of a singular K3 surface $X$ is the determinant of a Gram matrix of $T_X$. Since $T_X$ is an even positive definite lattice of rank 2, the discriminant $d$ of $X$ is a positive integer satisfying $d \equiv 0$ or $3 \mod 4$. Note that $T_X$ is naturally oriented by the Hodge structure. By the classical work of Shioda–Inose [32], we know that the isomorphism class of the oriented lattice $T_X$ determines $X$ up to $\mathbb{C}$-isomorphism.

An involution $\tilde{\varepsilon} : X \to X$ of a K3 surface $X$ is called an Enriques involution if $\tilde{\varepsilon}$ acts freely on $X$. Sertöz [25] gave a simple criterion to determine whether a singular K3 surface has an Enriques involution or not (see Theorem 3.2.1 and also Lee [18]). On the other hand, Ohashi [22] showed that each complex K3 surface $X$ (not necessarily singular) has only finitely many Enriques involutions up to conjugation in the automorphism group $\text{Aut}(X)$ of $X$, and that there exists no universal bound for the number of conjugacy classes of Enriques involutions.

Ohashi also gave a lattice theoretic method to enumerate Enriques involutions on certain K3 surfaces. In a subsequent paper [23] he classified all Enriques involutions on the Kummer surface $\text{Km}(\text{Jac}(C))$ associated with the jacobian variety of a generic curve $C$ of genus 2.

For some K3 surfaces $X$, the group $\text{Aut}(X)$ can be calculated by Borcherds method ([3], [4]); for instance, Kondo [16] implemented it in order to compute $\text{Aut}(\text{Km}(\text{Jac}(C)))$.

1.2. Main results. In this paper, we classify, up to conjugation in $\text{Aut}(X)$, all Enriques involutions $\tilde{\varepsilon}$ on the singular K3 surfaces $X$ whose discriminant $d$ satisfies...
d \leq 36. The classification is given in Table 3.1 and builds on a refinement and generalization of Ohashi’s method. Our main result, namely Theorem 3.1.9, applies to any K3 surface.

We then concentrate on 11 of these singular K3 surfaces, listed in Table 4.1, to which we can apply Borcherds method in order to compute the automorphism group. We first write the action of \( \text{Aut}(X) \) on the nef chamber of \( X \) explicitly. Building on this data, we re-enumerate all Enriques involutions up to conjugation. Using also a result of the preprint [6] (see Section 2.9), we are able to calculate the automorphism group of the Enriques surfaces covered by these K3 surfaces. The results are given in Theorem 5.4.1 and Table 5.1.

Note that the enumeration of Enriques involutions by Ohashi’s method and by Borcherds method are carried out independently. The results are, of course, consistent. We hope that these methods will be applied to many other K3 surfaces (with smaller Picard number) and Enriques surfaces covered by them, and that in these works, our general results on a K3 surface admitting an Enriques involution (Lemma 3.1.7 and Proposition 3.1.8) will be useful.

Recently, many studies on the automorphism groups \( \text{Aut}(Y) \) of Enriques surfaces \( Y \) have appeared ([1], [19], [30]). Our result gives a description of \( \text{Aut}(Y) \) in terms of its action on the lattice \( S_Y \) of numerical equivalence classes of divisors on \( Y \). We expect that this description is helpful in the search for a more geometric description of \( \text{Aut}(Y) \), that is, for writing elements of \( \text{Aut}(Y) \) as birational self-maps on some projective model of \( Y \).

Computations were carried out using \texttt{GAP} [9] and \texttt{sage} on SageMath [33]. Further computational data is provided on the web page [31].

As a corollary of our calculations, we obtain the following. For \( d = 3, 4 \) or 7, there exists exactly one singular K3 surface \( X_d \) of discriminant \( d \) up to \( \mathbb{C} \)-isomorphism. The K3 surfaces \( X_3, X_4 \), also known as “the two most algebraic K3 surfaces”, were studied by Vinberg [37]. Neither \( X_3 \) nor \( X_4 \) admits any Enriques involution, but \( X_7 \) does; following Vinberg, we call the Enriques surfaces covered by \( X_7 \) the most algebraic Enriques surfaces.

**Theorem 1.2.1.** The singular K3 surface \( X_7 \) of discriminant 7 has exactly two Enriques involutions \( \tilde{\varepsilon}_I \) and \( \tilde{\varepsilon}_{II} \) up to conjugation in \( \text{Aut}(X_7) \). Let \( Y_I \) and \( Y_{II} \) be the quotient Enriques surfaces corresponding to \( \tilde{\varepsilon}_I \) and \( \tilde{\varepsilon}_{II} \), respectively. Then \( \text{Aut}(Y_I) \) is finite of order 8, and \( \text{Aut}(Y_{II}) \) is finite of order 24.

Nikulin [21] and Kondo [15] classified all complex Enriques surfaces whose automorphism group is finite. It turns out that these Enriques surfaces are divided into 7 classes I, II, \ldots, VII, which we call Nikulin-Kondo type. See Kondo [15] for the properties of these Enriques surfaces.

**Corollary 1.2.2.** The most algebraic Enriques surfaces have finite automorphism groups and their Nikulin-Kondo types are I and II.

Mukai (private communication) had already realized this result previously. Answering a question by G. Kapustka, in Section 6 we give explicit models of the most algebraic Enriques surfaces \( Y_I \) and \( Y_{II} \) as Enriques sextic surfaces.

1.3. Contents. This paper is organized as follows. In Section 2, we recall basic facts about lattices, K3 surfaces and Enriques surfaces, and fix notions and notation. In Section 3, we classify all Enriques involutions on singular K3 surfaces with
discriminant $\leq 36$ by a generalization of Ohashi’s method. In Section 4, we recall Borcherds method, and apply it to the 11 singular K3 surfaces whose transcendental lattices are listed in Table 4.1. Recently, many geometric studies of singular K3 surfaces of small discriminant have appeared (see, for example, [2], [10], [17], [35]). We summarize the computational data for these 11 singular K3 surfaces in Table 4.2. In Section 5, we explain an algorithm to calculate Enriques involutions and the automorphism groups of the Enriques surfaces from the data obtained by Borcherds method, and apply this method to the 11 singular K3 surfaces. In Section 6, we study the most algebraic Enriques surfaces $Y_1$ and $Y_{11}$.

2. Preliminaries

2.1. Lattices. A lattice $L$ is a free $\mathbb{Z}$-module of finite rank with a $\mathbb{Z}$-valued non-degenerate symmetric form $\langle \cdot, \cdot \rangle$. The determinant $\det L$ of $L$ is the determinant of any Gram matrix of $L$. A lattice $L$ is unimodular if $\det L = \pm 1$. A lattice with the same underlying $\mathbb{Z}$-module as $L$ and symmetric form $n \cdot \langle \cdot, \cdot \rangle$ is denoted by $L(n)$.

The group of isometries of $L$ is denoted $O(L)$. We let $O(L)$ act on $L$ from the right. A vector $v$ of a lattice $L$ is called an $n$-vector if $\langle v, v \rangle = n$. We denote by $R_L$ the set of $(-2)$-vectors of a lattice $L$.

A lattice $L$ is even if $\langle v, v \rangle \in 2\mathbb{Z}$ for all $v \in L$; otherwise, it is odd. The signature of a lattice $L$ is the signature of $L \otimes \mathbb{R}$. Analogously, we say that $L$ is positive definite, negative definite or indefinite if $L \otimes \mathbb{R}$ is. A lattice $L$ of rank $n > 1$ is hyperbolic if the signature is $(1, n-1)$. A positive cone of a hyperbolic lattice $L$ is one of the two connected components of $\{ v \in L \otimes \mathbb{R} | \langle v, v \rangle > 0 \}$. For a hyperbolic lattice $L$ and a positive cone $P_L$ of $L$, we denote by $O(L, P_L)$ the group of isometries of $L$ that preserves $P_L$.

The standard positive definite lattices associated to Dynkin graphs will be denoted $A_n$ ($n \geq 1$), $D_n$ ($n \geq 4$), $E_6$, $E_7$, $E_8$.

2.2. Surfaces. Let $Z$ be a K3 surface or an Enriques surface. We denote by $S_Z$ the lattice of numerical equivalence classes of divisors on $Z$, and call it the Néron–Severi lattice of $Z$. Then $S_Z$ is an even hyperbolic lattice, provided that rank $S_Z > 1$. Let $P_Z$ denote the positive cone of $S_Z$ that contains an ample class, and let $R_Z$ be the set of $(-2)$-vectors of $S_Z$. For simplicity, we denote by $\text{aut}(Z)$ the the image of the natural representation

$$\rho_Z : \text{Aut}(Z) \to O(S_Z, P_Z).$$

We put

$$N_Z := \{ x \in P_Z | \langle x, [\Gamma] \rangle \geq 0 \text{ for all curves } \Gamma \text{ on } Z \},$$

and call it the nef chamber of $Z$. It is obvious that the action of $\text{aut}(Z)$ on $P_Z$ preserves $N_Z$.

2.3. Finite bilinear and quadratic forms. A finite quadratic form is a finite abelian group $G$ together with a function $q : G \to \mathbb{Q}/2\mathbb{Z}$ which satisfies

$$q(n\alpha) = n^2 q(\alpha) \text{ for every } \alpha \in G \text{ and } n \in \mathbb{Z}$$

such that the function $b(q) : G \times G \to \mathbb{Q}/\mathbb{Z}$ defined by

$$(\alpha, \beta) \mapsto \frac{q(\alpha + \beta) - q(\alpha) - q(\beta)}{2}$$

is a finite bilinear form with the property that for all $\alpha, \beta \in G$, we have

$$\frac{q(\alpha + \beta) - q(\alpha) - q(\beta)}{2} = \frac{q(\alpha) + q(\beta)}{2}.$$
is a finite symmetric bilinear form. For the sake of simplicity, we will denote by $q$ also the underlying finite abelian group $G$. The length, i.e. the minimal number of generators, of $G$ (resp. of the $p$-torsion part of $G$) is denoted by $\ell(G)$ (resp. $\ell_p(G)$).

A subgroup $\Gamma \subset G$ is called isotropic if $q|\Gamma = 0$, where $q|\Gamma$ denotes the restriction of $q$ to $\Gamma$. Given an isotropic subgroup $\Gamma$, the quadratic form $q$ descends to the quotient group $\Gamma^*/\Gamma$, where

$$\Gamma^* := \{\alpha \in G \mid b(q)(\alpha, \gamma) = 0 \text{ for every } \gamma \in \Gamma\};$$

we denote the resulting finite quadratic form by $q\Gamma^*/\Gamma$.

If $L$ is a lattice, then the group $L^*/L$, where $L^* := \text{Hom}(L, \mathbb{Z}) \subset L \otimes \mathbb{Q}$, is a finite abelian group of order $|\text{det } L|$. The discriminant bilinear form of a lattice $L$ is the finite symmetric bilinear form induced by $\langle , \rangle$

$$b(L): L^*/L \times L^*/L \to \mathbb{Q}/\mathbb{Z}.$$ 

If $L$ is even, the discriminant quadratic form of $L$ is the finite quadratic form induced by $\langle , \rangle$

$$q(L): L^*/L \to \mathbb{Q}/2\mathbb{Z}.$$ 

Let $O(q(L))$ denote the automorphism group of the finite quadratic form $q(L)$, which we let act on $q(L)$ from the right. There is a natural homomorphism

$$O(L) \to O(q(L)), \quad g \mapsto q(g).$$

Let $C_n(e)$ be the cyclic group of order $n$ generated by $e$. For $k \geq 1$, we denote by $u_k$ (resp. $v_k$) the finite quadratic form with underlying group $C_{2^k}(e) \times C_{2^k}(f)$ such that $\langle e, e \rangle = \langle f, f \rangle = 0$ (resp. $\langle e, e \rangle = \langle f, f \rangle = 1$) and $\langle e, f \rangle = \frac{1}{2^k}$. For $a,b \in \mathbb{Z}$ prime to each other, we denote by $\langle \frac{a}{b} \rangle$ the finite quadratic form with underlying group $C_b(e)$ such that $\langle e, e \rangle = \frac{a}{b}$.

### 2.4. Genera.

Given a pair of non-negative integers $(s_+, s_-)$ and a non-degenerate finite quadratic (resp. bilinear) form $h$, the genus $g(s_+, s_-, h)$ is the set of isometry classes of even (resp. odd) lattices of signature $(s_+, s_-)$ with discriminant quadratic (resp. bilinear) form isomorphic to $h$. If a genus contains only the isometry class of a lattice $L$, we say that $L$ is unique in its genus.

In general, enumerating all isometry classes in a given genus is a non-trivial problem. It is computationally easier to find lattices of smaller determinant, so the following elementary lemma can be very useful.

**Lemma 2.4.1.** Given a lattice $L$ and a prime number $p$, then $\ell_p(L^*/L) = \text{rank } L$ if and only if $L = L'(p)$ for some lattice $L'$. In this case, and if moreover $L$ is even and $p = 2$, then $L'$ is odd if and only if $q(L) = \langle \frac{1}{2} \rangle \oplus q'$ or $q(L) = \langle \frac{1}{2} \rangle \oplus q'$ for some finite quadratic form $q'$.

**Remark 2.4.2.** Suppose $q$ is a finite quadratic form admitting an isotropic subgroup $\Gamma$. In order to enumerate all isometry classes of even lattices in $g(s_+, s_-, q)$, we can take advantage of Proposition 1.4.1 in [20]: first we enumerate all lattices in $g(s_+, s_-, q|\Gamma^*/\Gamma)$, then we inspect all sublattices of index $|\Gamma|$.

Given a finite (bilinear or quadratic) form $h$ and $s \in \mathbb{N}$, the following algorithm, suggested by Degtyarev, finds all (odd or even) lattices in $g(s, 0, h)$. If $h$ is quadratic we put $b = b(h)$, otherwise we put $b = h$. 

Algorithm 2.4.3. Let \( r \) be the smallest possible rank for which there exists an (odd or even) positive definite lattice \( M \) of rank \( r \) and discriminant bilinear form \(-b\).

By results of Nikulin [20], for each \( N \in g(s, 0, h) \) there exists a primitive embedding \( \iota: M \hookrightarrow L \) into some positive definite unimodular lattice \( L \) of rank \( r + s \) such that \([i]^\perp \cong N\). Taking advantage of the classification of positive definite unimodular lattices of small rank (see, for instance, Table 16.7 in [7]), we list all such lattices \( L \). Using GAP and the function \texttt{ShortestVectors}, we list all primitive embeddings \( \iota: M \hookrightarrow L \) for all \( M \in g' \) and all \( L \). Then, we compute the lattices \([i]^\perp\) and select those ones which belong to \( g(s, 0, h) \). In order to eliminate pairs of isomorphic lattices, one can use the attribute \texttt{is_globally_equivalent_to} of the class \texttt{QuadraticForm} in \texttt{sage}.

The algorithm works provided that \( r + s \) is small enough and that we can find a lattice \( M \) explicitly. In order to find \( M \), we can apply the algorithm recursively to \( g(r, 0, -b) \). If \( r = 1 \) or \( 2 \), this genus can be enumerated a priori (see, for instance, Chapter 15 in [7]).

Remark 2.4.4. Another well-known way to enumerate lattices in a given genus is Kneser’s neighboring method [14]. This method has been implemented in \texttt{sage} by Brandhorst ([5] and private communication).

2.5. Primitive embeddings. Given an embedding of lattices \( \iota: M \hookrightarrow S \), we denote by \([i]\) its image and by \([i]^\perp\) the orthogonal complement of \([i]\) in \( S \). An embedding \( \iota: M \hookrightarrow S \) is called \textit{primitive} if \( S/[i] \) is a torsion-free group. All primitive embeddings are considered up to the action of \( O(M) \).

\textbf{Proposition 2.5.1} (Proposition 1.15.1 in [20]). If \( \iota: M \hookrightarrow S \) is a primitive embedding of even lattices, then there exist a subgroup \( H \subset M'/M \) and an isomorphism of finite quadratic forms \( \beta: q([i])|H \to q(S)/\beta(H) \) such that

\[
q([i]^\perp) \cong (\neg q([i])) \oplus q(S)|\Gamma_{\beta}^\perp/\Gamma_{\beta},
\]

where \( \Gamma_{\beta} \) is the push-out of \( \beta \) in \((-q([i])) \oplus q(S)).

Given a primitive embedding \( \iota: M \hookrightarrow S \), we put

\[
O(S, [i]) := \{ g \in O(S) \mid [i]^g = [i] \},
\]

and we denote by \( O(q(S), [i]) \) its image in \( O(q(S)) \) by the natural homomorphism \( O(S) \to O(q(S)) \).

Fix now two even lattices \( M, N \) and consider the set \( I(S, M, N) \) of primitive embeddings \( \iota: M \hookrightarrow S \) such that \([i]^\perp \cong N\). The group \( O(S) \) acts on \( I(S, M, N) \) in a natural way.

Consider also the set of pairs \((H, \gamma)\), where \( H \subset M'/M \) is a subgroup and \( \gamma: q(M)|H \to -q(N)\gamma(H) \) is an isomorphism of finite quadratic forms such that

\[
q(M) \oplus q(N)|\Gamma_{\gamma}^\perp/\Gamma_{\gamma} \cong q(S),
\]

where \( \Gamma_{\gamma} \) is the push-out of \( \gamma \) in \( q(M) \oplus q(N) \). We say that two such pairs \((H, \gamma)\) and \((H', \gamma')\) are \textit{equivalent} if there exist \( \varphi \in O(M) \) and \( \psi \in O(N) \) such that \( H^\varphi = H' \) and

\[
\gamma' \circ q(\varphi) = q(\psi) \circ \gamma.
\]

\textbf{Proposition 2.5.2} (Proposition 1.5.1 in [20]). In the above notation, there is a one-to-one correspondence between the elements of \( I(S, M, N) \) modulo the action of \( O(S) \) and the set of pairs \((H, \gamma)\) modulo equivalence.
Proposition 2.5.3 (Proposition 1.5.2 in [20]). For a fixed pair \((H, \gamma)\) corresponding to the orbit of a primitive embedding \(i: M \to S\), the subgroup \(O(q(S), [i])\) consists of those elements \(\xi \in O(q(S))\) for which there exist \(\varphi \in O(M)\) and \(\psi \in O(N)\) such that \(H^q(\varphi) = H\), equation (2.3) holds, and \(\xi\) corresponds under the isomorphism (2.2) to the automorphism induced by \(\varphi\) and \(\psi\) on \(\Gamma_{\gamma}/\Gamma_{\gamma}\).

2.6. Chambers and their faces. Let \(V\) be a \(\mathbb{Q}\)-vector space of dimension \(n > 1\) with a non-degenerate symmetric bilinear form \((,): V \times V \to \mathbb{Q}\) such that \(V \otimes \mathbb{R}\) is of signature \((1, n - 1)\). Let \(P_v\) be one of the two connected components of \(\{x \in V \otimes \mathbb{R} | \langle x, x \rangle > 0\}\). For \(v \in V\) with \(\langle v, v \rangle < 0\), we put
\[
(v)^\perp := \{x \in P_v | \langle x, v \rangle = 0\},
\]
which is a hyperplane of \(P_v\). For a set \(V\) of vectors \(v \in V\) with \(\langle v, v \rangle < 0\), we denote by \(V^\perp\) the family of hyperplanes \(\{v^\perp | v \in V\}\).

Let \(V\) be a set of vectors \(v \in V\) with \(\langle v, v \rangle < 0\) such that the family of hyperplanes \(V^\perp\) is locally finite. A \(V^\perp\)-chamber is the closure in \(P_v\) of a connected component of the complement
\[
P_v \setminus \bigcup_{H \in V^\perp} H.
\]
Let \(\overline{P}_v\) be the closure of \(P_v\) in \(V \otimes \mathbb{R}\), and \(\partial \overline{P}_v\) the boundary \(\overline{P}_v \setminus P_v\) of \(\overline{P}_v\). Let \(C\) be a \(V^\perp\)-chamber, and \(\overline{C}\) the closure of \(C\) in \(V \otimes \mathbb{R}\). We say that \(C\) is quasi-finite if \(\overline{C} \cap \partial \overline{P}_v\) is contained in a union of at most countably many real half-lines of \(V \otimes \mathbb{R}\).

Let \(C\) be a quasi-finite \(V^\perp\)-chamber. Suppose that we are given a set \(U_C\) of vectors \(v \in V\) with \(\langle v, v \rangle < 0\) such that
\[
C = \{x \in P_v | \langle x, v \rangle \geq 0 \text{ for all } v \in U_C\}.
\]
A wall of \(C\) is a closed subset \(w\) of \(C\) for which there exists a hyperplane \(H \in V^\perp\) with \(w = C \cap H\) such that \(w\) contains a non-empty open subset of \(H\). Let \(w\) be a wall of \(C\). A vector \(v \in V\) with \(\langle v, v \rangle < 0\) is said to define \(w\) if \(w\) is equal to \(C \cap (v)^\perp\) and \(\langle x, v \rangle > 0\) holds for all interior points \(x\) of \(C\). A vector \(v_0 \in U_C\) defines a wall of \(C\) if and only if there exists a point \(y \in P_v\) such that \(\langle y, v_0 \rangle < 0\) and that \(\langle y, v' \rangle > 0\) holds for all \(v' \in U_C\) with \(\langle v' \rangle \perp = \langle v_0 \rangle \perp\). Therefore, if \(U_C\) is finite, we can calculate the set of walls of \(C\) by means of linear programming.

A face is a closed subset of \(C\) that is the intersection of a finite number of walls of \(C\). Let \(f\) be a face of \(C\). We denote by \(\langle f \rangle\) the minimal linear subspace of \(V\) containing \(f\). The dimension of \(f\) is the dimension of \(\langle f \rangle\). Suppose that \(m := \dim f \geq 2\). Since \(f\) contains a non-empty open subset of \(\langle f \rangle\), the linear space \(\langle f \rangle\) contains a vector \(v\) with \(\langle v, v \rangle > 0\), and hence the restriction of \((,\)\) to \(\langle f \rangle\) is of signature \((1, m - 1)\). We denote by
\[
\iota_{\langle f \rangle}: \langle f \rangle \hookrightarrow V \text{ and } \text{pr}_{\langle f \rangle}: V \twoheadrightarrow \langle f \rangle
\]
the inclusion and the orthogonal projection, respectively, and let \(P_{\langle f \rangle}\) be the positive cone of \(\langle f \rangle\) that is mapped into \(P_v\) by \(\iota_{\langle f \rangle}\). We put
\[
\iota_{\langle f \rangle}^* V^\perp := \{\iota_{\langle f \rangle}^{-1}(H) | H \in V^\perp \text{ such that } \iota_{\langle f \rangle}^{-1}(H) \text{ is a hyperplane of } P_{\langle f \rangle}\},
\]
which is a locally finite family of hyperplanes of \(P_{\langle f \rangle}\). Note that \(\iota_{\langle f \rangle}^* V^\perp\) is equal to \((\text{pr}_{\langle f \rangle}^*)^* V^\perp\), where
\[
\text{pr}_{\langle f \rangle}^* V := \{\text{pr}_{\langle f \rangle}(v) | v \in V \text{ such that } \langle \text{pr}_{\langle f \rangle}(v), \text{pr}_{\langle f \rangle}(v) \rangle < 0\}.
\]
Then the face $f$ of $C$ is an $\iota^{\ast}_{(f)}V^\perp$-chamber in $\mathcal{P}_{(f)}$, and is equal to
\[
\{ z \in \mathcal{P}_{(f)} \mid \langle z, \text{pr}_{(f)}(v) \rangle \geq 0 \text{ for all } v \in U_C \text{ with } \langle \text{pr}_{(f)}(v), \text{pr}_{(f)}(v) \rangle < 0 \}.
\]
Therefore, if $U_C$ is finite, we can calculate the set of walls of the $\iota^{\ast}_{(f)}V^\perp$-chamber $f$, and hence we can calculate the set of all faces of $C$ by descending induction on the dimension of faces.

Let $w$ be a wall of $C$. Then there exists a unique $V^\perp$-chamber $C'$ such that $C \cap C' = w$. This $V^\perp$-chamber $C'$ is said to be adjacent to $C$ across the wall $w$.

### 2.7. Induced chambers

Let $L$ be an even hyperbolic lattice. We apply the above definitions to $L \otimes \mathbb{Q}$. Let $\mathcal{P}_L$ be a positive cone of $L$, and let $\mathcal{V}$ be a set of vectors $v \in L \otimes \mathbb{Q}$ with $\langle v, v \rangle < 0$ such that the family $V^\perp$ of hyperplanes of $\mathcal{P}_L$ is locally finite. Suppose that we have a primitive embedding
\[
\iota_S : S \hookrightarrow L
\]
of an even hyperbolic lattice $S$ of rank $m < n$, and let $\mathcal{P}_S$ be the positive cone of $S$ that is mapped into $\mathcal{P}_L$ by $\iota_S$. We use the same letter $\iota_S$ to denote the inclusion $\mathcal{P}_S \hookrightarrow \mathcal{P}_L$. We denote the orthogonal projection by $\text{pr}_S : L \otimes \mathbb{Q} \to S \otimes \mathbb{Q}$, and put
\[
\iota^{-1}_S V^\perp := \{ \iota^{-1}_S (H) \mid H \in V^\perp \text{ such that } \iota^{-1}_S (H) \text{ is a hyperplane of } \mathcal{P}_S \},
\]
\[
\text{pr}^{-1}_S V := \{ \text{pr}_S (v) \mid v \in V \text{ with } \langle \text{pr}_S (v), \text{pr}_S (v) \rangle < 0 \}.
\]
Then $\iota^{-1}_S V^\perp = (\text{pr}^{-1}_S V)^\perp$ is a locally finite family of hyperplanes of $\mathcal{P}_S$. A $V^\perp$-chamber $C \subset \mathcal{P}_L$ is said to be non-degenerate with respect to $\iota_S$ if the closed subset $\iota^{-1}_S (C)$ of $\mathcal{P}_S$ contains a non-empty open subset of $\mathcal{P}_S$. Suppose that $C$ is non-degenerate with respect to $\iota_S$. Then $\iota^{-1}_S (C)$ is an $\iota^{-1}_S V^\perp$-chamber, which we call the chamber induced by $C$. If $C$ is quasi-finite, then so is the induced chamber $\iota^{-1}_S (C)$.

### 2.8. Vinberg chambers and Conway chambers

Let $L$ be as above. Note that the family $\mathcal{R}_L^\perp$ of hyperplanes is locally finite, where $\mathcal{R}_L$ is the set of $(2)$-vectors. Each $r \in \mathcal{R}_L$ defines a reflection $x \mapsto x + \langle x, r \rangle r$. Let $W(L)$ be the subgroup of $O(L, \mathcal{P}_L)$ generated by reflections with respect to $(2)$-vectors. Then each $\mathcal{R}_L^\perp$-chamber is a standard fundamental domain of the action of $W(L)$ on $\mathcal{P}_L$.

For $n = 10$ and $n = 26$, let $L_n$ be an even unimodular hyperbolic lattice of rank $n$, which is unique up to isomorphism. We denote by $\mathcal{P}_n$ a positive cone of $L_n \otimes \mathbb{R}$, and by $\mathcal{R}_n$ the set of $(2)$-vectors of $L_n$.

An $\mathcal{R}_{10}^\perp$-chamber in $\mathcal{P}_{10}$ is called a Vinberg chamber. It is known that a Vinberg chamber is quasi-finite.

**Theorem 2.8.1** (Vinberg [36]). A Vinberg chamber has exactly 10 walls.

An $\mathcal{R}_{26}^\perp$-chamber in $\mathcal{P}_{26}$ is called a Conway chamber. It is known that a Conway chamber is quasi-finite. A non-zero primitive vector $w \in L_{26} \cap \partial \mathcal{P}_{26}$ is called a Weyl vector if the negative definite lattice $[w]^-/[w]$ is isomorphic to the negative definite Leech lattice, where $[w]^- := \{ v \in L_{26} \mid \langle v, w \rangle = 0 \}$.

**Theorem 2.8.2** (Conway [36]). For each Conway chamber $C$, there exists a unique Weyl vector $w_C$ such that the walls of $C$ are defined by $(2)$-vectors $r \in \mathcal{R}_{26}$ satisfying $\langle w, r \rangle = 1$. 

2.9. **Primitive embeddings of $L_{10}(2)$ into $L_{26}$**. In [6], we classified all primitive embeddings of $L_{10}(2)$ into $L_{26}$. It turns out that, up to the action of $O(L_{10}(2)) = O(L_{10})$ and $O(L_{26})$, there exist exactly 17 primitive embeddings, which are named as being of type $12A, 12B, 20A, \ldots, 20A, 20F, 40A, \ldots, 40E, 96A, \ldots, 96C, \text{infty}$.

Let $\iota: L_{10}(2) \hookrightarrow L_{26}$ be a primitive embedding. Identifying positive cones of $L_{10}(2)$ with positive cones of $L_{10}$ and replacing $\iota$ with $-\iota$ if necessary, we assume that $\iota$ maps $P_{10}$ into $P_{26}$. Then $P_{10}$ is covered by $\iota^* R_{26}^\perp$-chambers. Since Conway chambers are quasi-finite, every $\iota^* R_{26}^\perp$-chambers are quasi-finite. In [6], we have proved the following:

**Theorem 2.9.1.** Suppose that $\iota$ is not of type $\text{infty}$. Let $D$ and $D'$ be $\iota^* R_{26}^\perp$-chambers. Then there exists an isometry $g \in O^+(L_{10})$ that preserves the set of $\iota^* R_{26}^\perp$-chambers and maps $D$ to $D'$. Each $\iota^* R_{26}^\perp$-chamber has only a finite number of walls, and each wall is defined by a $(-2)$-vector. If $D \cap (r)^\perp$ is a wall of $D$ with $r \in R_{10}$, then the $\iota^* R_{26}^\perp$-chamber adjacent to $D$ across the wall $D \cap (r)^\perp$ is the image of the reflection of $D$ into the hyperplane $(r)^\perp$.

**Remark 2.9.2.** If a primitive embedding $\iota: L_{10}(2) \hookrightarrow L_{26}$ is of type $\text{infty}$, then the $\iota^* R_{26}^\perp$-chamber has infinitely many walls. The embedding $\iota$ is of type $\text{infty}$ if and only if $[\iota]^\perp$ contains no $(-2)$-vectors.

Let $Y$ be an Enriques surface. Then the Néron-Severi lattice $S_Y$ is isomorphic to $L_{10}$. It is known that the nef chamber $N_Y$ is bounded by hyperplanes $(r)^\perp$ defined by $(-2)$-vectors $r \in R_Y$. In [6], we have proved the following:

**Theorem 2.9.3.** Let $[\sigma, \tau]$ be one of the pairs $[12A, I], [12B, II], [20A, V], [20B, III], [20C, VII], [20D, VII], [20E, VI], [20F, IV]$. Then every $\iota^* R_{26}^\perp$-chamber $D$ for a primitive embedding $\iota: L_{10}(2) \hookrightarrow L_{26}$ of type $\sigma$ is equal to the nef chamber $N_Y$ of an Enriques surface $Y$ with finite automorphism group of Nikulin-Kondo type $\tau$ under an isomorphism $L_{10} \cong S_Y$. ■

2.10. **K3 surfaces**. Let $X$ be a complex projective K3 surface with transcendental lattice $T_X$. Then the nef chamber $N_X$ is an $R_X^\perp$-chamber, and each wall of $N_X$ is defined by the class of a smooth rational curve on $X$. We put

$$O(S_X, N_X) := \{ g \in O(S_X) \mid N_X^g = N_X \}.$$ 

Recall that $W_X := W(S_X)$ is the subgroup of $O(S_X, P_X)$ generated by reflections with respect to $(-2)$-vectors. The following relations hold (see [22]):

$$O(S_X, P_X) = W_X \rtimes O(S_X, N_X),$$

$$W_X \subset \ker(O(S_X) \to O(q(S_X))).$$

Let $O(T_X, \omega_X)$ be the group of isometries of $T_X$ that preserves the 1-dimensional subspace $H^{2,0}(X) \subset T_X \otimes \mathbb{C}$, and let $O(q(T_X), \omega_X)$ be the image of $O(T_X, \omega_X)$ by the natural homomorphism $O(T_X) \to O(q(T_X))$. The even unimodular overlattice $H^2(X, \mathbb{Z})$ of the orthogonal direct sum $S_X \oplus T_X$ induces an anti-isometry between the discriminant forms of $S_X$ and of $T_X$ (see [20]), and hence induces an isomorphism $O(q(S_X)) \cong O(q(T_X))$. Let $O(q(S_X), \omega_X)$ be the image of $O(q(T_X), \omega_X)$ through this isomorphism. We say that an isometry $g \in O(S_X)$ satisfies the period condition if $q(g) \in O(q(S_X), \omega_X)$. Let $O(S_X, \omega_X)$ denote the group of isometries
satisfying the period condition. Recall that \( \text{aut}(X) \subset O(S_X, P_X) \) is the image of \( \text{Aut}(X) \) by (2.1). The Torelli theorem for complex K3 surfaces asserts that
\[
\text{aut}(X) = O(S_X, N_X) \cap O(S_X, \omega_X).
\]
In particular, if \( g \in O(S_X, \omega_X) \) maps an interior point of \( N_X \) to an interior point of \( N_X \), then \( g \) belongs to \( \text{aut}(X) \).

**Remark 2.10.1.** By the Torelli theorem, the kernel of \( \rho_X : \text{Aut}(X) \to O(S_X) \) is isomorphic to the kernel of the natural homomorphism \( O(T_X, \omega_X) \to O(q(T_X)) \).

2.11. **Singular K3 surfaces.** Let \( X \) be a singular K3 surface. Its transcendental lattice \( T_X \) admits a basis with respect to which the Gram matrix is of the form
\[
[a, b, c] := \begin{bmatrix} a & b \\ b & c \end{bmatrix},
\]
with \( 0 \leq 2b \leq a \leq c \). We write \( X(T) \) for the K3 surface corresponding to an oriented positive definite even lattice \( T \) of rank 2. The lattice \( T = [a, -b, c] \) defines a distinct oriented isomorphism class if and only if \( 0 < 2b < a < c \).

**Remark 2.11.1.** If \( X \) is a singular K3 surface, the subgroup \( O(T_X, \omega_X) \) can be identified with the subgroup consisting of isometries of \( T_X \) of positive determinant. Its image \( O(q(T_X), \omega_X) \) depends only on the genus of \( T_X \).

3. **Classification of Enriques involutions up to conjugation**

Let \( X \) be a complex projective K3 surface. We are interested in classifying the images \( \varepsilon \) of Enriques involutions \( \tilde{\varepsilon} \) in \( \text{aut}(X) \) through the natural representation (2.1) up to conjugation in \( \text{aut}(X) \). The image \( \varepsilon \in \text{aut}(X) \) is also call an Enriques involution. This is essentially the same problem by the following observation due to Ohashi.

**Proposition 3.0.1 (Ohashi [22]).** Let \( \tilde{\varepsilon}_1, \tilde{\varepsilon}_2 : X \to X \) be two Enriques involutions. Then the quotients \( Y_i := X/\langle \tilde{\varepsilon}_i \rangle \), \( i = 1, 2 \), are isomorphic over \( \mathbb{C} \) if and only if \( \varepsilon_1, \varepsilon_2 \) are conjugate in \( \text{aut}(X) \).

In this section, after recalling part of Ohashi’s work, we refine and generalize his main Theorem 2.3 in [22].

3.1. **Main result.** Given an Enriques involution \( \varepsilon \in \text{aut}(X) \), we put
\[
S_X^{\varepsilon^{-1}} := \{ v \in S_X \mid v^\varepsilon = v \}.
\]
We have the following criterion by Keum.

**Theorem 3.1.1 (Keum [12]).** An involution \( \varepsilon \in \text{aut}(X) \) is an Enriques involution if and only if the following holds: the sublattice \( S_X^{\varepsilon^{-1}} \) is isomorphic to \( L_{10}(2) \) and its orthogonal complement in \( S_X \) contains no \((-2)\)-vectors.

Let \( I_X \) be the set of primitive embeddings \( \iota : L_{10}(2) \hookrightarrow S_X \) such that the orthogonal complement \( [\iota] \) of the image of \( \iota \) in \( S_X \) contains no \((-2)\)-vectors. The group \( O(S_X) \) acts on \( I_X \) in a natural way.

**Proposition 3.1.2 (Proposition 2.2 in [22]).** For every \( \iota \in I_X \) and \( g \in O(S_X) \) such that \( [\iota]^q \) intersects the interior of \( N_X \), there exists a unique \( \varepsilon \in \text{aut}(X) \) such that \( S_X^{\varepsilon^{-1}} = [\iota]^q \).
Corollary 3.1.3. Let $\varepsilon_1, \varepsilon_2 \in \text{aut}(X)$ be two Enriques involutions. Then, there exists $\gamma \in \text{aut}(X)$ such that $\varepsilon_2 = \gamma \circ \varepsilon_1 \circ \gamma^{-1}$ if and only if $(S_X^{\varepsilon_1=1}) \gamma = S_X^{\varepsilon_2=1}$. 

Proposition 3.1.4 (Step 1 of Theorem 2.3 in [22]). For every $i \in I_X$ there exists $h \in O(S_X)$ such that $[i]^h$ intersects the interior of $N_X$. 

Lemma 3.1.5 (Step 2 of Theorem 2.3 in [22]). Suppose $[i]$ intersects the interior of $N_X$. If there exist an Enriques involution $\varepsilon \in \text{aut}(X)$ and $g \in O(S_X)$ such that $S_X^{\varepsilon=1} = [i]^g$, then there exists $\tilde{g} \in O(S_X, N_X)$ such that $S_X^{\varepsilon=1} = [\tilde{g}]^g$. 

Proposition 3.1.6. Given $i \in I_X$, let $\varepsilon_1, \varepsilon_2 \in \text{aut}(X)$ be two Enriques involutions with $S_X^{\varepsilon_1=1} = [i]^{g_1}$ and $S_X^{\varepsilon_2=1} = [i]^{g_2}$ for some $g_1, g_2 \in O(S_X, N_X)$. Then the Enriques involutions $\varepsilon_1$ and $\varepsilon_2$ are conjugate in $\text{aut}(X)$ if and only if the natural images $q(g_1), q(g_2) \in O(q(S_X))$ belong to the same double coset with respect to $O(q(S_X), [i])$ and $O(q(S_X), \omega_X)$. 

Proof. Let $\ell_i := g_i \circ i$ for $i = 1, 2$. Suppose there exists $\gamma \in \text{aut}(X)$ with $\varepsilon_2 = \gamma \circ \varepsilon_1 \circ \gamma^{-1}$. Let $\varphi := g_2^{-1} \circ \gamma \circ g_1$, so that $\varphi \in O(S_X, [i])$. Indeed, by Corollary 3.1.3, 

$$[\varphi]^2 = [\gamma \circ \ell_i]^{g_2^{-1}} = [\varepsilon_2]^{g_2^{-1}} = [i].$$ 

As $g_1 = \varphi \circ g_2 \circ \gamma^{-1}$ and $\gamma \in O(S_X, \omega_X)$, the automorphisms $q(g_1), q(g_2)$ of $q(S_X)$ belong to the same double coset.

Conversely, assume that there exist $\varphi \in O(S_X, [i])$ and $\gamma' \in O(S_X, \omega_X)$ such that $q(g_2) = q(\varphi \circ g_1 \circ \gamma')$ in $O(q(S_X))$. Without loss of generality, we can suppose $\varphi \in O(S_X, N_X)$. In fact, we can first exchange $\varphi$ with $-\varphi$ if necessary and suppose that $\varphi \in O(S_X, \mathcal{P}_X)$. By (2.4) and (2.5), we can write $\varphi = w \circ \varphi'$, with $w \in W_X$ and $\varphi' \in O(S_X, K_X)$ and exchange $\varphi$ with $\varphi'$ if necessary. Define now $\gamma := g_2 \circ \gamma^{-1} \circ g_1^{-1}$. Then $\gamma \in O(S_X, N_X)$ and $q(\gamma) = q(\gamma')$, so $\gamma \in O(S_X, \omega_X)$. The Torelli Theorem (2.6) implies that $\gamma \in \text{aut}(X)$. Furthermore, we have 

$$[\ell_i]^{\gamma} = ([i]^{\varphi^{-1}})^{g_2} = [\varepsilon_2],$$

so $\varepsilon_1$ and $\varepsilon_2$ are conjugate in $\text{aut}(X)$ by Corollary 3.1.3. 

Lemma 3.1.7. If a K3 surface $X$ admits at least one Enriques involution, then the lattice $S_X$ is unique in its genus and the natural homomorphism $O(S_X) \rightarrow O(q(S_X))$ is surjective. 

Proof. Let $\iota : L_{10}(2) \rightarrow S_X$ be a primitive embedding. Then $q(S_X) \cong (q([i]) \oplus q([\iota^-])\Gamma^\perp/\Gamma$ for some isotropic subgroup $\Gamma$ of $q([i]) \oplus q([\iota^-])$. Since $q([i]) \cong q(L_{10}(2)) \cong u_1^{\oplus 5}$, this implies that 

$$\ell_p(S_X/\mathbb{Q}) \leq \text{rank}[i]^{\perp} = \text{rank} S_X - 10$$

for every odd prime $p$. Moreover, if $\ell_2(S_X/\mathbb{Q}) = \text{rank} S_X$, then $q(S_X) = q([i]) \oplus q'$. Therefore, we can conclude by Theorem 1.14.2 in [20]. 

Combining Lemma 3.1.7 and the same argument as in Step 5 of Theorem 2.3 in [22], we prove the following proposition.

Proposition 3.1.8. If a K3 surface $X$ admits at least one Enriques involution, then $O(S_X, N_X) \rightarrow O(q(S_X))$ is surjective. 

Our main result is the following theorem.
Theorem 3.1.9. Let $X$ be a K3 surface and $\iota_1, \ldots, \iota_r \in I_X$ be a complete set of representatives for the action of $O(S_X)$ on $I_X$. Then there exists a bijection between the set of Enriques involutions up to conjugation in $\text{aut}(X)$ and the disjoint union of the sets of double cosets

$$O(q(S_X), [\iota_i]) \backslash O(q(S_X))/O(q(S_X), \omega_X), \quad i = 1, \ldots, r.$$ 

Proof. Let $G = O(S_X)$, $H_i = O(q(S_X), [\iota_i])$ and $K = O(q(S_X), \omega_X)$. For each $i = 1, \ldots, r$, fix $h_i \in G$ such that $[\iota_i]^{h_i}$ intersects the interior of $N_X$ (Proposition 3.1.4). As exchanging $\iota_i$ with $h_i \circ \iota$ replaces $H_i$ with a conjugate subgroup, we can suppose without loss of generality that $[\iota_i]$ intersects the interior of $N_X$. For each Enriques involution $\varepsilon \in \text{aut}(X)$ there exists a unique $i \in \{1, \ldots, r\}$ such that there exists $g \in G$ with $S_X^{\varepsilon^{-1}} = [\iota_i]^{g}$. Moreover, by Lemma 3.1.5, we can suppose that $g \in O(S_X, N_X)$. We map such an $\varepsilon$ to the double coset $H_i q(g) K \subset H_i \backslash G/K$. This function is trivially well-defined and injective by Proposition 3.1.6.

To show surjectivity, take $i \in \{1, \ldots, r\}$ and $H_i \xi K \subset H_i \backslash G/K$, with $\xi \in G$. By Proposition 3.1.8, $\xi = q(g)$ for some $g \in O(S_X, N_X)$. As $[\iota_i]^{g}$ also intersects the interior of $N_X$, by Proposition 3.2.1 there is an Enriques involution $\varepsilon \in \text{aut}(X)$ which maps to $H_i \xi K$. This concludes the proof. \hfill $\Box$

Corollary 3.1.10. The number of Enriques involutions of a singular K3 surface $X$ up to conjugation in $\text{aut}(X)$ only depends on the genus of the transcendental lattice $T_X$.

Proof. The lattice $S_X$ is unique in its genus by Lemma 3.1.7, so it is completely determined by the genus of $T_X$. The subgroup $O(q(S_X), \omega_X)$ is also determined by the genus of $T_X$ when $X$ is singular (see Remark 2.11.1). The subgroups $O(q(S_X), [\iota])$ for $\iota \in I_X$ only depend on $S_X$, so in turn they depend only on the genus of $T_X$. \hfill $\Box$

Remark 3.1.11. Schütt [24] described a relation of two singular K3 surfaces whose transcendental lattices are in the same genus. See also [26].

3.2. Table 3.1. Table 3.1 contains the list of all singular K3 surfaces $X$ of discriminant $d$ with $d \leq 36$, given by their respective transcendental lattices $T_X$, together with the list of the Enriques involutions that they admit, up to conjugation in $\text{aut}(X)$. We will illustrate presently how this table was compiled.

The following theorem by Sertöz builds on work by Keum [12] and characterizes singular K3 surfaces without Enriques quotients.

Theorem 3.2.1 (Sertöz [25]; see also [11]). Let $X$ be a singular K3 surface of discriminant $d$. Then $X$ has no Enriques involution if and only if $d \equiv 3(8)$ or $T_X \in \{[2, 0, 2], [2, 0, 4], [2, 0, 8]\}$. \hfill $\blacksquare$

In all other cases, we determined the set of conjugacy classes of all Enriques involutions in $\text{aut}(X)$ by means of Theorem 3.1.9. The item $|\text{Enr}|$ in Table 3.1 indicates the number of such conjugacy classes.

First of all, one must determine a complete set of representatives for the action of $O(S_X)$ on $I_X$. Given a positive definite even lattice $N$ of rank 10 without 2-vectors (see Theorem 3.1.1), we put

$$I_X(N) := \{ \iota \in I_X \mid [\iota] \perp N(-1) \}.$$ 

Clearly, the sets $I_X(N)$ form a partition of $I_X$ which respects the $O(S_X)$-action, so we reduce the problem to computing a complete set of representatives for the action of $O(S_X)$ on $I_X(N)$, for each $N$ such that $I_X(N) \neq \emptyset$. 

### Table 3.1. Enriques involutions up to conjugation of singular K3 surfaces of discriminant \( d \leq 36 \) (see Section 3.2).

| \( d \) | \( T_X \) | \(|\text{Enr}|\) | \( q(N) \) | \( N \) | \( |I_X(N)| \) |
|-------|------|------|------|------|------|
| 3     | \([2, 1, 2]\) | 0    | –    | –    | –    |
| 4     | \([2, 0, 2]\) | 0    | –    | –    | –    |
| 7     | \([2, 1, 4]\) | 2    | \(u_1^{\mathbb{G}_5} \oplus \left\langle \frac{2}{3} \right\rangle\) | \(N_{144}^{10,15}(2)\) | 1 |
| 8     | \([2, 0, 4]\) | 0    | –    | –    | –    |
| 11    | \([2, 1, 6]\) | 0    | –    | –    | –    |
| 12    | \([2, 0, 6]\) | 1    | \(u_1^{\mathbb{G}_4} \oplus \left\langle \frac{1}{2} \right\rangle \oplus \left\langle \frac{1}{4} \right\rangle\) | \(M_{144}^{10,14}(2)\) | 1 |
| 12    | \([4, 2, 4]\) | 3    | \(u_1^{\mathbb{G}_4} \oplus v_1 \oplus \left\langle \frac{1}{3} \right\rangle\) | \(N_{244}^{10,9}(2)\) | 3 \(\times\) 1 |
| 15    | \([2, 1, 8]\) | 5    | \(u_1^{\mathbb{G}_5} \oplus \left\langle \frac{2}{15} \right\rangle\) | \(N_{90}^{90,10,15}(2)\) | 1 |
| 15    | \([4, 1, 4]\) | 4    | \(u_1^{\mathbb{G}_5} \oplus \left\langle \frac{4}{15} \right\rangle\) | \(N_{112}^{90,10,15}(2)\) | 1 |
| 16    | \([2, 0, 8]\) | 0    | –    | –    | –    |
| 16    | \([4, 0, 4]\) | 9    | \(u_1^{\mathbb{G}_4} \oplus \left\langle \frac{1}{1} \right\rangle \oplus \left\langle \frac{1}{4} \right\rangle\) | \(D_{10}^{10,4}(2)\) | 3 \(\times\) 1 |
| 19    | \([2, 1, 10]\) | 0    | –    | –    | –    |
| 20    | \([2, 0, 10]\) | 1    | \(u_1^{\mathbb{G}_4} \oplus \left\langle \frac{1}{2} \right\rangle \oplus \left\langle \frac{1}{10} \right\rangle\) | \(M_{132}^{10,13}(2)\) | 1 |
| 20    | \([4, 2, 6]\) | 2    | \(u_1^{\mathbb{G}_4} \oplus \left\langle \frac{4}{3} \right\rangle \oplus \left\langle \frac{1}{11} \right\rangle\) | \(M_{92}^{10,5,15}(2)\) | 1 |
| 23    | \([2, 1, 12]\), \([4, \pm 1, 6]\) | 7    | \(u_1^{\mathbb{G}_5} \oplus \left\langle \frac{2}{23} \right\rangle\) | \(N_{74}^{10,23}(2)\) | 1 |
| 24    | \([2, 0, 12]\) | 1    | \(u_1^{\mathbb{G}_4} \oplus \left\langle \frac{1}{2} \right\rangle \oplus \left\langle \frac{1}{11} \right\rangle\) | \(M_{90}^{10,6}(2)\) | 1 |
| 24    | \([4, 0, 6]\) | 1    | \(u_1^{\mathbb{G}_4} \oplus \left\langle \frac{3}{2} \right\rangle \oplus \left\langle \frac{11}{12} \right\rangle\) | \(M_{242}^{10,5}(2)\) | 1 |
| 27    | \([2, 1, 14]\) | 0    | –    | –    | –    |

Continued on next page
Table 3.1 – continued from previous page

| $d$  | $T$           | $|\text{Enr}|$ | $q(N)$  | $N$ | $|\text{IX}(N)|$ |
|------|---------------|----------------|---------|-----|-----------------|
| 27   | $[6, 3, 6]$   | 0              | –       | –   | –               |
| 28   | $[2, 0, 14]$  | 1 $u_1^{\oplus 4} \oplus \langle \frac{1}{2} \rangle \oplus \langle \frac{1}{14} \rangle$ | $M_{10,1}^{114}(2)$ | 1   |
| 28   | $[4, 2, 8]$   | 24 $u_1^{\oplus 5} \oplus \langle \frac{2}{7} \rangle$ | $N_{10,2}^{144}(2)$ $N_{242,1}^{24}(2)$ | $3 \times 1 + 4 \times 2$ $4 \times 1 + 4 \times 2$ |
|      |               |                | $u_1^{\oplus 4} \oplus \langle \frac{2}{7} \rangle$ | $N_{10,1792}^{0.274}$ | 1   |
| 31   | $[2, 1, 16]$, $[4, \pm 1, 8]$ | 9 $u_1^{\oplus 5} \oplus \langle \frac{4}{29} \rangle$ | $N_{10,31}^{60}(2)$ $N_{10,31}^{72}(2)$ $N_{10,31}^{86}(2)$ $N_{10,31}^{90}(2)$ $N_{10,31}^{112}(2)$ $N_{10,31}^{128}(2)$ $N_{10,31}^{144}(2)$ $N_{10,31}^{230}(2)$ $N_{10,31}^{242}(2)$ | 1   |
| 32   | $[2, 0, 16]$  | 1 $u_1^{\oplus 4} \oplus \langle \frac{1}{2} \rangle \oplus \langle \frac{1}{16} \rangle$ | $M_{10,8}^{110}(2)$ | 1   |
| 32   | $[4, 0, 8]$   | 33 $u_1^{\oplus 4} \oplus \langle \frac{1}{4} \rangle \oplus \langle \frac{1}{18} \rangle$ | $N_{10,8}^{138}(2)$ $N_{10,8}^{144}(2)$ $N_{10,8}^{144}(2)$ $N_{10,8}^{144}(2)$ | $2 \times 1 + 4 \times 2$ $3 \times 1 + 2 \times 2$ $3 \times 1 + 5 \times 2$ |
|      |               |                | $u_1^{\oplus 3} \oplus \langle \frac{1}{4} \rangle \oplus \langle \frac{1}{8} \rangle$ | $N_{10,2048}^{0.210}$ $N_{10,2048}^{0.210}$ $N_{10,2048}^{0.210}$ $N_{10,2048}^{0.210}$ | 1   |
| 32   | $[6, 2, 6]$   | 3 $u_1^{\oplus 4} \oplus \langle \frac{3}{7} \rangle \oplus \langle \frac{3}{15} \rangle$ | $M_{10,8}^{112}(2)$ $M_{10,8}^{114}(2)$ $M_{10,8}^{114}(2)$ $M_{10,8}^{114}(2)$ | 1   |
| 35   | $[2, 1, 18]$  | 0              | –       | –   | –               |
| 35   | $[6, 1, 6]$   | 0              | –       | –   | –               |
| 36   | $[2, 0, 18]$  | 3 $u_1^{\oplus 4} \oplus \langle \frac{1}{2} \rangle \oplus \langle \frac{1}{10} \rangle$ | $M_{10,9}^{74}(2)$ $M_{10,9}^{70}(2)$ $M_{10,9}^{70}(2)$ $M_{10,9}^{70}(2)$ | 1   |
| 36   | $[4, 2, 10]$  | 2 $u_1^{\oplus 4} \oplus \langle \frac{1}{2} \rangle \oplus \langle \frac{4}{10} \rangle$ | $M_{10,9}^{80}(2)$ $M_{10,9}^{80}(2)$ $M_{10,9}^{80}(2)$ $M_{10,9}^{80}(2)$ | 1   |
| 36   | $[6, 0, 6]$   | 3 $u_1^{\oplus 4} \oplus \langle \frac{1}{6} \rangle \oplus \langle \frac{1}{8} \rangle$ | $M_{10,9}^{60}(2)$ $M_{10,9}^{60}(2)$ $M_{10,9}^{60}(2)$ $M_{10,9}^{60}(2)$ | 1   |
We find all such lattices in the following way. Using Proposition 2.5.1, we list all possible finite quadratic forms $q$, such that $q \cong q(N)$. For each form $q$, we determine all lattices $N$ in the genus $g(10,0,q)$ without 2-vectors (see Algorithm 2.4.3).

All possible finite quadratic forms $q = q(N)$ and orthogonal complements $N$ have been listed in Table 3.1 in the items $q(N)$ and $N$. The name $N_{r,d}^{p_2,p_4}$ (resp. $M_{r,d}^{p_2,p_4}$) denotes a positive definite even (resp. odd) lattice of rank $r$, determinant $d$, with $p_2$ 2-vectors and $p_4$ 4-vectors ($p_4$ omitted if not needed to distinguish two lattices). A Gram matrix for each of these lattices can be found in [31].

Since $I_X(N) = I(S_X, L_{10}(2), N(-1))$ as defined in Section 2.5, a complete set of representatives $\iota_1, \ldots, \iota_r$ up to the action of $O(S_X)$ on $I_X(N)$ can be enumerated using Proposition 2.5.2. For each $i \in \{1, \ldots, r\}$, the subgroup $H_i = O(q(S_X), [\iota_i])$ of $G = O(q(S_X))$ can be determined using Proposition 2.5.3. On the other hand, the subgroup $K = O(q(S_X), \omega_X)$ can be computed using Remark 2.1.1.1.

**Remark 3.2.2.** In order to apply Proposition 2.5.2, it is worth mentioning that for $L = L_{10}(2)$ the natural homomorphism $O(L) \to O(q(L))$ is surjective and that, up to the action of $O(q(L))$, there are only two subgroups of $L^\perp/L$ of order 2.

On the other hand, since $N$ is positive definite, we can compute $O(N)$ by the attribute automorphism group of the class QuadraticForm in sage; hence, we can compute its image in $O(q(N))$.

The item $|I_X(N)|$ gives the cardinalities of the sets of double cosets $H_i \backslash G/K$. For instance, the entry “$3 \times 1 + 4 \times 2$” means that $r = 7$, $|H_i \backslash G/K| = 1$ for $i = 1, 2, 3$ and $|H_i \backslash G/K| = 2$ for $i = 4, \ldots, 7$. Note that the item $|\text{Enr}|$ is the sum of the items $|I_X(N)|$ over the lattices $N$.

4. Automorphism groups of singular K3 surfaces

4.1. Borcherds method. We explain Borcherds method ([3], [4]) to calculate $\text{Aut}(X)$ of a K3 surface $X$ and its action on $N_X$. The details of the algorithms in the computation below are explained in [27]. Suppose that we have a primitive embedding

$$\iota_X : S_X \hookrightarrow L_{26}.$$ 

We assume that $\iota_X$ maps $P_X$ to the positive cone $P_{26}$ of $L_{26}$, and consider the decomposition of $P_X$ by $\iota_X^* R_{26}^\perp$-chambers, that is, by chambers induced by Conway chambers non-degenerate with respect to $\iota_X$. Since $\iota_X$ maps $R_X$ to $R_{26}$, every $R_{26}^\perp$-chamber is a union of $\iota_X^* R_{26}^\perp$-chambers. In particular, the nef chamber $N_X$ is a union of $\iota_X^* R_{26}^\perp$-chambers. Since a Conway chamber is quasi-finite, every $\iota_X^* R_{26}^\perp$-chamber is quasi-finite.

The orthogonal complement $[\iota_X]^\perp$ of the image of $\iota_X$ is an even negative definite lattice. The even unimodular overlattice $L_{26}$ of $S_X \oplus [\iota_X]^\perp$ induces an anti-isometry $q(S_X) \cong -q([\iota_X]^\perp)$, and hence an isomorphism $O(q(S_X)) \cong O(q([\iota_X]^\perp))$. We assume the following condition:

- the image of $O(q(S_X), \omega_X)$ by the isomorphism $O(q(S_X)) \cong O(q([\iota_X]^\perp))$ above is contained in the image of the natural homomorphism $O([\iota_X]^\perp) \to O(q([\iota_X]^\perp))$.

Since $O([\iota_X]^\perp)$ and $O(q(S_X), \omega_X)$ are finite, we can determine whether this condition is fulfilled or not. Suppose that Condition (A) is satisfied. Then every isometry
$g \in O(S_X, \omega_X) \cap O(S_X, \mathcal{P}_X)$ extends to an isometry $\tilde{g} \in O(L_{26}, \mathcal{P}_{26})$, which preserves the set of Conway chambers. Therefore every isometry of $S_X$ satisfying the period condition preserves the set of Conway chambers. Therefore every isometry of $S_X$ satisfying the period condition preserves the set of $\mathcal{R}^{\perp}_{26}$-chambers.

We also assume the following condition:

(B) \quad $[\iota_X]^{\perp}$ cannot be embedded into the negative definite Leech lattice.

For example, if $[\iota_X]^{\perp}$ contains a $(-2)$-vector, then this condition is fulfilled. Condition (B) implies that each $\mathcal{R}^{\perp}_{26}$-chamber $D$ in $\mathcal{P}_X$ has only a finite number of walls (see [27]). More precisely, if $D$ is induced by a Conway chamber $C$, then the set of vectors defining walls of $D$ can be calculated from the Weyl vector $w_C$ corresponding to $C$ by Theorem 2.8.2. By this finiteness, we can calculate, for two $\mathcal{R}^{\perp}_{26}$-chambers $D$ and $D'$, the set of all isometries $g \in O(S_X)$ such that $D^g = D'$. In particular, the group

$$O(S_X, D) := \{ g \in O(S_X) | D^g = D \}$$

is finite, and can be calculated explicitly. If $D \subset N_X$, then

$$\text{aut}(X, D) := O(S_X, D) \cap O(S_X, \omega_X)$$

is contained in $\text{aut}(X)$, and can be calculated explicitly.

**Definition 4.1.1.** Let $D$ be an $\mathcal{R}^{\perp}_{26}$-chamber contained in $N_X$. A wall $D \cap (v)^{\perp}$ of $D$ is called an outer wall if it is defined by a $(-2)$-vector, that is, if there exists a rational number $\lambda$ such that $-2/(v, v) = \lambda^2$ and $\lambda v \in S_X$. Otherwise, we say that $D \cap (v)^{\perp}$ is an inner wall.

A wall $D \cap (v)^{\perp}$ is an outer wall if and only if $N_X \cap (v)^{\perp}$ is a wall of $N_X$. The $\mathcal{R}^{\perp}_{26}$-chamber $D'$ adjacent to $D$ across a wall $D \cap (v)^{\perp}$ of $D$ is contained in $N_X$ if and only if $D \cap (v)^{\perp}$ is an inner wall.

Let $D$ be an $\mathcal{R}^{\perp}_{26}$-chamber, and let $w_C$ be the Weyl vector corresponding to a Conway chamber $C$ inducing $D = \iota_X^{-1}(C)$. Let $D \cap (v)$ be a wall of $D$, and let $D'$ be the $\mathcal{R}^{\perp}_{26}$-chamber adjacent to $D$ across the wall $D \cap (v)^{\perp}$. Then we can calculate the Weyl vector $w_{C'}$ corresponding to a Conway chamber $C'$ inducing $D' = \iota_X^{-1}(C')$ (see [27]), and hence we can calculate the set of walls of $D'$, which is again finite. Therefore we can determine whether there exists an isometry $g \in O(S_X, \omega_X)$ that maps $D$ to $D'$.

**Definition 4.1.2.** Let $D \cap (v)^{\perp}$ be an inner wall of an $\mathcal{R}^{\perp}_{26}$-chamber $D$ contained in $N_X$. An isometry $g \in O(S_X, \omega_X)$ is said to be an extra automorphism associated with $D \cap (v)^{\perp}$ if $g$ maps $D$ to the $\mathcal{R}^{\perp}_{26}$-chamber adjacent to $D$ across the wall $D \cap (v)^{\perp}$.

Let $g$ be an extra automorphism as above. Since $g$ satisfies the period condition, Condition (A) implies that $g$ preserves the set of $\mathcal{R}^{\perp}_{26}$-chambers. Moreover $g$ maps an interior point of $N_X$ to the interior of $N_X$, and hence $g \in \text{aut}(X)$. We consider the following condition:

(IX) \quad There exists an $\mathcal{R}^{\perp}_{26}$-chamber $D_0$ contained in $N_X$ such that every inner wall of $D_0$ has an extra automorphism.

**Definition 4.1.3.** We say that an embedding $\iota_X$ satisfying Conditions (A), (B) and (IX) is of simple Borcherds type.

**Theorem 4.1.4 ([27]).** Suppose that $\iota_X$ is of simple Borcherds type.
Table 4.1. The 11 K3 surfaces to which we can apply Borcherds method (see Section 4.2).

<table>
<thead>
<tr>
<th>No.</th>
<th>$T_X$</th>
<th>root type</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$m_3$</th>
<th>$m_4$</th>
<th>$k_1$</th>
<th>$k_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[2, 1, 2]</td>
<td>$E_6$</td>
<td>12</td>
<td>6</td>
<td>6</td>
<td>3</td>
<td>103680</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>[2, 0, 2]</td>
<td>$D_6$</td>
<td>8</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>46080</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>[2, 1, 4]</td>
<td>$A_6$</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>10080</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>[2, 0, 4]</td>
<td>$D_5 + A_1$</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>7680</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>[2, 0, 6]</td>
<td>$A_5 + A_1$</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2880</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>[4, 2, 4]</td>
<td>$D_4 + A_2$</td>
<td>12</td>
<td>6</td>
<td>1</td>
<td>1</td>
<td>13824</td>
<td>12</td>
</tr>
<tr>
<td>7</td>
<td>[2, 1, 8]</td>
<td>$A_4 + A_2$</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2880</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>[4, 0, 4]</td>
<td>$2A_3$</td>
<td>8</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>4608</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>[4, 2, 6]</td>
<td>$A_4 + 2A_1$</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1920</td>
<td>4</td>
</tr>
<tr>
<td>10</td>
<td>[2, 0, 12]</td>
<td>$A_3 + A_2 + A_1$</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1152</td>
<td>4</td>
</tr>
<tr>
<td>11</td>
<td>[6, 0, 6]</td>
<td>$2A_2 + 2A_1$</td>
<td>8</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>2304</td>
<td>16</td>
</tr>
</tbody>
</table>

(1) For any point $v$ of $N_X$, there exists an automorphism $g$ of $X$ such that $v^g \in D_0$.

(2) Let $o_1, \ldots, o_m$ be the orbits of the action of $\text{aut}(X, D_0)$ on the set of inner walls of $D_0$, and, for $i = 1, \ldots, m$, let $g(o_i)$ be an extra automorphism associated with an inner wall $D_0 \cap (v_i)^\perp$ belonging to $o_i$. Then $\text{aut}(X)$ is generated by $\text{aut}(X, D_0)$ and the extra automorphisms $g(o_1), \ldots, g(o_m)$. 

4.2. Application to certain singular K3 surfaces. We consider singular K3 surfaces with transcendental lattice $T_X = [a, b, c]$ in Table 4.1. These transcendental lattices are characterized among all even binary positive definite lattices by the following properties: there exists a primitive embedding $\iota_X: S_X \hookrightarrow L_{26}$ of simple Borcherds type such that the orthogonal complement $[\iota_X]^\perp$ is generated by $(-2)$-vectors. In particular, Condition (B) is satisfied. The column root type in Table 4.1 indicates the $ADE$-type of the standard fundamental root system of $[\iota_X]^\perp$. For these cases, the natural homomorphism $O([\iota_X]^\perp) \to O(q([\iota_X]^\perp))$ is surjective and hence Condition (A) is satisfied. The following data are also given in Table 4.1.

- $m_1$ is the order of $O(T_X)$, $m_2$ is the order of $O(T_X, \omega_X)$, $m_3$ is the order of the kernel $K$ of the homomorphism $O(T_X) \to O(q(T_X))$, and $m_4$ is the order of $O(T_X, \omega_X) \cap K$. Then $m_4$ is the order of the kernel of $\rho_X$ by Remark 2.10.1, and the order of $O(q(T_X), \omega_X) = O(q(S_X), \omega_X)$ is $m_4/m_4$.
- $k_1$ is the order of $O([\iota_X]^\perp)$, and $k_2$ is the order of $O(q(T_X)) = O(q(S_X)) = O(q([\iota_X]^\perp))$.

We have a Conway chamber $C_0$ that induces an $\iota_X^\perp R_{26}^\perp$-chamber $D_0$ contained in $N_X$. Let $w \in L_{26}$ be the Weyl vector corresponding to $C_0$, and let $w_S \in S_X \otimes \mathbb{Q}$ be the image of $w$ by the orthogonal projection $\text{pr}_S: L_{26} \otimes \mathbb{Q} \to S_X \otimes \mathbb{Q}$. For each of the 11 cases, we can confirm that $w_S$ belongs to the interior of $D_0$ and that $w_S$ is invariant under the action of $\text{aut}(X, D_0)$. Let $o$ be an orbit of the action of $\text{aut}(X, D_0)$ on the set of walls of $D_0$, and let $D_0 \cap (v)^\perp$ be a member of $o$. We choose the defining vector $v$ of this wall in such a way that $v$ is primitive in $S_X$. Then $v$ is unique. The values $n := \langle v, v \rangle$ and $a := \langle v, w_S \rangle$ are independent of the choice of the wall $D_0 \cap (v)^\perp \in o$. Suppose that the orbit $o$ consists of inner walls. Then we can find an extra automorphism $g \in \text{aut}(X)$ associated with $D_0 \cap (v)^\perp$ by a direct
calculation. Hence $\iota_X$ is of simple Borcherds type. The degree $d_g := \langle w^g_S, w^S \rangle$ is also independent of the choice of $D_0 \cap (v)^\perp$ and $g$. Table 4.2 contains the data of walls and extra automorphisms of $D_0$. If $D_0 \cap (v)^\perp$ is an inner wall, the $(-2)$-vectors $r$ of $L_{26}$ such that $(r)^\perp$ passes through $\iota_X(D_0 \cap (v)^\perp) \subset P_{26}$ form a root system, whose ADE-type is also given below.

**Remark 4.2.1.** Almost all results in Table 4.2 have already appeared in previous works. See Vinberg [37] for Nos. 1 and 2 of Table 4.1, Ujikawa [34] for No. 3, Keum and Kondo [13] for Nos. 6 and 8, [27] for Nos. 4, 5 and 6, [28] for Nos. 7, 9 and 11.

**Remark 4.2.2.** In Table 4.2, the order of the finite group $\text{aut}_0 := \text{aut}(X, D_0)$ is given. The list of all elements of $\text{aut}_0$ is given in [31].

**Table 4.2.** Walls and extra automorphisms of $D_0$.

| $T_X$ | $|\text{aut}_0|$ | $\langle w^S, w^S \rangle$ | No. | $|\alpha|$ | $n$ | $a$ | $d_g$ | root type |
|-------|--------------------|-----------------------------|-----|-----------|-----|-----|-------|------------|
| [2, 1, 2] | 72 78 | $w^S_S$ | 6 | outer | $-2$ | 1 | |
| & | & | & | 18 | outer | $-2$ | 1 | |
| & | & | & | 1 | inner | $-2/3$ | 9 | 321 | $E_7$ |
| [2, 0, 2] | 120 55 | $w^S_S$ | 10 | outer | $-2$ | 1 | |
| & | & | & | 15 | outer | $-2$ | 1 | |
| & | & | & | 20 | outer | $-1/2$ | 17/2 | |
| & | & | & | 1 | inner | $-1$ | 6 | 127 | $D_7$ |
| [2, 1, 4] | 336 28 | $w^S_S$ | 28 | outer | $-2$ | 1 | |
| & | & | & | 1 | inner | $-8/7$ | 4 | 56 | $A_7$ |
| & | & | & | 2 | inner | $-4/7$ | 6 | 154 | $D_7$ |
| & | & | & | 3 | inner | $-2/7$ | 7 | 371 | $E_7$ |
| [2, 0, 4] | 48 61/2 | $w^S_S$ | 6 | outer | $-2$ | 1 | |
| & | & | & | 8 | outer | $-2$ | 1 | |
| & | & | & | 12 | outer | $-2$ | 1 | |
| & | & | & | 2 | outer | $-1/2$ | 11/2 | |
| & | & | & | 1 | inner | $-3/4$ | 6 | 253/2 | $A_1 + E_6$ |
| & | & | & | 5 | inner | $-3/4$ | 6 | 253/2 | $A_1 + E_6$ |
| & | & | & | 6 | inner | $-1/4$ | 13/2 | 737/2 | $E_7$ |
| [2, 0, 6] | 144 18 | $w^S_S$ | 12 | outer | $-2$ | 1 | |
| & | & | & | 18 | outer | $-2$ | 1 | |
| & | & | & | 12 | outer | $-1/2$ | 11/2 | |
| & | & | & | 36 | outer | $-1/2$ | 11/2 | |
| & | & | & | 1 | inner | $-3/2$ | 3/2 | 67/2 | $A_2 + D_7$ |
| & | & | & | 2 | inner | $-1$ | 5 | 161/2 | $A_1 + D_6$ |
| & | & | & | 3 | inner | $-1$ | 5 | 161/2 | $A_1 + D_6$ |
| & | & | & | 4 | inner | $-3/4$ | 6 | 253/2 | $A_1 + E_6$ |
| & | & | & | 5 | inner | $-3/4$ | 6 | 253/2 | $A_1 + E_6$ |
| & | & | & | 6 | inner | $-1/4$ | 13/2 | 737/2 | $E_7$ |
| [4, 2, 4] | 1152 16 | $w^S_S$ | 32 | outer | $-2$ | 1 | |
| & | & | & | 1 | inner | $-4/3$ | 2 | 22 | $A_3 + D_4$ | Continued on next page |
| $T_X$ | $|\text{auto}|$ | $(w_S, w_S)$ | No. | $|o|$ | $n$ | $a$ | $d_o$ | root type |
|---|---|---|---|---|---|---|---|---|
| 2 | 72 | inner | $-1$ | 4 | 48 | $A_2 + D_5$ |
| 3 | 96 | inner | $-1/3$ | 5 | 166 | $D_7$ |
| $[2, 1, 8]$ | 720 | 12 | 36 | outer | $-2$ | 1 | |
| 1 | 12 | inner | $-4/3$ | 2 | 18 | $A_3 + A_4$ |
| 2 | 40 | inner | $-6/5$ | 3 | 27 | $A_2 + A_5$ |
| 3 | 90 | inner | $-4/5$ | 4 | 52 | $A_2 + D_5$ |
| 4.5 | 30 | inner | $-8/15$ | 4 | 72 | $A_7$ |
| 6.7 | 120 | inner | $-2/15$ | 5 | 387 | $E_7$ |
| $[4, 0, 4]$ | 3840 | 10 | 40 | outer | $-2$ | 1 | |
| 1 | 64 | inner | $-5/4$ | 5/2 | 20 | $A_3 + A_4$ |
| 2 | 40 | inner | $-1$ | 3 | 28 | $A_3 + D_4$ |
| 3 | 160 | inner | $-1/2$ | 4 | 74 | $A_7$ |
| 4 | 320 | inner | $-1/4$ | 9/2 | 172 | $D_7$ |
| $[4, 2, 6]$ | 120 | 11 | 5 | outer | $-2$ | 1 | |
| 1 | 6 | inner | $-3/2$ | 3/2 | 14 | $A_1 + A_2 + A_4$ |
| 3 | 20 | inner | $-6/5$ | 3 | 26 | $2A_1 + A_5$ |
| 4 | 30 | inner | $-6/5$ | 3 | 26 | $2A_1 + A_5$ |
| 5 | 1 | inner | $-1$ | 2 | 19 | $A_3 + A_4$ |
| 6 | 30 | inner | $-4/5$ | 4 | 51 | $2A_1 + D_5$ |
| 7 | 40 | inner | $-4/5$ | 4 | 51 | $2A_1 + D_5$ |
| 8 | 60 | inner | $-4/5$ | 4 | 51 | $2A_1 + D_5$ |
| 9, 10 | 20 | inner | $-7/10$ | 7/2 | 46 | $A_1 + A_6$ |
| 11, 12 | 20 | inner | $-3/10$ | 9/2 | 146 | $A_1 + E_6$ |
| 13, 14 | 60 | inner | $-3/10$ | 9/2 | 146 | $A_1 + E_6$ |
| 15 | 10 | inner | $-1/5$ | 4 | 171 | $D_7$ |
| $[2, 0, 12]$ | 720 | 15/2 | 45 | outer | $-2$ | 1 | |
| 1 | 10 | inner | $-3/2$ | 3/2 | 21/2 | $2A_2 + A_3$ |
| 2 | 30 | inner | $-4/3$ | 2 | 27/2 | $A_1 + 2A_3$ |
| 3 | 72 | inner | $-5/4$ | 5/2 | 35/2 | $A_1 + A_2 + A_4$ |
| 4 | 60 | inner | $-1$ | 3 | 51/2 | $A_1 + A_2 + D_4$ |
| 5 | 12 | inner | $-5/6$ | 5/2 | 45/2 | $A_3 + A_4$ |
| 6 | 40 | inner | $-3/4$ | 3 | 63/2 | $A_2 + A_5$ |
| 7.8 | 120 | inner | $-7/12$ | 7/2 | 99/2 | $A_1 + A_6$ |
| 9 | 120 | inner | $-1/3$ | 4 | 207/2 | $A_1 + D_6$ |
| 10 | 180 | inner | $-1/3$ | 4 | 207/2 | $A_1 + D_6$ |
| 11, 12 | 120 | inner | $-1/12$ | 4 | 783/2 | $E_7$ |
| $[6, 0, 6]$ | 1440 | 5 | 60 | outer | $-2$ | 1 | |
| 1 | 40 | inner | $-3/2$ | 3/2 | 8 | $A_1 + 3A_2$ |
| 2 | 180 | inner | $-4/3$ | 2 | 11 | $2A_1 + A_2 + A_3$ |
| 3 | 10 | inner | $-1$ | 2 | 13 | $2A_1 + D_3$ |
| 4.5 | 144 | inner | $-5/6$ | 5/2 | 20 | $A_1 + A_2 + A_4$ |
| 6 | 240 | inner | $-2/3$ | 3 | 32 | $2A_1 + A_5$ |
| 7 | 360 | inner | $-2/3$ | 3 | 32 | $2A_1 + A_5$ |
| 8 | 180 | inner | $-1/3$ | 3 | 59 | $A_2 + D_5$ |
| 9, 10 | 240 | inner | $-1/6$ | 7/2 | 152 | $A_1 + E_6$ |
| 11, 12 | 720 | inner | $-1/6$ | 7/2 | 152 | $A_1 + E_6$ |
5. ENRIQUES INVOLUTIONS AND BORCHERDS METHOD

In this section, we assume that $X$ is a complex K3 surface admitting a primitive embedding $\iota_X : S_X \hookrightarrow \mathcal{L}_{26}$ of simple Borcherds type and, in addition, that

\begin{itemize}
  \item[(C)] the natural homomorphism $\rho_X : \text{Aut}(X) \rightarrow \text{O}(S_X, \mathcal{P}_X)$ is injective.
\end{itemize}

5.1. Inner faces. Let $D_0$ be an $\iota_X^* \mathcal{L}_{26}^+$-chamber contained in $N_X$. Let $w_1, \ldots, w_k$ be the inner walls of $D_0$. For each $w_i$, we calculate an extra automorphism $g_i \in \text{aut}(X)$ associated with $w_i$ (see Definition 4.1.2).

**Definition 5.1.1.** A face $f$ of $D_0$ is said to be $D_0$-inner if $f$ is not contained in any outer wall of $D_0$, whereas $f$ is said to be $N_X$-inner if $f$ is not contained in any wall of $N_X$.

**Remark 5.1.2.** An $N_X$-inner face is always $D_0$-inner. The converse is, however, not true in general as illustrated in Figure 5.1, in which a black circle indicates a $D_0$-inner face of codimension 2 that is not $N_X$-inner.

Let $f$ be a $D_0$-inner face of dimension $> 0$. We put

\[
\mathcal{D}(f) := \{ D \mid D \text{ is an } \iota_X^* \mathcal{L}_{26}^+\text{-chamber contained in } N_X \text{ and containing } f \},
\]

\[
\mathcal{A}(X, f) := \{ g \in \text{aut}(X) \mid D_0^g \in \mathcal{D}(f) \} = \{ g \in \text{aut}(X) \mid f \subset D_0^g \},
\]

\[
\text{aut}(X, f) := \{ g \in \text{aut}(X) \mid f^g = f \}.
\]

The set $\mathcal{D}(f)$ is calculated by the following method.

**Algorithm 5.1.3.** We set $\mathcal{D} = \{ D_0 \}$, $\gamma_0 = \text{id}$, $\Gamma = \{ \gamma_0 \}$, and $i = 0$. During the calculation, the ordered set $\mathcal{D}$ is a subset of $\mathcal{D}(f)$, and the $(i + 1)$st member $\gamma_i$ of $\Gamma$ is an element of $\text{aut}(X)$ that maps $D_0$ to the $(i + 1)$st member $D_i$ of $\mathcal{D}$. While $i < |\mathcal{D}|$, we execute the following. We calculate the set \{w_{\nu(1)}, \ldots, w_{\nu(m)}\} of inner walls $w_{\nu(j)}$ of $D_0$ such that $f \subset w_{\nu(j)}^\gamma$. Let $g_{\nu(j)} \in \text{aut}(X)$ be an extra automorphism associated with $w_{\nu(j)}$. For each $j = 1, \ldots, m$, we calculate the induced chamber $D' := D_0^{g_{\nu(j)} \gamma_i}$, which is adjacent to $D_i = D_0^{\gamma_i}$ across $w_{\nu(j)}^\gamma$ and contains $f$. If $D'$ has not yet been added to $\mathcal{D}$, we add $D'$ to $\mathcal{D}$ and $g_{\nu(j)} \gamma_i$ to $\Gamma$. Then we increment $i$ to $i + 1$.

When this algorithm terminates, the list $\mathcal{D}$ is equal to $\mathcal{D}(f)$. Moreover, we have calculated $\Gamma = \{ g_D \mid D \in \mathcal{D}(f) \}$, where $g_D \in \text{aut}(X)$ maps $D_0$ to $D \in \mathcal{D}(f)$. Note that the action of $g_D \in \Gamma$ preserves the walls of $N_X$. The following is obvious from the definition.

**Criterion 5.1.4.** The $D_0$-inner face $f$ is $N_X$-inner if and only if, for any $g_D \in \Gamma$ and any outer wall $D_0 \cap (r)^\perp$ of $D_0$, the wall $(D_0 \cap (r)^\perp)^{g_D}$ of $D = D_0^{g_D}$ does not contain $f$. 

\[ \text{D0-}\text{inner face of } D_0 \text{ and any outer wall of } D_0 \]

\[ \text{D0-}\text{inner face of } D_0 \text{ and any outer wall of } D_0 \]

\[ \text{a wall of } N_X \]

**Figure 5.1.** A $D_0$-inner face that is not $N_X$-inner.
Suppose that $f$ is $N_X$-inner and $D$ is an element of $\mathcal{D}(f)$. Note that the set of all elements $g \in \text{aut}(X)$ that maps $D_0$ to $D$ is equal to $\text{aut}(X, D_0) \cdot g_D$. Therefore we can calculate $\mathcal{A}(X, f)$ by

$$\mathcal{A}(X, f) = \bigsqcup_{D \in \mathcal{D}(f)} \text{aut}(X, D_0) \cdot g_D.$$ 

The subgroup $\text{aut}(X, f)$ of $\text{aut}(X)$ is contained in the finite set $\mathcal{A}(X, f)$, and thus we can calculate $\text{aut}(X, f)$.

**Definition 5.1.5.** Let $f$ and $f'$ be $N_X$-inner faces of $D_0$. We say that $f$ and $f'$ are $\text{aut}(X)$-equivalent (resp. $\text{aut}(X, D_0)$-equivalent) if there exists an element $g \in \text{aut}(X)$ (resp. $g \in \text{aut}(X, D_0)$) such that $f^g = f'$.

Even though $\text{aut}(X)$ is infinite in general, we can calculate the $\text{aut}(X)$-equivalence classes by the following:

**Criterion 5.1.6.** The faces $f$ and $f'$ are $\text{aut}(X)$-equivalent if and only if there exists an element $g \in \mathcal{A}(X, f')$ such that $f^g = f'$.

### 5.2. An algorithm to classify all Enriques involutions.

Let $\bar{\xi}: X \to X$ be an Enriques involution, and $\pi: X \to Y := X/\langle \bar{\xi} \rangle$ the quotient morphism to the Enriques surface $Y$. Let $\varepsilon \in \text{aut}(X)$ denote the image of $\bar{\xi}$ by the natural homomorphism (2.1). Then $\pi$ induces a primitive embedding $\pi^*: S_Y(2) \to S_X$. We have canonical identifications $S_Y(2) \otimes \mathbb{R} = S_Y \otimes \mathbb{R}$ and $O(S_Y(2)) = O(S_Y)$. In particular, we regard the positive cone $\mathcal{P}_Y$ of $S_Y$ as a positive cone of $S_Y(2)$. The embedding $\pi^*$ induces an embedding

$$\pi^*: \mathcal{P}_Y \hookrightarrow \mathcal{P}_X.$$ 

Henceforth, we regard $S_Y(2)$ as a primitive sublattice of $S_X$ and $\mathcal{P}_Y$ as a subspace of $\mathcal{P}_X$ by $\pi^*$. Note that $S_Y(2)$ is equal to $\{ v \in S_X \mid \varepsilon v = v \}$, and $\mathcal{P}_Y$ is equal to $\{ x \in \mathcal{P}_X \mid x^\varepsilon = x \}$.

**Proposition 5.2.1.** We have $N_Y = N_X \cap \mathcal{P}_Y$. Let $y$ be a point of $N_Y$. Then $y$ is an interior point of $N_Y$ if and only if $y$ is an interior point of $N_X$.

**Proof.** The first equality is obvious. By Theorem 3.1.1, the orthogonal complement of $S_Y(2)$ in $S_X$ contains no $(-2)$-vectors, and a line bundle of $Y$ is ample if and only if its pull-back to $X$ is ample. \hfill \Box

Let $y$ be a sufficiently general point of $N_Y$. By Theorem 4.1.4, there exists an automorphism $g \in \text{aut}(X)$ such that $y^g \in D_0$, and hence $D_0 \cap N_Y^\perp$ contains a non-empty open subset of $\mathcal{P}_Y^\perp$. Therefore, replacing $\varepsilon$ by $g^{-1}\varepsilon g$, we can assume that $E_0 := D_0 \cap N_Y$ contains a non-empty open subset of $\mathcal{P}_Y$. Consider the composite

$$\iota_Y := \iota_X \circ \pi^*: S_Y(2) \hookrightarrow L_{26}$$

of primitive embeddings. Then $\mathcal{P}_Y$ is decomposed into the union of $\iota_Y^* R_{26}^\perp$-chambers. Since every wall of $N_Y$ is defined by a $(-2)$-vector, it follows that $N_Y$ is decomposed into a union of $\iota_Y^* R_{26}^\perp$-chambers. Note that $E_0$ is one of the $\iota_Y^* R_{26}^\perp$-chambers in $N_Y$.

**Definition 5.2.2.** For a closed subset $A$ of $D_0$, the minimal face of $D_0$ for $A$ is the face of $D_0$ containing $A$ with the minimal dimension.
Let \( f_\varepsilon \) be the minimal face of \( D_0 \) for \( E_0 \). Since the orthogonal complement of \( S_Y(2) \) in \( S_X \) contains no \((-2)\)-vector, the face \( f_\varepsilon \) is \( N_X \)-inner. Moreover, the involution \( \varepsilon \in \text{aut}(X) \) belongs to \( \text{aut}(X, f_\varepsilon) \). Let \( \varepsilon' \) be an Enriques involution such that \( f_{\varepsilon'} \) is a face of \( D_0 \). If \( \varepsilon' \) is conjugate to \( \varepsilon \), then \( f_{\varepsilon'} \) is \( \text{aut}(X) \)-equivalent to \( f_{\varepsilon'} \). If \( f_\varepsilon = f_{\varepsilon'} \), then \( \varepsilon \) and \( \varepsilon' \) are conjugate if and only if \( \varepsilon \) and \( \varepsilon' \) are conjugate in \( \text{aut}(X, f_\varepsilon) \).

We calculate all \( N_X \)-inner faces of \( D_0 \) of dimension \( \geq 10 \) by descending induction of the dimension of faces (see Section 2.6), and compute a complete set of representatives of the \( \text{aut}(X) \)-equivalence classes. For each representative \( f \), we calculate \( \text{aut}(X, f) \). We then calculate the set of Enriques involutions \( \varepsilon \) contained in \( \text{aut}(X, f) \) such that \( f_\varepsilon = f \) by Keum’s criterion (Theorem 3.1.1), and thus we obtain a set of complete representatives of Enriques involutions in \( \text{aut}(X) \) modulo conjugation.

### 5.3. Computation of \( \text{Aut}(Y) \)

Let \( \varepsilon \) be a representative of \( \text{aut}(X) \)-conjugacy classes of Enriques involutions obtained by the method above. In particular, we have an \( \iota_Y^* R^\perp_{26} \)-chamber \( E_0 = D_0 \cap N_Y \), the minimal face \( f_\varepsilon \) of \( D_0 \) for \( E_0 \), and the associated data \( D(f_\varepsilon), A(X, f_\varepsilon), \text{aut}(X, f_\varepsilon) \). We put

\[
\text{aut}(X, \varepsilon) := \{ g_X \in \text{aut}(X) | g_X \varepsilon = \varepsilon \} = \{ g_X \in \text{aut}(X) | S_Y(2)^{g_X} = S_Y(2) \},
\]

where the second equality follows from \( S_Y(2) = \{ v \in S_X | v^f = v \} \). We have a natural restriction homomorphism \( \text{aut}(X, \varepsilon) \rightarrow \text{O}(S_Y) \), which is denoted by \( g_X \mapsto g_X|S_Y \). By Condition (C), we have a natural identification

\[
(5.1) \quad \text{Aut}(Y) \cong \text{aut}(X, \varepsilon)/\langle \varepsilon \rangle.
\]

Under the identification (5.1), the homomorphism \( \rho_Y : \text{Aut}(Y) \rightarrow \text{O}(S_Y, \mathcal{P}_Y) \) is identified with the homomorphism \( g_X \mod \langle \varepsilon \rangle \mapsto g_X|S_Y \). The method below, when it works, gives us a finite set of generators of \( \text{aut}(X, \varepsilon) \), and hence a finite set of generators of \( \text{Aut}(Y) \).

Recall that \( \text{aut}(Y) \) is the image of \( \text{Aut}(Y) \) by \( \rho_Y \). We put

\[
\text{aut}(Y, E_0) := \{ g \in \text{aut}(Y) | E_0^g = E_0 \},
\]

and let \( \text{Aut}(Y, E_0) \) denote the inverse image of \( \text{aut}(Y, E_0) \) by \( \rho_Y \).

**Proposition 5.3.1.** The action of \( \text{aut}(Y) \) on \( N_Y \) preserves the set of \( \iota_Y^* R^\perp_{26} \)-chambers contained in \( N_Y \).

**Proof.** Let \( g \) be an element of \( \text{aut}(Y) \). Then \( g \) extends to \( g_X \in \text{aut}(X, \varepsilon) \). By Condition (A), this isometry \( g_X \in \text{O}(S_X, \omega_X) \cap \text{O}(S_X, \mathcal{P}_X) \) extends to an isometry \( \tilde{g}_X \) of \( L_{26} \), which preserves the set of Conway chambers. Hence its restriction \( g \) to \( S_Y(2) \) preserves the set of chambers induced by Conway chambers. \( \square \)

We put

\[
\text{aut}(X, \varepsilon, f_\varepsilon) := \text{aut}(X, \varepsilon) \cap \text{aut}(X, f_\varepsilon).
\]

**Proposition 5.3.2.** The identification (5.1) induces \( \text{Aut}(Y, E_0) \cong \text{aut}(X, \varepsilon, f_\varepsilon)/\langle \varepsilon \rangle \).

**Proof.** Note that \( E_0 = f_\varepsilon \cap N_Y \). Since \( E_0 \) contains an interior point of the face \( f_\varepsilon \), an element \( g_X \) of \( \text{aut}(X, \varepsilon) \) fixes \( E_0 \) if and only if \( g_X \) fixes \( f_\varepsilon \). \( \square \)

**Corollary 5.3.3.** By the identification (5.1), the kernel of \( \rho_Y : \text{Aut}(Y) \rightarrow \text{O}(S_Y, \mathcal{P}_Y) \) is equal to

\[
\{ g_X \in \text{aut}(X, \varepsilon, f_\varepsilon) | g_X|S_Y = \text{id} \}/\langle \varepsilon \rangle.
\]
Recall from Section 2.9 that we have classified primitive embeddings of $S_Y(2) \cong L_{10}(2)$ into $L_{26}$. The $i_Y^* \mathcal{R}_{26}^+$-chamber $E_0$ has only finitely many walls. By Remark 2.9.2, the primitive embedding $i_Y: S_Y(2) \hookrightarrow L_{26}$ is not of type infinity. By Theorem 2.9.1, every $i_Y^* \mathcal{R}_{26}^+$-chamber $E$ has only a finite number of walls, and each wall of $E$ is defined by a $(-2)$-vector $r \in \mathcal{R}_Y$.

**Definition 5.3.4.** A wall $w$ of $E_0$ is said to be outer if $w$ is contained in a wall of $N_Y$. Otherwise $w$ is said to be inner.

There are several criteria to determine whether a given wall $w$ of $E_0$ is outer or inner.

**Criterion 5.3.5.** Suppose that the wall $w$ of $E_0$ is defined by $r \in \mathcal{R}_Y$. Then $w$ is outer if and only if there exists a $(-2)$-vector $u$ in the orthogonal complement $[\pi^*]_{\perp}$ of $S_Y(2)$ in $S_X$ such that $(u + r)/2 \in S_X$.

Indeed, the condition in the statement is equivalent to the condition that $r$ is the class of an effective divisor of $Y$ (see [21]).

**Criterion 5.3.6.** Let $f_\varepsilon(w)$ be the minimal face of $D_0$ for the closed subset $w$ of $D_0$. Then $w$ is inner if and only if $f_\varepsilon(w)$ is $N_X$-inner.

Indeed, by minimality of $f_\varepsilon(w)$, there exists an interior point $y$ of $w$ that is an interior point of $f_\varepsilon(w)$. Then the statement follows from Proposition 5.2.1.

When $E_0$ has no inner walls, we have $E_0 = N_Y$ and $|\text{Aut}(Y)| < \infty$, and the Nikulin-Kondo type of $Y$ is obtained by comparing the configuration of $(-2)$-vectors defining the walls of $E_0$ with the dual graphs of smooth rational curves given in [15].

We consider $\text{Aut}(Y)$ when $E_0$ has an inner wall. Let $I_0$ denote the set of inner walls of $E_0$. For each $w = E_0 \cap (r)_{\perp} \in I_0$ with $r \in \mathcal{R}_Y$, we put $E(w) := E_0^w$, where $s_r: \mathcal{P}_Y \to \mathcal{P}_Y$ is the reflection into the hyperplane $(r)_{\perp} \subset \mathcal{P}_Y$. Theorem 2.9.1 implies that $E(w)$ is the $i_Y^* \mathcal{R}_{26}^+$-chamber adjacent to $E_0$ across $w$. Recall that $\mathcal{A}(X, f_\varepsilon(w))$ is the set of $g_X \in \text{Aut}(X)$ such that $D_0^X$ contains $f_\varepsilon(w)$. If the restriction $g_X|S_Y$ to $S_Y(2)$ of $g_X \in \text{Aut}(X, \varepsilon)$ maps $E_0$ to $E(w)$, then $g_X \in \mathcal{A}(X, f_\varepsilon(w))$ holds.

**Definition 5.3.7.** An element $g_X$ of $\text{Aut}(X, \varepsilon) \cap \mathcal{A}(X, f_\varepsilon(w))$ is an extra automorphism for the inner wall $w \in I_0$ if the restriction $g_X|S_Y$ of $g_X$ to $S_Y(2)$ maps $E_0$ to $E(w)$.

Since $\mathcal{A}(X, f_\varepsilon(w))$ is finite, we can determine the existence of an extra automorphism for each inner wall of $E_0$.

**Theorem 5.3.8.** Suppose that Condition (C) is satisfied. Suppose also that the following holds:

\begin{itemize}
  \item [(IY)] there exists an extra automorphism $g_X(w)$ for each inner wall $w \in I_0$.
\end{itemize}

Then $\text{Aut}(X, \varepsilon)$ is generated by the finite subgroup $\text{Aut}(X, \varepsilon, f_\varepsilon)$ and the extra automorphisms $g_X(w)$ ($w \in I_0$).

**Proof.** Let $\Gamma$ denote the subgroup of $\text{Aut}(X, \varepsilon)$ generated by the extra automorphisms $g_X(w)$ ($w \in I_0$). First we prove the following claim. For any $i_Y^* \mathcal{R}_{26}^+$-chamber $E$ contained in $N_Y$, there exists an element $\gamma \in \Gamma$ such that $\gamma|S_Y$ maps $E_0$ to $E$. There exists a chain $E_0, E_1, \ldots, E_m = E$ of $i_Y^* \mathcal{R}_{26}^+$-chambers contained in $N_Y$ such that $E_{i-1}$ and $E_i$ is adjacent for $i = 1, \ldots, m$. We prove the claim by
induction on the length $m$ of the chain with the case $m = 0$ being trivial. Suppose that $m > 0$. There exists an element $\gamma' \in \Gamma$ such that $\gamma'|S_Y$ maps $E_0$ to $E_{m-1}$. Let $E'$ be the $i^\ast Y|\mathcal{R}_{26}'$-chamber that is mapped to $E_m$ by $\gamma'|S_Y$. Then $E'$ is adjacent to $E_0$. Note that $\gamma'|S_Y \in \text{aut}(Y)$ preserves $N_Y$. Therefore $E'$ is contained in $N_Y$. In particular, the wall $w$ between $E_0$ and $E'$ is inner, and hence there exists an extra automorphism $g_X(w)$ such that $g_X(w)|S_Y$ maps $E_0$ to $E'$. We put $\gamma := g_X(w) \cdot \gamma' \in \Gamma$. Then $\gamma|S_Y$ maps $E_0$ to $E_m$.

Next we show that $\Gamma$ and $\text{aut}(X, \varepsilon, f_\varepsilon)$ generate $\text{aut}(X, \varepsilon)$. Let $g$ be an arbitrary element of $\text{aut}(X, \varepsilon)$. We apply the claim above to the $i^\ast Y|\mathcal{R}_{26}'$-chamber $E_0^{11|S_Y}$, and obtain an element $\gamma \in \Gamma$ such that $(g\gamma^{-1})|S_Y$ is an element of $\text{aut}(Y, E_0)$. By Proposition 5.3.2, we have $g\gamma^{-1} \in \text{aut}(X, \varepsilon, f_\varepsilon)$.

**Definition 5.3.9.** We say that a triple $(X, \iota_X, \varepsilon)$ of a K3 surface $X$, a primitive embedding $\iota_X : S_X \hookrightarrow L_{26}$, and an Enriques involution $\varepsilon$ of $X$ is of simple Borcherds type if $X$ satisfies Condition (C), $(X, \iota_X)$ is of simple Borcherds type in the sense of Definition 4.1.3, and $\varepsilon$ satisfies Condition (IY).

**Remark 5.3.10.** The notion of simple Borcherds type was introduced in [29] for K3 surfaces. We hope that we can find a bound on the degrees of polarizations similar to that of [29] for Enriques surfaces.

### 5.4. Enriques involutions of the 11 singular K3 surfaces

We apply the method in the previous section to the singular K3 surfaces in Section 4.2. First remark that Condition (C) holds for the 11 cases except for the cases $T_X = [2, 1, 2]$ and $T_X = [2, 0, 2]$ (see Remark 2.10.1 and Table 4.1). Note that in these two cases, and also in the case $T_X = [2, 0, 4]$, there exist no Enriques involutions by Theorem 3.2.1.

Our main result is as follows.

**Theorem 5.4.1.** Let $X$ be one of the singular K3 surfaces of No. $\neq 1, 2, 4$ in Table 4.1, and let $\iota_X : S_X \hookrightarrow L_{26}$ be the primitive embedding given in Section 4.2. Then the Enriques involutions of $X$ modulo conjugation in $\text{Aut}(X) \cong \text{aut}(X)$ are given in Table 5.1. For each Enriques involution $\varepsilon$ on $X$, the triple $(X, \iota_X, \varepsilon)$ is of simple Borcherds type.

We explain the contents of Table 5.1. The item $\iota_Y$ is the type of the primitive embedding $\iota_Y : S_Y(2) \hookrightarrow L_{26}$ given in [6]. The item NK is the Nikulin-Kondo type of the $i^\ast Y|\mathcal{R}_{26}$-chamber $E_0$ (see Theorem 2.9.3). The item $m_4$ is the number of $(-4)$-vectors in the orthogonal complement of $S_Y(2)$ in $S_X$. The item $|ws|$ is the number of walls of $E_0$. The item $|G_\varepsilon|$ is the order of

$$G_\varepsilon := \text{aut}(X, \varepsilon, f_\varepsilon).$$

The item $|I_0|$ is the number of inner walls of $E_0$.

**Remark 5.4.2.** For the Enriques involution No. 24 on $X$ with $T_X = [6, 0, 6]$, the $i^\ast Y|\mathcal{R}_{26}$-chamber $E_0$ has 40 walls and the configuration of the walls is not of Nikulin-Kondo type. The dual graph is too complicated to be presented here. See [31] for the matrix presentation of this configuration.

The item $|K_\rho|$ is the order of the kernel of $\rho_Y : \text{Aut}(Y) \to \text{aut}(Y)$, and the item $|\text{aut}|$ is the order of $\text{aut}(Y)$. The fact that $\text{aut}(Y)$ is infinite when $I_0$ is non-empty was confirmed by selecting elements of $\text{aut}(Y)$ randomly by means of the finite
Table 5.1. Enriques involutions of the 11 singular K3 surfaces (see Section 5.4).

| No. | $T_X$ | dim $f_ε$ | $ε$ | $ι$ | $Y$ | NK | $m_4$ | $|ws|$ | $|G_ε|$ | $|I_0|$ | $|K_ρ|$ | $|\text{aut}|$ |
|-----|------|----------|-----|----|-----|-----|------|-----|------|------|------|------|
| 1   | $[2, 1, 4]$ | 19 | $12B$ | II | 144 | 12 | 48 | 0 | 1 | 24 |
| 2   | 18 | $12A$ | I | 242 | 12 | 16 | 0 | 2 | 4 |
| 3   | $[2, 0, 6]$ | 19 | $12B$ | II | 144 | 12 | 48 | 0 | 1 | 24 |
| 4   | $[4, 2, 4]$ | 18 | $12A$ | I | 246 | 12 | 16 | 0 | 2 | 4 |
| 5   | 18 | $20B$ | III | 246 | 20 | 64 | 4 | 2 | $\infty$ |
| 6   | 17 | $20A$ | V | 246 | 20 | 96 | 0 | 2 | 24 |
| 7   | $[2, 1, 8]$ | 19 | $20D$ | VII | 90 | 20 | 120 | 5 | 1 | $\infty$ |
| 8   | 19 | $20D$ | VII | 180 | 20 | 120 | 5 | 1 | $\infty$ |
| 9   | 19 | $12B$ | II | 144 | 12 | 48 | 0 | 1 | 24 |
| 10  | 18 | $12A$ | I | 240 | 12 | 8 | 2 | 2 | $\infty$ |
| 11  | 17 | $20A$ | V | 132 | 20 | 48 | 4 | 1 | $\infty$ |
| 12  | $[4, 0, 4]$ | 20 | $20F$ | IV | 180 | 20 | 640 | 0 | 1 | 320 |
| 13  | 19 | $20D$ | VII | 180 | 20 | 120 | 5 | 1 | $\infty$ |
| 14  | 19 | $12B$ | II | 180 | 12 | 48 | 0 | 1 | 24 |
| 15  | 18 | $12A$ | I | 244 | 12 | 16 | 0 | 2 | 4 |
| 16  | 18 | $12A$ | I | 244 | 12 | 16 | 0 | 2 | 4 |
| 17  | 18 | $20B$ | III | 244 | 20 | 64 | 8 | 2 | $\infty$ |
| 18  | 18 | $20B$ | III | 244 | 20 | 64 | 4 | 2 | $\infty$ |
| 19  | 18 | $20B$ | III | 308 | 20 | 256 | 0 | 2 | 64 |
| 20  | 17 | $20A$ | V | 244 | 20 | 32 | 2 | 2 | $\infty$ |
| 21  | $[4, 2, 6]$ | 19 | $20D$ | VII | 92 | 20 | 240 | 0 | 1 | 120 |
| 22  | 18 | $12A$ | I | 242 | 12 | 16 | 0 | 2 | 4 |
| 23  | $[2, 0, 12]$ | 19 | $20D$ | VII | 90 | 20 | 120 | 5 | 1 | $\infty$ |
| 24  | $[6, 0, 6]$ | 20 | $40E$ | 140 | 40 | 1440 | 10 | 1 | $\infty$ |
| 25  | 18 | $12A$ | I | 240 | 12 | 16 | 2 | 4 | $\infty$ |
| 26  | 17 | $20A$ | V | 132 | 20 | 48 | 4 | 1 | $\infty$ |

![Figure 5.2. Configuration of Nikulin-Kondo type I](image-url)

Remark 5.4.3. Consider the Enriques involutions of Nos. 10, 16 and 25, that is, the cases where the Nikulin-Kondo type is I and $\text{Aut}(Y)$ is infinite. In these cases, we...
Table 5.2. $N_X$-inner faces corresponding to Enriques involutions.

| $T_X$   | dim | numb | $pws$          | $|D|$ | $|\operatorname{aut}(X,f)|$ | $\varepsilon$ |
|---------|-----|------|----------------|------|--------------------------|--------------|
| [2, 1, 4] | 19  | 14   | $1^1$          | 2    | 48                       | No. 1        |
|         | 18  | 42 + 84 | $1^2, 1^1 2^1$ | 6    | 16                       | No. 2        |
| [2, 0, 6] | 19  | 6    | $3^1$          | 2    | 48                       | No. 3        |
| [4, 2, 4] | 18  | 288 $\times$ 2 | $1^2 2^1, 1^1 3^1$ | 8    | 16                       | No. 4        |
|         | 18  | 12   | $1^2$          | 6    | 576                      | No. 5        |
|         | 17  | 144  | $1^2 2^1$      | 12   | 96                       | No. 6        |
| [2, 1, 8] | 19  | 12   | $1^1$          | 2    | 120                      | No. 7        |
|         | 19  | 30   | $4^1$          | 2    | 48                       | No. 8        |
|         | 19  | 30   | $5^1$          | 2    | 48                       | No. 9        |
|         | 18  | 180 $\times$ 4 | $1^2 4^1, 1^1 5^1, 3^1 4^1, 3^1 5^1$ | 8    | 8                        | No. 10       |
|         | 17  | 90 $\times$ 2 | $1^2 3^1$      | 12   | 48                       | No. 11       |
| [4, 0, 4] | 20  | 1    |                | 1    | 3840                     | No. 12       |
|         | 19  | 64   | $1^1$          | 2    | 120                      | No. 13       |
|         | 19  | 160  | $3^1$          | 2    | 48                       | No. 14       |
|         | 18  | 960 $\times$ 2 | $1^2 2^1, 1^1 4^1$ | 8    | 16                       | No. 15       |
|         | 18  | 960 $\times$ 2 | $2^1 3^1$      | 8    | 16                       | No. 16       |
|         | 18  | 60   | $2^2$          | 4    | 256                      | Nos. 17, 18, 19 |
|         | 17  | 480 + 960 | $1^2 3^1, 1^1 2^2$ | 12   | 32                       | No. 20       |
| [4, 2, 6] | 19  | 1    | $5^1$          | 2    | 240                      | No. 21       |
|         | 18  | 30 $\times$ 2 | $4^1 5^1, 4^1 15^1$ | 8    | 16                       | No. 22       |
| [2, 0, 12] | 19  | 12   | $5^1$          | 2    | 120                      | No. 23       |
| [6, 0, 6] | 20  | 1    | $5^1$          | 1    | 1440                     | No. 24       |
|         | 18  | 360 $\times$ 2 | $7^1 8^1$      | 8    | 16                       | No. 25       |
|         | 17  | 180 $\times$ 2 | $2^2 8^1$      | 12   | 48                       | No. 26       |

have $|I_0| = 2$. The configuration of Nikulin-Kondo type I is as in Figure 5.2, and the inner walls are defined by the $(-2)$-vectors $\mathbb{I}$ and $\mathbb{I}'$.

See [31] for the inner walls of $E_0$ for the other Enriques involutions. The finite generating sets of $\operatorname{aut}(X, \varepsilon)$ and of $\operatorname{aut}(Y)$ are also given explicitly in [31].

Table 5.2 is a list of $N_X$-inner faces of $D_0$ that corresponds to Enriques involutions. Note that an $\operatorname{aut}(X)$-equivalence class of $N_X$-inner faces is a union of orbits of the action of $\operatorname{aut}(X, D_0)$ on the set of $N_X$-inner faces.

The item numb gives the number of faces in the $\operatorname{aut}(X)$-equivalence class. The formula in this column shows the decomposition of the $\operatorname{aut}(X)$-equivalence class into a union of $\operatorname{aut}(X, D_0)$-orbits. The item $pws$ indicates the types of inner walls of $D_0$ passing through the face. The type of an inner wall of $D_0$ is given by No. in Table 4.2.

For example, take the case $T_X = [2, 1, 4]$. For a face $f$ in the $\operatorname{aut}(X)$-equivalence class corresponding to the Enriques involution No. 2, there exist exactly two inner walls of $D_0$ passing through $f$, and they are both of type 1, whereas for another face $f'$ in this $\operatorname{aut}(X)$-equivalence class, there exist exactly two inner walls of $D_0$ passing through $f'$, and they are of type 1 and 2.
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We explain how the data \( pws \) depends on the choice of a representative of an \( \text{aut}(X) \)-equivalence class. Let \( f \) be a face in this \( \text{aut}(X) \)-equivalence class. Then there exists exactly three members \( (v_1)^+, (v_1')^+, (v_2)^\perp \) in the family \( \iota_{X}^* \mathcal{R}_{26}^\perp \) of hyperplanes that pass through \( f \), where \( v_1, v_1', v_2 \) are primitive vectors of \( S_X^* \) such that \( \langle v_1, v_1 \rangle = \langle v_1', v_1' \rangle = -8/7 \) and \( \langle v_2, v_2 \rangle = -4/7 \). See Figure 5.3. If \( D_0 \) is located in the region \( D^{(\pm 1)} \), then the data \( pws \) for \( f \) is \( 1^2 \), whereas if \( D_0 \) is located in the region \( D^{(\pm 2)} \) or \( D^{(\pm 3)} \), then the data \( pws \) for \( f \) is \( 1^12^1 \). The item \( |D| \) is the size of \( D(f) \) and \( |\text{aut}(X,f)| \) is the order of the group \( \text{aut}(X,f) \). The item \( \varepsilon \) shows the Nos. of the Enriques involutions given in Table 5.1.

6. The two most algebraic Enriques surfaces

In this section, we study the two most algebraic Enriques surfaces, that is, Enriques surfaces covered by the singular K3 surface \( X_7 \) of discriminant 7.

We recall that the Néron–Severi lattice and the automorphism group of \( X_7 \) were determined by Ujikawa [34]. Elliptic fibrations on \( X_7 \) were studied by Harrache–Lecacheux [10] and Lecacheux [17].

6.1. Conjugacy classes of Enriques involutions. We exemplify Theorem 3.1.9 for the case \( X_7 \). Let \( T = TX_7 = [2, 1, 4] \) and \( S = S_X \). Let \( \iota \in I_X \), and put \( N := |\iota|\perp(-1) \). Let \( q := q(T) = \langle \frac{7}{2} \rangle \), so that \( q(S) \cong -q \cong \langle \frac{7}{2} \rangle \). In the notation of Proposition 2.5.1, the subgroup \( H \subset q([\iota]) \) must be trivial, so \( N \) is an even lattice of genus \( g(10, 0, u_1^{105} \oplus q) \). By Lemma 2.4.1, \( N \cong N'(2) \), with \( N' \) an even lattice of genus \( g(10, 0, q) \).

Lemma 6.1.1. The genus \( g(10, 0, q) \) contains exactly two isomorphism classes, namely \( N_{10,7}^{242} \) and \( N_{10,7}^{144} \) (see [31]).

Proof. Let \( N' \) be a lattice in this genus. The smallest lattice with bilinear form \( b = -b(q) \) is the odd lattice \( M_{3,7} := [2, 1, 2, 1, 1, 3] \), which is unique in its genus. Thus, by [20], \( N' \cong [\iota]^{\perp} \) for some primitive embedding \( \iota : M_{3,7} \hookrightarrow L \) into a unimodular lattice \( L \) of rank 13. Inspecting all such embeddings, we find exactly two non-isomorphic even orthogonal complements.

By Proposition 2.5.2, for both \( N = N_{10,7}^{242}(2) \) and \( N = N_{10,7}^{144}(2) \), the set \( I_{X_7}(N) \) has exactly one \( \text{O}(S) \)-orbit. Thus, \( r = 2 \) in Theorem 3.1.9. Since \( \text{O}(g(S), \omega_{X_7}) = \text{O}(q(S)) \), there is exactly 1 double coset in both cases. Hence, \( X_7 \) admits exactly
two Enriques involutions up to conjugation in aut(\(X\)). The two involutions can be distinguished by the number of \((-4)\)-vectors in the orthogonal complements of their fixed lattices.

6.2. Models of the two Enriques quotients. By the results of Section 5.4, the two quotients \(Y_1\) and \(Y_{11}\) of \(X_7\) have Nikulin-Kondo type I and II. Kondo [15] gives two explicit 1-dimensional families containing all Enriques surfaces of Nikulin-Kondo type I and II. Each family depend on one parameter \(\alpha\); in this section we determine which values of \(\alpha\) give \(Y_1\) and \(Y_{11}\). We first summarize Kondo’s construction.

Let \(\phi\) be the involution on \(\mathbb{P}^1 \times \mathbb{P}^1\) defined by
\[
([u_0, u_1], [v_0, v_1]) \mapsto ([u_0, -u_1], [v_0, -v_1]),
\]
and consider the curves \(L_1: u_0 = u_1, L_2: u_0 = -u_1, L_3: v_0 = v_1, L_4: v_0 = -v_1\). Let \(C\) be a curve of bidegree \((2, 2)\), defined by a polynomial \(f(u_0, u_1, v_0, v_1)\), which is invariant with respect to \(\phi\), and consider the divisor \(B = C + \sum L_i\).

Let \(\pi_i: X \to \mathbb{P}^1 \times \mathbb{P}^1\) be the minimal resolution of the double covering ramified over \(B\). In Kondo’s families, \(C\) is chosen so that \(X\) is a K3 surface and \(\phi\) lifts to an Enriques involution \(\bar{\varepsilon}\) of \(X\). We let \(Y\) be the quotient of \(X\) by \(\bar{\varepsilon}\).

For \(i = 1, 2\), the composite morphism \(\pi_i = \text{pr}_i \circ \pi_i: X \to \mathbb{P}^1\) is an elliptic fibration on \(X\), which induces an elliptic fibration \(\bar{\pi}_i\) on \(Y\). There is a third elliptic fibration \(\bar{\pi}_3: X \to \mathbb{P}^1\), one of whose fiber is the strict transform of \(C\) on \(X\). The half pencils of \(\bar{\pi}_3\): \(Y \to \mathbb{P}^1\) correspond to the fibers over \(C\) and over \(\sum L_i\).

For \(i = 1, 2, 3\), we choose coordinates so that the half-pencils of \(\bar{\pi}_i\) are mapped to \([0, 1], [1, 0] \in \mathbb{P}^1\). The image of the morphism \(\bar{\pi}_1 \times \bar{\pi}_2 \times \bar{\pi}_3: X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\) is then defined by the tridegree \((2, 2, 2)\) polynomial
\[
(u_0^2 - u_1^2)(v_0^2 - v_1^2)w_0^2 = f(u_0, u_1, v_0, v_1)w_0^2.
\]
Consider the Segre embedding \(\Sigma: \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^7\), defined by
\[
([u_0, u_1], [v_0, v_1], [w_0, w_1]) \mapsto [x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7] = [u_0v_0w_0u_0v_1w_1, u_0v_0w_1u_1v_1w_0, u_0v_0w_1u_1v_1w_1, u_0v_1w_0u_1v_0w_1, u_0v_1w_0u_1v_1w_1, u_0v_1w_1u_1v_0w_0, u_0v_1w_1u_1v_1w_1].
\]
The involution on \(\mathbb{P}^7\) given by \([x_0, \ldots, x_7] \mapsto [x_0, \ldots, x_3, -x_4, \ldots, -x_7]\) induces the Enriques involution \(\bar{\varepsilon}\) on \(X\). Hence, we have the following commuting diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\bar{\pi}_1 \times \bar{\pi}_2 \times \bar{\pi}_3} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\
\downarrow & & \downarrow \Sigma \\
Y & \xrightarrow{\text{pr}_{0123}} & \mathbb{P}^3
\end{array}
\]
where \(\text{pr}_{0123}\) is the projection \([x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7] \mapsto [x_0, x_1, x_2, x_3]\). Note that the half-pencils on \(Y\) are mapped onto the coordinate tetrahedron in \(\mathbb{P}^3\), so the image of \(Y\) in \(\mathbb{P}^3\) is defined by an Enriques sextic surface, i.e. a non-normal surface of degree 6 in \(\mathbb{P}^3\) that passes doubly through the edges of the coordinate tetrahedron (see [8]).

6.2.1. Nikulin-Kondo type I. For \(\alpha \in \mathbb{C} \setminus \{1, \frac{1}{2}, \frac{3}{2}\}\), let \(C\) be the curve defined by
\[
C: (2u_0^2 - u_1^2)(v_0^2 - v_1^2) = (2\alpha v_0^2 + (1 - 2\alpha)v_1^2)(u_0^2 - u_1^2).
\]
Put $B = C + \sum_{i=1}^{4} L_i$. Then, the minimal resolution of the double covering of \( \mathbb{P}^1 \times \mathbb{P}^1 \) ramified over $B$ is a K3 surface $X$ endowed with an Enriques involution $\tilde{\epsilon}$ such that the quotient $X/\langle \tilde{\epsilon} \rangle$ has Nikulin-Kondo type I.

Consider the curves
\[
Q_1: u_1 v_0 + u_0 v_1 = 0; \quad Q_2: u_1 v_0 - u_0 v_1 = 0; \\
Z: (u_0 + 3 u_1) v_0^2 + (3 u_0 + u_1) v_1^2 = 0.
\]

The curve $Z$ intersects $Q_1$ and $Q_2$ in one point with multiplicity 3, and intersects $C$ with even multiplicities if and only if
\[
\alpha = \frac{15}{16} \quad \text{or} \quad \alpha = \frac{17}{16}.
\]
(The two cases differ only by a relabeling of the variables.)

In these cases, consider the sublattice $S' \subset S_X$ generated by the classes of the strict transforms of $C, L_1, \ldots, L_4, Q_1, Q_2, Z$ and of the exceptional divisors. Then, rank $S' = 20$ and $\det S' = 7$, hence the same holds for $S_X$. This implies that $X$ is isomorphic to $X_7$, so the quotient $X/\langle \tilde{\epsilon} \rangle$ is isomorphic to $Y_1$.

An Enriques sextic model for $Y_1$ is given by
\[
(2 \alpha - 2) x_0^2 x_1^2 x_2^2 + x_0^2 x_1 x_2 x_3^2 + x_0 x_2^2 x_3^2 + (2 \alpha - 2) x_1^2 x_2^2 x_3^2 = 0.
\]

6.2.2. \textit{Nikulin-Kondo type II}. For $\alpha \in \mathbb{C} \setminus \{0, -1\}$, let $C$ be the curve defined by
\[
C: (v_0^2 - v_1^2) u_0^2 - (v_0^2 + \alpha v_1^2) v_1^2 = 0.
\]

Put $B = C + \sum_{i=1}^{4} L_i$. Then, the minimal resolution of the double covering of \( \mathbb{P}^1 \times \mathbb{P}^1 \) ramified over $B$ is a K3 surface $X$ endowed with an Enriques involution $\tilde{\epsilon}$ such that the quotient $X/\langle \tilde{\epsilon} \rangle$ has Nikulin-Kondo type II.

Consider the curves
\[
F_1: u_1 = 0; \quad F_2: v_1 = 0; \\
Z: (u_0 - u_1) v_0 + (u_0 + 3 u_1) v_1 = 0
\]

The curve $Z$ intersects $C$ in a third point of multiplicity 2 exactly when
\[
\alpha = 63.
\]

In this case, consider the sublattice $S' \subset S_X$ generated by the classes of the strict transforms of $C, L_1, \ldots, L_4, F_1, F_2, Z$ and of the exceptional divisors. Then, rank $S' = 20$ and $\det S' = 7$, hence the same holds for $S_X$. This implies that $X$ is isomorphic to $X_7$, so the quotient $X/\langle \tilde{\epsilon} \rangle$ is isomorphic to $Y_{11}$.

An Enriques sextic model for $Y_{11}$ is given by
\[
-x_0^2 x_1^2 x_2^2 + x_0^2 x_1 x_2 x_3^2 + x_0 x_2^2 x_3^2 + \alpha x_1^2 x_2^2 x_3^2 = x_0 x_1 x_2 x_3 (x_0^2 - x_1^2 - x_2^2 + x_3^2).
\]

\textbf{Acknowledgements.} Both authors warmly thank Hisanori Ohashi and the other organizers of the 3rd edition of the Japanese-European Symposium on Symplectic Varieties and Moduli Spaces at Tokyo University of Science in August 2018, where their collaboration started.

The first author would like to thank Igor Dolgachev, Shigeyuki Kondo and Shigeru Mukai for discussions.

The second author would like to thank Fabio Bernasconi, Chiara Camere, Alberto Cattaneo, Alex Degtyarev, Dino Festi, Grzegorz Kapustka, Roberto Laface
and Matthias Schütt for their support and interest in this work. A special acknowledgement goes to Simon Brandhorst for his help with sage.

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