ON AN ENRIQUES SURFACE ASSOCIATED WITH A QUARTIC HESSIAN SURFACE: COMPUTATIONAL DATA

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1. Introduction

We explain the contents of the text file `compdataEnriquesQH.txt`, which presents the computational data for the results of the paper [1]. This text file is available from

`http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3.html`

as a zipped file. The items below can also be obtained separately from the folder

`EnriquesQHFolder`,

whose `zip` file is also at the webpage above.

2. The data

We use the notions and notation of [1]. In particular, we use the bases of the lattices L_{26}, S_X, and $L_{10} = S_Y$ that are fixed in the paper [1].

2.1. The data on L_{10} and L_{26}.

- `GramL10` is the Gram matrix of L_{10}, which is the standard Gram matrix of $U \oplus E_8$.
- `WeylVectorL10` is the Weyl vector w_{10} of L_{10}.
- `WallsVinberg` is the list of the primitive defining vectors $e_1, \dots, e_{10}$ of the walls of the Vinberg chamber $D_{10} = V_0$ corresponding to the Weyl vector `WeylVectorL10`.
- `BasisLeechGolay` is the basis of the Leech lattice Λ (Table 3.1 of [1]).
- `GramL26` is the Gram matrix of $L_{26} = U \oplus \Lambda$.

2.2. The data on S_X and D_X.

- `EmbSXinL26` is the 16×26 matrix M such that $v \mapsto vM$ is the primitive embedding of S_X into L_{26}.
- `ProjL26toSX` is the 26×16 matrix N such that $v \mapsto vN$ is the orthogonal projection $\mathrm{pr}_S \colon L_{26} \otimes \mathbb{R} \to S_X \otimes \mathbb{R}$.
- `GramSX` is the Gram matrix of S_X.

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- `DiscGroupSX` is $[2, 2, 2, 6]$, which describes the discriminant group

$$S_X^\vee/S_X \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \tag{2.1}$$

of S_X.
- `DiscFormSX` is the discriminant form q_{S_X} of S_X. We fix a basis a_1, a_2, a_3, a_4 of $S_X^\vee/S_X$ that gives the isomorphism (2.1); that is, a_i is the generator of the ith cyclic factor in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. `DiscFormSX` is a 4×4 matrix whose ith diagonal component is $q_{S_X}(a_i) \in \mathbb{Q}/2\mathbb{Z}$, and whose off-diagonal (i, j)-component is $b_{S_X}(a_i, a_j) \in \mathbb{Q}/\mathbb{Z}$, where

$$b_{S_X}(x, y) := \frac{1}{2}(q_{S_X}(x + y) - q_{S_X}(x) - q_{S_X}(y)).$$

- `ProjDiscFormSX` is the 16×4 matrix P that gives the natural projection

$$\mathrm{pr}_{S_X^\vee} : S_X^\vee \to S_X^\vee/S_X.$$

We write an element $v \in S_X^\vee$ as a vector of $S_X \otimes \mathbb{Q}$ with respect to the fixed basis of S_X. Then

$$vP = [x_1, x_2, x_3, x_4] = x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4$$

is equal to $\mathrm{pr}_{S_X^\vee}(v)$.
- `LiftDiscFormSX` is the 4×16 matrix Q whose ith row vector is an element v_i of $S_X^\vee \subset S_X \otimes \mathbb{Q}$ such that $\mathrm{pr}_{S_X^\vee}(v_i) = a_i \in S_X^\vee/S_X$.

Using $P =$ `ProjDiscFormSX` and $Q =$ `LiftDiscFormSX`, we can compute the natural homomorphism

$$\eta_{S_X} : \mathrm{O}(S_X) \to \mathrm{O}(q_{S_X}).$$

If an element $g \in \mathrm{O}(S_X)$ is given as a 16×16 matrix M with integer components, then the 4×4 matrix QMP gives the image $\eta_{S_X}(g) \in \mathrm{O}(q_{S_X})$.

- `alphas` is the list A.
- `betas` is the list B. The ith element of `betas` is the complement of the ith element of `alphas` in $\{1, \ldots, 5\}$.
- `alphasbetas` is the concatenation of the lists A and B. If $i \le 10$, then the ith element of `alphasbetas` is an element α_i of `alphas`. If $i > 10$, then the ith element is the complement of α_{i-10} in $\{1, \ldots, 5\}$.
- `hQ` is $h_Q \in S_X$.
- `hX` is $h_X \in S_X$.
- `autDX` is the subgroup $\mathrm{aut}(D_X) \subset \mathrm{O}^+(S_X)$, which is the list of 240 square matrices of size 16 belonging to $\mathrm{O}^+(S_X)$.
- `WallsOfDXTypea` is the list of vectors $[E_\alpha] \in S_X$ and $[L_\beta] \in S_X$, which define the outer walls (walls of type (a)) of D_X. These vectors are sorted according to the list of indices `alphasbetas`.

- `WallsOfDXTypeb` is the list of primitive defining vectors v_α of inner walls of D_X of type (b). These vectors are sorted according to the list of indices `alphas`.
- `WallsOfDXTypec` is the list of primitive defining vectors of inner walls of D_X of type (c).
- `WallsOfDXTyped` is the list of primitive defining vectors of inner walls of D_X of type (d).
- `OuterReflectsDX` is the list of reflections with respect to the defining roots of the outer walls of D_X (that is, the elements of `WallsOfDXTypea`). This list is sorted according to `WallsOfDXTypea`.
- `InvolsAutXTypeb` is the list of lists of involutions in $\operatorname{aut}(X)$ that map D_X to the induced chamber adjacent to D_X across a wall of type (b). This list is sorted according to `WallsOfDXTypeb`. Each item of `InvolsAutXTypeb` is a list consisting of two matrices belonging to $\mathrm{O}^+(S_X)$, the first of which is the involution g_α we constructed in Proposition 6.8 of [1], and the second of which is the involution $g_\alpha g_\varepsilon = g_\varepsilon g_\alpha$.
- `InvolsAutXTypec` is the list of lists of involutions in $\operatorname{aut}(X)$ that map D_X to the induced chamber adjacent to D_X across a wall of type (c). This list is sorted according to `WallsOfDXTypec`. Each item of `InvolsAutXTypec` is a list consisting of only one matrix.
- `InvolsAutXTyped` is the list of lists of involutions in $\operatorname{aut}(X)$ that map D_X to the induced chamber adjacent to D_X across a wall of type (d). This list is sorted according to `WallsOfDXTyped`. Each item of `InvolsAutXTyped` is a list consisting of only one matrix.

2.3. **The data on S_Y and D_Y.**

- `EnriquesInvol` is the matrix representation g_ε of the Enriques involution $\varepsilon\colon X \to X$.
- `SXplus` is the basis of the sublattice S_X^+ of S_X.
- `SXminus` is the basis of the sublattice S_X^- of S_X.
- `EmbSYinSX` is the matrix M such that $v \mapsto vM$ is the embedding of S_Y into S_X. This matrix is identical with `SXplus`.
- `MinusFourVectorsInSXminus` is the list of all vectors $t \in S_X^-$ such that $\langle t, t\rangle = -4$. This list consists of 72 vectors, and each of them is written as a row vector with respect to the basis of S_X (not of S_X^-).
- `ProjSXtoSY` is the 16×10 matrix M such that $v \mapsto vM$ is the orthogonal projection $\mathrm{pr}^+\colon S_X \otimes \mathbb{R} \to S_Y \otimes \mathbb{R}$.
- `GramSY` is the Gram matrix of $S_Y = S_X^+(1/2)$. By the choice of the basis of S_Y, this Gram matrix is identical with `GramL10`.
- `hY` is $h_Y \in S_Y$.

- `OuterWallsOfDY` is the list of primitive defining vectors $u_\alpha := 2\,\mathrm{pr}^+([E_\alpha]) = 2\,\mathrm{pr}^+([L_{\bar\alpha}])$ of the outer walls of D_Y. These vectors are sorted according to the list of indices `alphas`.
- `InnerWallsOfDY` is the list of primitive defining vectors $\bar v_\alpha := 2\,\mathrm{pr}^+(v_\alpha)$ of the inner walls of D_Y. These vectors are sorted according to the list of indices `alphas`.
- `WallsOfDY` is the concatenation of `OuterWallsOfDY` and `InnerWallsOfDY`. This list is useful in presenting the lists `FacesOfDY` and `FacesOfDYWithGeomData` below.
- `autDY` is the subgroup $\mathrm{aut}(D_Y)$ of $\mathrm{O}^+(S_Y)$, which is the list of 120 square matrices of size 10 belonging to $\mathrm{O}^+(S_Y)$.
- `OuterReflectsDY` is the list of reflections with respect to the defining roots of the outer walls of D_Y (that is, the elements of `OuterWallsOfDY`). This list is sorted according to `OuterWallsOfDY`.
- `InvolsAutY` is the list of involutions in $\mathrm{aut}(Y)$ that map D_Y to the induced chamber adjacent to D_Y across an inner wall of D_Y. This list is sorted according to `InnerWallsOfDY`. The involutions in this list generate the group $\mathrm{aut}(Y)$.
- `SmoothRationalCurvesOnY` consists of 46 lists. For $d = 1, \dots, 46$, the dth item of `SmoothRationalCurvesOnY` is the list of the classes of all smooth rational curves C on Y such that $\langle [C], h_Y\rangle = d$.

2.4. **Data of faces of D_Y with geometric data.**

- `FacesOfDY` is the list of faces of D_Y. Each item of this list is of the form

$$[n, \{i_1, \dots, i_m\}].$$

Let F be the face of D_Y corresponding to this item. Then n is the dimension of F, and the set $\{i_1, \dots, i_m\}$ indicates that the set of all walls of D_Y containing the face F consists of the i_νth member of `WallsOfDY` for $\nu = 1, \dots, m$.
- `FacesOfDYWithGeomData` is the list of faces of D_Y and their geometric data. Each item of this list is of the form

$$[\ n,\ \{i_1, \dots, i_m\},\ \texttt{geomdata}\ ].$$

Let F be the face of D_Y corresponding to this item. Then n and $\{i_1, \dots, i_m\}$ are the same as `FacesOfDY`.

If F is an ideal face, then `geomdata` is the following data that describe the elliptic fibration $\phi\colon Y \to \mathbb{P}^1$ corresponding to the face F.

$$[\ \texttt{"ellfib"},\ f,\ \texttt{types},\ [\texttt{Rfull}, \texttt{Rhalf}]\ ].$$

The first item is the string `"ellfib"`, which shows that F is an ideal face. The second item f is the primitive vector in S_Y such that $F = \mathbb{R}_{\ge 0}\, f$. Thus

$2f \in S_Y$ is the class of a fiber of ϕ. The third item `types` is an ordered pair of lists of indecomposable ADE-types, which indicate the ADE-type of non-multiple reducible fibers and of multiple reducible fibers. (For example, `types` $= [[$`"A5"`$,$ `"A1"`$], [\,]]$ means that ϕ has exactly two reducible fibers, both of which is non-multiple, one of which is of type A_5, and the other of which is of type A_1.) The first member `Rfull` of the fourth item [`Rfull`, `Rhalf`] is the data of reducible fibers of $\phi\colon Y \to \mathbb{P}^1$. The list `Rfull` consists of items

[`ADEtype`, `irreds`],

each of which describes a non-multiple reducible fiber. Here `ADEtype` is the indecomposable ADE-type of a non-multiple reducible fiber $\phi^{-1}(p)$ and `irreds` is the list of classes of irreducible components of $\phi^{-1}(p)$. The list `Rhalf` is the data of the divisors E such that $2E$ is a multiple reducible fiber of ϕ. The contents of `Rhalf` have the same structure and the meaning as those of `Rfull`.

If F is *not* an ideal face, then `geomdata` is the data

[`"RDPs"`, $\mathcal{G}(F)$, `types`, `singpts` , `ismaximal`],

which describe a birational morphism $\Phi_{|L_F|}\colon Y \to \overline{Y}$ to a surface $\overline{Y}$ with only rational double points such that the pull-back of the class of a hyperplane section of $\overline{Y}$ is a point of F that is not contained in any wall of F. The first item is the string `"RDPs"`, which shows that F is not an ideal face. The second item $\mathcal{G}(F)$ is the list of all $\bar{g} \in \mathrm{aut}(Y)$ such that $F \subset D_Y^{\bar{g}}$. The third item `types` is the list of indecomposable ADE-types that gives the ADE-type of the configuration of smooth rational curves contracted by $\Phi_{|L_F|}$. (Note that, if `types` is the empty list [ ], then F is an inner face. If, moreover, $n = \dim F$ is equal to 8, then we can obtain from the list $\mathcal{G}(F)$ the defining relation of $\mathrm{aut}(Y)$ with respect to the generators $\bar{g}(\alpha)$ ($\alpha \in A$) corresponding to the face F.) The fourth item `singpts` is the list that describes singular points of $\overline{Y}$, each item of which is the following data on a singular point $p \in \mathrm{Sing}(\overline{Y})$;

[`ADEtype`, `irreds`],

where `ADEtype` is the indecomposable ADE-type of the exceptional divisor over p, and `irreds` is the list of classes of the irreducible components of the exceptional divisor. The last item `ismaximal` is either

[`true`] or [`false`, ν].

Let $\mathcal{R}(F)$ denote the set of the classes of smooth rational curves contracted by $\Phi_{|L_F|}$; that is, $\mathcal{R}(F)$ is the union of the second items `irreds` of the items

[`ADEtype`, `irreds`] of all `singpts` in `geomdata`. If there exists another face F' of D_Y that satisfies

$$F \subset F', \quad F \neq F', \quad \mathcal{R}(F) = \mathcal{R}(F'),$$

then `ismaximal` is [`false`, ν], and an example of such a face F' is given by the ν th element of the list `FacesOfDYWithGeomData`. Otherwise, `ismaximal` is [`true`].

- `autYClassesOfFacesOfDY` is the list of $\mathrm{aut}(Y)$-equivalence classes of faces of D_Y. Each item of this list is of the form

$$[\ n,\ \{k_1, \dots, k_N\}\ ],$$

where n is the dimension of the faces in this class, and $\{k_1, \dots, k_N\}$ indicates that this $\mathrm{aut}(Y)$-equivalence class consists of the k_νth member of `FacesOfDY` for $\nu = 1, \dots, N$.

From these two lists `FacesOfDYWithGeomData` and `autYClassesOfFacesOfDY`, we can make Tables 1.1 and 1.2 of [1].

References

[1] Ichiro Shimada. On an Enriques surface associated with a quartic Hessian surface, preprint, 2016, `http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3.html`.

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