LATTICES OF ALGEBRAIC CYCLES ON FERMAT VARIETIES IN POSITIVE CHARACTERISTICS

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Dedicated to Professor Tetsuji Shioda for his 60th birthday

Abstract. Let X be the Fermat hypersurface of dimension $2m$ and of degree $q+1$ defined over an algebraically closed field of characteristic $p>0$, where q is a power of p, and let $NL^m(X)$ be the free abelian group of numerical equivalence classes of linear subspaces of dimension m contained in X. By the intersection form, we regard $NL^m(X)$ as a lattice. Investigating the configuration of these linear subspaces, we show that the rank of $NL^m(X)$ is equal to the $2m$ th Betti number of X, that the intersection form multiplied by $(-1)^m$ is positive definite on the primitive part of $NL^m(X)$, and that the discriminant of $NL^m(X)$ is a power of p. Let $\mathcal{L}^m(X)$ be the primitive part of $NL^m(X)$ equipped with the intersection form multiplied by $(-1)^m$. In the case $p=q=2$, the lattice $\mathcal{L}^m(X)$ is described in terms of certain codes associated with the unitary geometry over $\mathbb{F}_2$. The lattice $\mathcal{L}^2(X)$ is isomorphic to the laminated lattice of rank 22. This explains Conway's identification $\cdot 222 \cong PSU(6,2)$ geometrically. The lattice $\mathcal{L}^3(X)$ is of discriminant $2^{16}\cdot 3$, minimal norm 8, and kissing number 109421928.

1. Introduction

Let p be a prime integer, and $q=p^\nu$ a power of p, where ν is a positive integer. Let X be the Fermat hypersurface of dimension $2m$ and of degree $q+1$ defined over an algebraically closed field $\bar{k}$ of characteristic $p>0$; that is, the hypersurface X in $\mathbb{P}^{2m+1}$ is defined by the homogeneous equation

$$x_0^{q+1}+\cdots+x_{2m+1}^{q+1}=0.$$

Let $\mathrm{CH}^m(X)$ be the Chow group of algebraic cycles on X with codimension m modulo the rational equivalence. We have the intersection form on $\mathrm{CH}^m(X)$, which is $\mathbb{Z}$-valued, bilinear and symmetric. We define $N^m(X)$ to be the quotient group of $\mathrm{CH}^m(X)$ by the numerical equivalence. Here an element α of $\mathrm{CH}^m(X)$ is said to be numerically equivalent to zero if $\alpha.\beta=0$ holds for any $\beta\in\mathrm{CH}^m(X)$. Tate [16] proved that X is supersingular; that is, each element of the middle cohomology group $H^{2m}(X,\mathbb{Q}_l)(m)$ of X is represented by an element of $\mathrm{CH}^m(X)\otimes_{\mathbb{Z}}\mathbb{Q}_l$. (See also Shioda and Katsura [15].) In particular, the numerical equivalence on $\mathrm{CH}^m(X)$ coincides with the homological equivalence, and $N^m(X)$ is embedded in $H^{2m}(X,\mathbb{Q}_l)(m)$. Moreover, because the intersection form is $\mathbb{Z}$-valued, the subgroup $N^m(X)$ of $H^{2m}(X,\mathbb{Q}_l)(m)$ is finitely generated, and its rank is equal to the $2m$ th Betti number $b_{2m}(X)$ of X.

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On the other hand, the hypersurface X contains many linear subspaces of dimension m. The number of these linear subspaces is given by $\prod_{\nu=0}^{m}(q^{2\nu+1}+1)$ (cf. Corollary 2.22). Let $NL^m(X)$ be the subgroup of $N^m(X)$ generated by the numerical equivalence classes of these linear subspaces. Our first result is as follows.

Theorem 1.1. (1) *The rank of $NL^m(X)$ is equal to $b_{2m}(X)$.*

(2) *The signature of the intersection form on $NL^m(X)\otimes_{\mathbb{Z}}\mathbb{R}$ is $(1, b_{2m}(X)-1)$ when m is odd, while it is $(b_{2m}(X), 0)$ when m is even.*

(3) *The discriminant of the intersection form on $NL^m(X)$ is a power of p.*

This result not only gives a new proof to the supersingularity of X but also implies that $N^m(X)/NL^m(X)$ is at most a finite p-group. It is quite plausible that $N^m(X)$ actually coincides with $NL^m(X)$, but this conjecture is not yet verified. Shioda [13] studied the Néron-Severi group of a complex Fermat surface, and asked whether it is generated by the numerical equivalence classes of lines on the Fermat surface (cf. [13, Question 7.4]).

In $N^m(X)$, we have the numerical equivalence class h of the intersection of X with a general linear subspace of $\mathbb{P}^{2m+1}$ of dimension $m+1$. We define the primitive part $N^m_{\mathrm{prim}}(X)$ of $N^m(X)$ to be the orthogonal complement $(h)^{\perp}$ of h. The assertion (2) of Theorem 1.1 implies the following.

Corollary 1.2. *The intersection form multiplied by $(-1)^m$ is positive definite on $N^m_{\mathrm{prim}}(X)$.*

When X is a surface, this corollary follows from the Hodge index theorem, which is valid in any characteristics. Over the complex number field, the corresponding result follows immediately from the Hodge theory.

We put

$$\mathcal{L}^m(X) := (-1)^m(NL^m(X)\cap N^m_{\mathrm{prim}}(X)),$$

where the factor $(-1)^m$ means that the intersection form is multiplied by $(-1)^m$. Then $\mathcal{L}^m(X)$ is a positive definite lattice. Our second result describes the structure of this lattice when $p=q=2$.

Suppose that $p=q=2$. Let k be the finite field $\mathbb{F}_4$, and let T be the set of k-rational points of the projective space $\mathbb{P}^m$ defined over k. We identify the power set 2^T of T with the vector space $\mathbb{F}_2^T$ of $\mathbb{F}_2$-valued functions on T in such a way that a subset S of T corresponds to the function from T to $\mathbb{F}_2$ whose pre-image of $1\in\mathbb{F}_2$ is exactly S. Via this identification, an addition is defined on 2^T by the symmetric difference;

$$S_1+S_2 := (S_1\cup S_2)\setminus(S_1\cap S_2).$$

A square matrix $A=(a_{ij})$ of size $m+1$ with coefficients in k is said to be *Hermitian* if it satisfies ${}^tA=A^{(2)}$, where $A^{(2)} := (a_{ij}^2)$. For a Hermitian matrix A, let $Y_A(k)$ be the set of k-rational points of the subvariety Y_A of $\mathbb{P}^m$ defined by the homogeneous equation

$$\sum_{i,j=0}^{m} a_{ij}x_ix_j^2 = 0.$$

When A is the zero matrix O, we have $Y_O(k)=T$. Let R be the finite ring $\mathbb{Z}/(2^{m+1})$, and R^T the R-module of R-valued functions on T. For a subset S of T, we define $\overline{V}_S \in R^T$ to be the function given by

$$\overline{V}_S(H) := \begin{cases} 1 & \text{if } H \in S, \\ 0 & \text{if } H \in T \setminus S. \end{cases}$$

Let $\widetilde{T}$ be the disjoint union of T and $\{\varphi\}$, where φ is a formal element, and $R^{\widetilde{T}}$ the R-module of R-valued functions on $\widetilde{T}$. For a function $\bar{v} \in R^T$, we define its *extension* $\bar{v}^{\sim} \in R^{\widetilde{T}}$ to be the unique function from $\widetilde{T}$ to R that satisfies

$$\bar{v}^{\sim}|_T = \bar{v} \quad \text{and} \quad \bar{v}^{\sim}(\varphi) = \sum_{H \in T} \bar{v}(H).$$

For an R-submodule M of R^T, we define $M^{\sim}$ to be the set $\{\bar{v}^{\sim} : \bar{v} \in M\}$, which is obviously an R-submodule of $R^{\widetilde{T}}$ isomorphic to M as an R-module. Let $\mathbb{Z}^{\widetilde{T}}$ be the free abelian group of $\mathbb{Z}$-valued functions on $\widetilde{T}$. We equip $\mathbb{Z}^{\widetilde{T}}$ with a $\mathbb{Q}$-valued positive definite symmetric bilinear form defined by

$$(v, w)_T := \frac{1}{2^{m+1}} \Big(\sum_{H \in T} v(H)w(H) + 3\, v(\varphi)w(\varphi) \Big). \tag{1.1}$$

Definition 1.3. We define $\mathcal{H}_m$ to be the linear subspace of $\mathbb{F}_2^T$ generated by the subsets $Y_A(k)$ of T, where A runs through the set of all Hermitian matrices. Let $\overline{L}^m$ be the submodule of R^T generated by the vectors $\overline{V}_T$ and $2\,\overline{V}_S$, where S runs through $\mathcal{H}_m$. We define $\widetilde{L}^m$ to be the pull-back of the extension $(\overline{L}^m)^{\sim}$ of $\overline{L}^m$ by the natural homomorphism $\mathbb{Z}^{\widetilde{T}} \to R^{\widetilde{T}}$.

It turns out that the symmetric bilinear form $(\ ,\ )_T$ takes values in $\mathbb{Z}$ on $\widetilde{L}^m$, so that $\widetilde{L}^m$ becomes a lattice. Note that the rank $|T| + 1$ of $\widetilde{L}^m$ is equal to the rank $b_{2m}(X) - 1$ of $\mathcal{L}^m(X)$.

Theorem 1.4. *The lattice $\mathcal{L}^m(X)$ is isomorphic to $\widetilde{L}^m$.*

We will study the structure of $\mathcal{L}^m(X)$ in detail for $m \le 3$.

When $\dim X = 2$, the lattice $\mathcal{L}^1(X) \cong \widetilde{L}^1$ is easily seen to be isomorphic to the root lattice of type E_6. In other words, the primitive part of the Néron-Severi lattice of the cubic Fermat surface in characteristic 2 does not differ from that of a nonsingular cubic surface in characteristic 0, which has been studied for many years in the relation with the configuration of the twenty-seven lines on a cubic surface. (See Manin [10].)

Theorem 1.5. *Suppose that $\dim X = 4$. Then the lattice $\mathcal{L}^2(X)$ is isomorphic to the laminated lattice Λ_{22} of rank 22.*

See the book by Conway and Sloane [2, Chapter 6] for the definition of the laminated lattice Λ_{22}. This lattice is obtained as a section of the Leech lattice, and the subgroup $\cdot 222$ of the automorphism group $\cdot 0$ of the Leech lattice acts on Λ_{22} (cf. [2, Chapter 10]). In search for the explanation of Conway's identification $\cdot 222 \cong PSU_6(2)$ (cf. [2, Chapter 10, Table 10.4]), Edge [4] suggested that there should be some correspondence between the planes contained in the cubic Fermat 4-fold X in characteristic 2 and certain vectors in the Leech lattice, and presented some numerical evidences. Using this putative correspondence, Jónsson and McKay constructed an embedding of the Mathieu group M_{22} into $PSU_6(2)$ explicitly in [9].

In the course of the proof of Theorem 1.5, we construct an embedding of $\widetilde{L}^2$ into the Leech lattice. Combining this embedding with the isomorphism $\mathcal{L}^2(X) \cong \widetilde{L}^2$ in Theorem 1.4, we can make a correspondence between *pairs* of planes contained in X and certain vectors of the Leech lattice. This correspondence explains the numerical coincidences given by Edge [4].

Theorem 1.6. *Suppose that* $\dim X = 6$. *Then the lattice* $\mathcal{L}^3(X)$ *of rank* 86 *has discriminant* $2^{16} \cdot 3$, *minimal norm* 8, *and kissing number* 109421928.

Recall that the normalized center density of a positive definite lattice L is defined to be

$$(\operatorname{disc} L)^{-1/2} \cdot (N_{\min}/4)^{r/2},$$

where $\operatorname{disc} L$, $N_{\min}$ and r are the discriminant, the minimal norm and the rank of L, respectively. By Theorem 1.6, the normalized center density of $\mathcal{L}^3(X)$ is equal to $2^{35}/\sqrt{3} = 2^{34.2075\dots}$. The Minkowski-Hlawka theorem (cf. [2, Chapter 1]) says that there is a lattice of rank 86 with normalized center density at least

$$\zeta(86) \cdot 2^{-85} \cdot V_{86}^{-1} = 2^{19.3208\dots},$$

where ζ is the Riemann zeta function and V_{86} is the volume of the 86-dimensional unit ball. Thus the lattice $\mathcal{L}^3(X)$ gives us a new example of sphere packing whose center density exceeds the Minkowski-Hlawka bound (cf. [2, Chapter 1, Table 1.3]).

Computing the norms of geometrically natural generators of $\mathcal{L}^m(X)$ and looking at the results for $m \le 3$, we are led to a guess that the minimal norm of $\mathcal{L}^m(X)$ is 2^m for every m. Shioda [14] and, independently, Elkies [5, 6] obtained many lattices of high center density as Mordell-Weil lattices of elliptic surfaces in positive characteristics. The minimal norm of a Mordell-Weil lattice is easily calculated from geometric invariants of the elliptic surface (cf. [14, Section 2]). For our lattices, unfortunately, we do not know any geometric method for determining the minimal norm. The proof of Theorem 1.6 is carried out using a computer, and the computation becomes intractable when $m \ge 4$.

Theorems 1.1 and 1.4 are proved by looking at the configuration of m-dimensional linear subspaces contained in X. In order to study the configuration, we investigate the action of the projective automorphism group of X on the set of linear subspaces on X. Our new tool for carrying out this investigation is the notion of *p-quadric hypersurfaces* and *special linear subspaces* (cf. Definitions 2.1 and 2.11). The notion of p-quadric hypersurfaces has been introduced by Shimada in [12, Introduction] in the study of unirationality of complete intersections in positive characteristics. Some of the results about the configuration we prove in this paper have been obtained by Segre [11] in the context of Hermitian hypersurfaces over finite fields.

We give a brief outline of our paper. In Section 2, we prove elementary facts about p-quadric hypersurfaces, Hermitian hypersurfaces, and special linear subspaces. A nonsingular hypersurface of degree $q + 1$ is p-quadric if and only if it is projectively isomorphic to the Fermat hypersurface of the same degree (cf. Proposition 2.3), and any m-dimensional linear subspace contained in a nonsingular p-quadric hypersurface of dimension $2m$ is always a special linear subspace (cf. Corollary 2.17). Therefore we are allowed to replace the Fermat hypersurface by any nonsingular p-quadric hypersurface. In Section 3, we investigate the configuration of special linear subspaces on a nonsingular p-quadric hypersurface X_J of dimension $2m$ defined by the equation

$$\sum_{i=0}^{m}(x_i y_i^q - x_i^q y_i) = 0.$$

We show that, when $p = q = 2$, each m-dimensional linear subspace contained in X_J is labeled in a one-to-one way by a pair of a k-rational linear subspace of $\mathbb{P}^m$ and a Hermitian hypersurface on it (cf. Corollary 3.11). In Section 4, we prove Theorem 1.1 by writing down explicitly an embedding of $NL^m(X_J)$ into an $\mathbb{R}$-vector space of dimension $b_{2m}(X_J)$ equipped with a symmetric bilinear form of signature $(b_{2m}(X_J), 0)$ or $(1, b_{2m}(X_J)-1)$, according to the parity of m. In Section 5, we prove Theorem 1.4. Using the labeling obtained in Section 3, we assign to each generator of $\mathcal{L}^m(X_J)$ a vector of $\widetilde{L}^m$, and show that this assignment yields an isomorphism of lattices between $\mathcal{L}^m(X_J)$ and $\widetilde{L}^m$. In Section 6, we show that the linear code $\mathcal{H}_2$ of length 21 is related to a certain subcode of the Golay code $\mathcal{C}_{24}$, and construct an embedding of $\widetilde{L}^2$ into the Leech lattice, whose image is the laminated lattice Λ_{22}. This proves Theorem 1.5. In Section 7, we evaluate the discriminant, the minimal norm and the kissing number of the lattice $\widetilde{L}^3$. In the last section, we give a geometric explanation for the kissing number of $\widetilde{L}^m \cong \mathcal{L}^m(X)$ for $m = 1, 2, 3$, and present a conjectural formula of the kissing number of $\widetilde{L}^m$ for general m.

After the first version of this paper was finished, the author was informed that Dummigan and Tiep [3] have also considered the lattices $N^m_{\mathrm{prim}}(X)$ from a group-theoretic point of view using an idea of Gross [8].

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Conventions.

(1) We fix a prime integer p and its power $q = p^\nu$, where ν is a positive integer. (From Section 5 onwards, we put $p = q = 2$.) Let k be the finite field $\mathbb{F}_{q^2}$, and $\bar{k}$ its algebraic closure.

(2) For an algebraic variety V defined over a field K, we denote by $V(K)$ the set of K-rational points of V.

(3) The homogeneous coordinates of a projective space $\mathbb{P}^n$ are expressed as a *column* vector ${}^t(x_0, \dots, x_n)$, so that the group $GL(n+1)$ of $(n+1) \times (n+1)$ invertible matrices acts on $\mathbb{P}^n$ from the left.

(4) We consider the empty set $\emptyset$ as a linear subspace of $\mathbb{P}^n$ with dimension -1, and understand that every subvariety of $\mathbb{P}^n$ contains $\emptyset$.

(5) Let $r = p^\mu$ be a power of p, where $\mu \in \mathbb{Z}$. For a matrix $\Gamma = (\gamma_{ij})$ with coefficients in $\bar{k}$, we denote by $\Gamma^{(r)}$ the matrix (γ_{ij}^r). Then we have $(\Gamma_1 \cdot \Gamma_2)^{(r)} = \Gamma_1^{(r)} \cdot \Gamma_2^{(r)}$. The transpose of a matrix Γ is denoted by ${}^t\Gamma$.

(6) For a set S, the cardinality of S is denoted by $|S|$. For an abelian group A, the A-module of A-valued functions on S is denoted by A^S. The power set 2^S is identified with $\mathbb{F}_2^S$ in such a way that the addition of subsets of S is defined by the

symmetric difference. The disjoint union of two disjoint sets S_1 and S_2 is written by $S_1 \sqcup S_2$.

2. Projective geometry of p-quadric hypersurfaces

Let n be a positive integer. We denote by $M(n+1, K)$ the set of square matrices of size $n+1$ with coefficients in a field K, where K is k or $\bar{k}$.

First we work over $\bar{k}$. We consider the projective space $\mathbb{P}^n$ of dimension n defined over $\bar{k}$ with homogeneous coordinates $x = {}^t(x_0, \dots, x_n)$. For a non-zero square matrix $A = (a_{ij}) \in M(n+1, \bar{k})$, we define a homogeneous polynomial f_A of $2n+2$ variables by

$$f_A(x, y) := {}^t x \cdot A \cdot y^{(q)} = \sum_{i,j=0}^{n} a_{ij} x_i y_j^q,$$

where $x = {}^t(x_0, \dots, x_n)$ and $y = {}^t(y_0, \dots, y_n)$. We denote by X_A the hypersurface of degree $q+1$ defined in $\mathbb{P}^n$ by the homogeneous equation

$$f_A(x, x) = 0.$$

When A is the identity matrix I, the hypersurface X_I is nothing but the Fermat hypersurface of degree $q+1$.

Definition 2.1. We say that a hypersurface of $\mathbb{P}^n$ is a *p-quadric hypersurface* if it is written as X_A by some non-zero matrix $A \in M(n+1, \bar{k})$.

Note that the definition of p-quadric hypersurfaces does not depend on the choice of homogeneous coordinates of $\mathbb{P}^n$, because we have $g^{-1}(X_A) = X_{A'}$ for any $g \in GL(n+1, \bar{k})$, where $A' = {}^t g A g^{(q)}$. We define an action ρ of the group $GL(n+1, \bar{k})$ on $M(n+1, \bar{k})$ from the left by

$$\rho(g^{-1})(A) := {}^t g A g^{(q)}, \tag{2.1}$$

so that $g(X_A) = X_{\rho(g)(A)}$ holds for any $g \in GL(n+1, \bar{k})$. The following lemma will be used frequently throughout this paper.

Lemma 2.2. *Let Z be a reduced irreducible locally closed subvariety of $GL(n+1, \bar{k})$, and let H be a connected reduced algebraic subgroup of $GL(n+1, \bar{k})$. Suppose that $\rho(h)(Z) \subseteq Z$ holds for any $h \in H$. If $\dim Z \leq \dim H$, then the action of $H(\bar{k})$ on the set $Z(\bar{k})$ by ρ is transitive.*

Proof. For a point $A \in Z(\bar{k})$, we define a morphism $\psi_A : H \to GL(n+1, \bar{k})$ by $\psi_A(h) := \rho(h)(A)$. Let ϵ be the dual number; $\epsilon^2 = 0$. Then we have

$$\psi_A(I + \epsilon a) - \psi_A(I) = -\epsilon\, {}^t a A$$

for all $a \in \mathrm{Lie}(H) \subseteq M(n+1, \bar{k})$ because $(I + \epsilon a)^{-1} = I - \epsilon a$ and $(I - \epsilon a)^{(q)} = I$. Since A is invertible, we see that ψ_A is an immersion locally at $I \in H$. On the other hand, the image of ψ_A is in Z by assumption. Hence $\dim Z \leq \dim H$ implies that $\dim Z = \dim H$, and that ψ_A is dominant onto Z. In particular, for any two points A and A' of $Z(\bar{k})$, we have $\mathrm{Im}\, \psi_A \cap \mathrm{Im}\, \psi_{A'} \neq \emptyset$, which means that there exist elements $h, h' \in H(\bar{k})$ such that $\rho(h)(A) = \rho(h')(A')$. Hence we have $\rho(h'^{-1} h)(A) = A'$. *q.e.d.*

Note that X_A is nonsingular if and only if $\det A \neq 0$. Applying Lemma 2.2 to $Z = GL(n+1, \bar{k})$ and $H = GL(n+1, \bar{k})$, we obtain the following proposition, which is a small part of the main result of Beauville [1].

Proposition 2.3. *A nonsingular hypersurface of degree $q+1$ is p-quadric if and only if it is projectively isomorphic over $\bar{k}$ to the Fermat hypersurface X_I of degree $q+1$.*

By virtue of this proposition, we are allowed to replace the Fermat hypersurface of degree $q+1$ by an arbitrary nonsingular p-quadric hypersurface in the proof of our main results.

Next we recall the definition of Hermitian hypersurfaces, which is a notion over the finite field k of q^2 elements.

Definition 2.4. Suppose that $\mathbb{P}^n$ is defined over the finite field k. A p-quadric hypersurface X_A is said to be *Hermitian* if ${}^tA = A^{(q)}$ holds. In this case, we have $A^{(q^2)} = A$, and hence X_A is defined over k. We define the rank of a Hermitian hypersurface X_A to be the rank of A. By abuse of language, we define a Hermitian hypersurface of rank 0 to be the whole projective space $\mathbb{P}^n$.

Let $H(n+1, r)$ be the set of matrices $A \in M(n+1, k)$ satisfying ${}^tA = A^{(q)}$ and $\operatorname{rank} A = r$. Note that, if $A \in H(n+1, r)$, then ${}^tgAg^{(q)}$ is also an element of $H(n+1, r)$ for any linear transformation $g \in GL(n+1, k)$ with coefficients in k. Hence the finite group $GL(n+1, k)$ acts on the set $H(n+1, r)$ by ρ. Let $GU(r, k)$ be the finite group $\{g \in GL(r, k) \,:\, {}^tgg^{(q)} = I\}$. We understand that $GL(0, k)$ and $GU(0, k)$ are the group of order 1. We have

$$|GL(r,k)| = \prod_{j=0}^{r-1}(q^{2r} - q^{2j}) \quad \text{and} \quad |GU(r,k)| = q^{(r-1)r/2} \cdot \prod_{j=1}^{r}(q^j - (-1)^j).$$

The following is due to Segre [11, n. 3].

Proposition 2.5. (1) *The action of $GL(n+1, k)$ on $H(n+1, r)$ by ρ is transitive for each r.*

(2) *The stabilizer subgroup of an element of $H(n+1, r)$ in $GL(n+1, k)$ is of order $|GU(r, k)| \cdot |GL(n+1-r, k)| \cdot q^{2r(n+1-r)}$.*

Proof. Suppose that $r \neq 0$. Let A be an element of $H(n+1, r)$. Then there exist vectors $v, w \in k^{n+1}$ such that $f_A(v, w) \neq 0$. Because the homomorphism $k \to \mathbb{F}_q$ of additive groups given by $z \mapsto z + z^q$ is surjective, there is a linear combination $u = \lambda v + \mu w$ ($\lambda, \mu \in k$) such that $f_A(u, u) \neq 0$. Because the homomorphism $N : k^\times \to \mathbb{F}_q^\times$ of multiplicative groups given by $z \mapsto z^{q+1}$ is also surjective, there is a multiplicative constant $\gamma \in k^\times$ such that $u_0 := \gamma u$ satisfies $f_A(u_0, u_0) = 1$. The k-rational linear subspace $(u_0)^\perp := \{x \in k^{n+1} \,:\, f_A(u_0, x) = f_A(x, u_0) = 0\}$ of k^{n+1} is of codimension 1, and the restriction of f_A to $(u_0)^\perp$ is of rank $r-1$. Hence the assertion (1) is proved by induction on n. The assertion (2) is obvious. *q.e.d.*

Let H_{n+1} be the set of matrices $A \in M(n+1, k)$ satisfying ${}^tA = A^{(q)}$. Then H_{n+1} carries a natural structure of the vector space over the subfield $\mathbb{F}_q$ of k such that $\dim_{\mathbb{F}_q} H_{n+1} = (n+1)^2$. The set of Hermitian hypersurfaces in $\mathbb{P}^n$ is then identified with $H_{n+1}/\mathbb{F}_q^\times$.

Corollary 2.6. (1) *The group $GL(n+1,k)$ acts on the set of Hermitian hypersurfaces of rank r transitively for each r.*

(2) *The number $h(n,r)$ of Hermitian hypersurfaces of rank r in $\mathbb{P}^n$ is*

$$\frac{1}{q-1}\cdot q^{(r-1)r/2}\cdot\prod_{j=1}^{r}\Big(\frac{q^{2(n+1-r)+2j}-1}{q^j-(-1)^j}\Big)$$

if $r>0$, while it is 1 if $r=0$.

We shall study the set $X_A(k)$ of k-rational points of a Hermitian hypersurface X_A. The following is also due to Segre [11, n. 30].

Proposition 2.7. *Suppose that X_A is a Hermitian hypersurface of rank r. Then $|X_A(k)|$ is equal to $F(n,r)$, where*

$$F(n,r):=\frac{q^{2n-2r+2}-1}{q^2-1}+q^{2n-2r+1}\cdot\Big(\frac{q^{2r}-1}{q^2-1}+\frac{(-q)^r-1}{q+1}\Big).$$

Proof. When $r=0$, we have $X_A(k)=\mathbb{P}^n(k)$, and hence the assertion holds obviously. Suppose that $r>0$. By Corollary 2.6, the hypersurface X_A is projectively isomorphic over k to the hypersurface defined by

$$x_0^{q+1}+\cdots+x_{r-1}^{q+1}=0,$$

which is a cone over the Fermat hypersurface Y of degree $q+1$ in $\mathbb{P}^{r-1}$ with vertex being a k-rational linear subspace of dimension $n-r$. Therefore we have

$$|X_A(k)|=|Y(k)|\cdot|\mathbb{A}^{n-r+1}(k)|+|\mathbb{P}^{n-r}(k)|,$$

where $\mathbb{A}^{n-r+1}$ is the affine space of dimension $n-r+1$ defined over k. The number of k-rational points of the Fermat hypersurface is classically known. We use Edge's argument (Edge [4]) to calculate $|Y(k)|$. Let $\nu(t)$ be the number of t-tuples $(\zeta_1,\dots,\zeta_t)$ of elements of $\mathbb{F}_q^\times$ satisfying $\zeta_1+\cdots+\zeta_t=0$, which is determined by the initial condition $\nu(0)=1$ and the recursive relation

$$q^{t-1}=\sum_{s=0}^{t}\binom{t}{s}\nu(s).$$

Because each fiber of the norm map $N: k^\times\to\mathbb{F}_q^\times$ consists of $q+1$ elements, we have

$$|Y(k)|=\frac{1}{q^2-1}\sum_{s=1}^{r}\binom{r}{s}(q+1)^s\nu(s)=\frac{1}{q}\Big(\frac{q^{2r}-1}{q^2-1}+\frac{(-q)^r-1}{q+1}\Big).$$

Thus we obtain the formula for $|X_A(k)|$. *q.e.d.*

Proposition 2.8. *Let X_A and X_B be two Hermitian hypersurfaces in $\mathbb{P}^n$. If $X_A(k)=X_B(k)$, then there is a non-zero scalar $\lambda\in\mathbb{F}_q^\times$ such that $A=\lambda B$. In other words, the Hermitian hypersurface is determined by the set of its k-rational points.*

Proof. Suppose that $n=1$. Using Proposition 2.7, we can check that $|X_A(k)|=|X_B(k)|$ implies $\operatorname{rank}A=\operatorname{rank}B$. In particular, the assertion is proved when $\operatorname{rank}A=0$. Suppose that A is of positive rank. By Corollary 2.6, the defining equation of X_A can be written as either $x_0^{q+1}=0$ or $x_0^{q+1}+x_1^{q+1}=0$, if we choose appropriate k-rational homogeneous coordinates ${}^t(x_0,x_1)$ of $\mathbb{P}^1$. It follows that $X_A(k)=X_A(\bar{k})$. (Note that the equation $x^{q+1}+1=0$ has $q+1$ distinct roots in the finite field k.) Hence we have $X_A(\bar{k}) = X_B(\bar{k})$. Therefore f_A and f_B are proportional over $\bar{k}$. Because A and B are Hermitian, this implies that A and B are proportional over $\mathbb{F}_q$. When $n \geq 2$, the restrictions of f_A and f_B to any k-rational linear subspace of k^{n+1} of dimension 2 are proportional over $\mathbb{F}_q$ by the above observation, whence so are A and B. *q.e.d.*

Next we introduce a notion of special linear subspaces of nonsingular p-quadric hypersurfaces. From now to the end of this section, we always assume that A is invertible, so that X_A is nonsingular. We will work over $\bar{k}$ unless otherwise stated.

Let a be a $\bar{k}$-rational point of a nonsingular p-quadric hypersurface X_A (not necessarily Hermitian). It is easy to see that the tangent space $T(a, X_A)$ to X_A at a is given by the linear equation $f_A(x, a) = 0$.

Definition 2.9. The *q-tangent space* $qT(a, X_A)$ to X_A at a is the reduced part of the subvariety defined by $f_A(a, x) = 0$. Because $f_A(a, x)$ is the q th power of the linear form $f_{{}^tA^{(1/q)}}(x, a^{(1/q^2)})$, the q-tangent space $qT(a, X_A)$ is a hyperplane for any a.

It is easy to check that the definition of the q-tangent space does not depend on the choice of homogeneous coordinates of $\mathbb{P}^n$; that is, we have

$$g(qT(a, X_A)) = qT(g(a), g(X_A)) = qT(g(a), X_{\rho(g)(A)})$$

for all $g \in GL(n+1, \bar{k})$.

Proposition 2.10. *Let L be a linear subspace of $\mathbb{P}^n$ contained in X_A, and let a be a $\bar{k}$-rational point of L. Then L is contained in $T(a, X_A) \cap qT(a, X_A)$.*

Proof. We choose homogeneous coordinates ${}^t(x_0, \dots, x_n)$ of $\mathbb{P}^n$ such that the point a is ${}^t(1, 0, \dots, 0)$. Let $\sum_{i,j=0}^n a_{ij} x_i x_j^q = 0$ be the defining equation of X_A. We have $a_{00} = 0$ because $a \in X_A(\bar{k})$. In terms of the affine coordinates $u_i := x_i/x_0$ $(i = 1, \dots, n)$, the defining equation of X_A is written as follows:

$$\sum_{i=1}^n a_{i0} u_i + \Big(\sum_{j=1}^n a_{0j}^{1/q} u_j\Big)^q + \sum_{i,j=1}^n a_{ij} u_i u_j^q = 0.$$

It is easy to see that $T(a, X_A)$ is defined by $\sum_{i=1}^n a_{i0} u_i = 0$, and that $qT(a, X_A)$ is defined by $\sum_{j=1}^n a_{0j}^{1/q} u_j = 0$. On the other hand, each of the homogeneous parts

$$\sum_{i=1}^n a_{i0} u_i, \quad (\sum_{j=1}^n a_{0j}^{1/q} u_j)^q \quad \text{and} \quad \sum_{i,j=1}^n a_{ij} u_i u_j^q$$

of the defining equation of X_A must vanish on L, because the linear subspace L is contained in X_A and contains the origin a. *q.e.d.*

Definition 2.11. A $\bar{k}$-rational point s of a nonsingular p-quadric hypersurface X_A is said to be *a special point* if $T(s, X_A) = qT(s, X_A)$ holds. A linear subspace L contained in X_A is said to be *a special linear subspace* of X_A if L is spanned by special points of X_A. We denote by $\Sigma^l(X_A)$ the set of special linear subspaces of X_A with dimension l. We consider the empty set $\emptyset$ as a special linear subspace of X_A so that $\Sigma^{-1}(X_A) = \{\emptyset\}$. (See Convention (4).) We denote by $\Sigma(X_A)$ the disjoint union of $\Sigma^l(X_A)$ for all $l \geq -1$, and define a structure of the poset on $\Sigma(X_A)$ by

$$L_1 \leq L_2 \Longleftrightarrow L_1 \subseteq L_2.$$

Remark again that the definition of special linear subspaces does not depend on the choice of homogeneous coordinates of $\mathbb{P}^n$. If $L \in \Sigma^l(X_A)$, then $g(L) \in \Sigma^l(X_{\rho(g)(A)})$ holds for any $g \in GL(n+1, \bar{k})$. The map $L \mapsto g(L)$ induces an isomorphism of posets between $\Sigma(X_A)$ and $\Sigma(X_{\rho(g)(A)})$. Hence, by Proposition 2.3, the isomorphism class of the poset $\Sigma(X_A)$ depends only on q and n, and is independent of the choice of the invertible matrix A.

Proposition 2.12. *Suppose that $\mathbb{P}^n$ is defined over k and that X_A is Hermitian. Then a linear subspace L contained in X_A is special if and only if L is k-rational.*

Proof. It is enough to prove the assertion when $\dim L = 0$. A $\bar{k}$-rational point a of X_A is special if and only if the two linear forms $f_A(x, a)$ and $f_{{}^tA^{(1/q)}}(x, a^{(1/q^2)})$ are linearly dependent. When ${}^tA = A^{(q)}$, this is equivalent to saying that $Aa^{(q)}$ and $Aa^{(1/q)}$ are linearly dependent. Because $\det A \neq 0$, this is equivalent to saying that a and $a^{(1/q^2)}$ are linearly dependent; that is, the point a is k-rational. *q.e.d.*

Remark 2.13. There is another characterization of special points of a nonsingular p-quadric hypersurface (not necessarily Hermitian). It is known that the dual hypersurface $X_A^\vee$ of X_A is again a nonsingular p-quadric hypersurface. Let $\delta : X_A \to X_A^\vee$ and $\delta^\vee : X_A^\vee \to X_A$ be the natural morphisms. Then a point $a \in X_A(\bar{k})$ is a special point if and only if $\delta^\vee(\delta(a)) = a$.

If X_A is Hermitian and s is a k-rational point of X_A, then the hyperplane $T(s, X_A)$ is also k-rational. Using this, we get the following.

Corollary 2.14. *Let X_A be a nonsingular p-quadric hypersurface.*
(1) *If $L_1, L_2 \in \Sigma(X_A)$, then $L_1 \cap L_2 \in \Sigma(X_A)$.*
(2) *If $L \in \Sigma(X_A)$ and $s \in \Sigma^0(X_A)$, then $L \cap T(s, X_A) \in \Sigma(X_A)$.*
(3) *If $L \in \Sigma^l(X_A)$, then the number of special points of X_A contained in L is $(q^{2(l+1)} - 1)/(q^2 - 1)$. In particular, if $L_1, L_2 \in \Sigma(X_A)$ and $L_1 \setminus (L_2 \cap L_1) \neq \emptyset$, then there is at least one special point of X_A on $L_1 \setminus (L_2 \cap L_1)$.*

The assertion (1) of Corollary 2.14 implies that any two elements L_1 and L_2 of the poset $\Sigma(X_A)$ have the greatest common lower bound

$$L_1 \wedge L_2 := L_1 \cap L_2.$$

Proposition 2.15. *Let L_0 be the linear subspace of dimension $l \geq 0$ defined by $x_{l+1} = \cdots = x_n = 0$, and let X_A be a nonsingular p-quadric hypersurface associated with a matrix*

$$A := \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right),$$

where A_{11} is a square matrix of size $l+1$. Then X_A contains L_0 as a special linear subspace if and only if the following hold:

- (i) *the submatrix A_{11} is the zero matrix, and*
- (ii) *there is an invertible square matrix Γ of size $l+1$ such that $A_{12} = \Gamma\, {}^tA_{21}^{(q)}$.*

Proof. The condition (i) is equivalent to saying that $L_0 \subset X_A$. We assume that L_0 is contained in X_A. If $a = {}^t(a_0, \ldots, a_l, 0, \ldots, 0)$ is a $\bar{k}$-rational point of L_0, then $T(a, X_A)$ is defined by the linear equation

$$(x_{l+1}, \ldots, x_n) \cdot A_{21} \cdot a' = 0,$$

and $qT(a, X_A)$ is defined by by the linear equation

$$(x_{l+1}, \dots, x_n) \cdot {}^tA_{12}^{(1/q)} \cdot a'^{(1/q)} = 0,$$

where $a' = {}^t(a_0, \dots, a_l)$. Suppose that L_0 is special with respect to X_A. Then there are special points $s_0, \dots, s_l$ of X_A spanning L_0. Let s'_i be the vector of the first $l+1$ components of s_i. Then $A_{21}s'_i$ and ${}^tA_{12}^{(1/q)} s'^{(1/q)}_i$ are linearly dependent, and hence there is $\lambda_i \in \bar{k}^\times$ such that $\lambda_i A_{21} s'_i = {}^tA_{12}^{(1/q)} s'^{(1/q)}_i$. Because the points s_i span L_0, the square matrix $\sigma := (s'_0, \dots, s'_l)$ of size $l+1$ is invertible. Let Λ be the diagonal matrix with diagonal entries $\lambda_0, \dots, \lambda_l$. Then we have $A_{12} = \Gamma\, {}^tA_{21}^{(q)}$, where $\Gamma = {}^t(\sigma^{(q)}\Lambda^{(q)}\sigma^{-1})$. Conversely, suppose that $A_{12} = \Gamma\, {}^tA_{21}^{(q)}$ holds for some $\Gamma \in GL(l+1, \bar{k})$. By the same argument as in the proof of Lemma 2.2, we can prove that the action of $GL(l+1, \bar{k})$ on itself given by $(\gamma, \beta) \mapsto {}^t(\gamma^{(q)}\beta\gamma^{-1})$ is transitive. In particular, the morphism $GL(l+1, \bar{k}) \to GL(l+1, \bar{k})$ given by $\gamma \mapsto {}^t(\gamma^{(q)}\gamma^{-1})$ is surjective. Hence there is an invertible matrix $\sigma = (s'_0, \dots, s'_l)$ such that $\Gamma = {}^t(\sigma^{(q)}\sigma^{-1})$. Then we have ${}^tA_{12}^{(1/q)}\sigma^{(1/q)} = A_{21}\sigma$, which means that the $\bar{k}$-rational points s_i of L_0 whose first $l+1$ coordinates form the $(i+1)$ th column vector s'_i of σ are special points of X_A. Because $\det \sigma \neq 0$, these points span L_0. Therefore L_0 is a special linear subspace of X_A. *q.e.d.*

Corollary 2.16.

$$\Sigma^l(X_A) \neq \emptyset \iff -1 \leq l \leq (n-1)/2.$$

If $\dim X_A = 2m$ and $l = m$, then $\det A \neq 0$ and $A_{11} = O$ imply $\det A_{12} \neq 0$ and $\det A_{21} \neq 0$. Hence we obtain the following.

Corollary 2.17. *Suppose that a nonsingular p-quadric hypersurface X_A is of dimension $2m$. Then any linear subspace contained in X_A of dimension m is special with respect to X_A.*

Corollary 2.18. *Every m-dimensional linear subspace contained in the Fermat hypersurface of degree $q+1$ and of dimension $2m$ is k-rational.*

Next we investigate the action of an automorphism group of X_A on the poset $\Sigma(X_A)$. We put

$$G_A := \{\, g \in GL(n+1, \bar{k}) \;:\; \rho(g)(A) = A \,\}.$$

If $g \in G_I$, then $g^{(q^2)} = g$ holds, and hence g is contained in $GL(n+1, k)$. Therefore G_I coincides with the group $GU(n+1, k)$. Then it follows from Proposition 2.3 that, for any $A \in GL(n+1, \bar{k})$, the group G_A is conjugate to $GU(n+1, k)$ in $GL(n+1, \bar{k})$. The group G_A acts on X_A projectively, and hence G_A acts on the poset $\Sigma(X_A)$.

Proposition 2.19. *The action of G_A on $\Sigma^l(X_A)$ is transitive for each l.*

Proof. Let L_0 be the linear subspace of dimension l defined in the statement of Proposition 2.15. We put

$$Z_0 := \{\, A' \in GL(n+1, \bar{k}) \;:\; L_0 \in \Sigma^l(X_{A'}) \,\},$$
$$H_0 := \{\, g \in GL(n+1, \bar{k}) \;:\; g(L_0) = L_0 \,\}.$$

By definition, the group H_0 acts on Z_0 by ρ. Proposition 2.15 implies that Z_0 is irreducible, and that

$$\dim Z_0 = (l+1)(n-l) + (l+1)^2 + (n-l)^2 = \dim H_0.$$

By Lemma 2.2, the action of $H_0(\bar{k})$ on the set $Z_0(\bar{k})$ is transitive. Suppose that two elements L_1 and L_2 of $\Sigma^l(X_A)$ are given. There are elements $g_1, g_2 \in GL(n+1, \bar{k})$ such that $g_1(L_1) = L_0$ and $g_2(L_2) = L_0$. Then both of $\rho(g_1)(A)$ and $\rho(g_2)(A)$ are in $Z_0(\bar{k})$, and hence there is an element $h \in H_0(\bar{k})$ such that $\rho(h)\rho(g_1)(A) = \rho(g_2)(A)$. Then we have $g_2^{-1}hg_1 \in G_A$. Thus we obtain an element $g_2^{-1}hg_1$ of G_A which satisfies $g_2^{-1}hg_1(L_1) = L_2$. *q.e.d.*

Using Proposition 2.12, we can also deduce Proposition 2.19 from Segre's result [11, n. 50].

Next we consider the projection from a special point of a nonsingular p-quadric hypersurface, which enables us to investigate the structure of the poset $\Sigma(X_A)$ by means of induction on $\dim X_A$.

Let s be a special point of X_A. For simplicity, we denote by T_s the hyperplane $T(s, X_A) = qT(s, X_A)$, and by $\mathbb{P}T_s$ the projective space of lines in T_s passing through s. Then we have the projection $T_s \setminus \{s\} \to \mathbb{P}T_s$ with the center s. We put $\widetilde{X}_A[s] := (X_A \cap T_s) \setminus \{s\}$.

Proposition 2.20. *Suppose that* $\dim X_A \geq 2$.

(1) *There is a nonsingular p-quadric hypersurface* $X_A[s]$ *of dimension* $\dim X_A - 2$ *in* $\mathbb{P}T_s$ *such that* $X_A \cap T_s$ *is the cone over* $X_A[s]$ *with the vertex* s.

(2) *Let* $\phi_s : \widetilde{X}_A[s] \to X_A[s]$ *be the projection with the center* s*, every fiber of which is isomorphic to an affine line. If* L *is a special linear subspace of* X_A*, then the image of* $L \cap \widetilde{X}_A[s]$ *by* ϕ_s *is a special linear subspace of* $X_A[s]$.

(3) *If* L' *is a special linear subspace of* $X_A[s]$*, then the union of* $\phi_s^{-1}(L')$ *and* $\{s\}$ *is a special linear subspace of* X_A *passing through* s.

Proof. By Proposition 2.3, we may assume that X_A is a Hermitian hypersurface defined by the equation

$$x_0^q x_1 + x_0 x_1^q + x_2^{q+1} + \cdots + x_n^{q+1} = 0,$$

and by Proposition 2.19, we may assume that s is the k-rational point ${}^t(1, 0, \ldots, 0)$. Let $u_i := x_i/x_0$ $(i = 1, \ldots, n)$ be the affine coordinates of $\mathbb{P}^n$ with the origin s. Then X_A is defined by $u_1 + u_1^q + u_2^{q+1} + \cdots + u_n^{q+1} = 0$, and T_s is defined by $u_1 = 0$. We can consider ${}^t(u_2, \ldots, u_n)$ as homogeneous coordinates of $\mathbb{P}T_s$. Then $X_A \cap T_s$ is the cone over a hypersurface of $\mathbb{P}T_s$ defined by

$$u_2^{q+1} + \cdots + u_n^{q+1} = 0,$$

which is nonsingular and Hermitian. We write this hypersurface in $\mathbb{P}T_s$ by $X_A[s]$. The projection from $T_s \setminus \{s\}$ to $\mathbb{P}T_s$ with the center s is given by

$${}^t(1, 0, a_2, \ldots, a_n) \mapsto {}^t(a_2, \ldots, a_n),$$

which is obviously defined over k. Hence, by Proposition 2.12 and Corollary 2.14, we see that ϕ_s satisfies (2) and (3). (Note that $\phi_s(\emptyset) = \emptyset \in \Sigma^{-1}(X_A[s])$, and that $\phi_s^{-1}(\emptyset) \cup \{s\} = \{s\} \in \Sigma^0(X_A)$.) *q.e.d.*

For a special point s of X_A, we denote by $\Sigma(X_A; \geq s)$ the subset of $\Sigma(X_A)$ consisting of elements L such that $s \in L$. We can define maps

$$\Phi_s : \Sigma(X_A) \to \Sigma(X_A[s]) \quad \text{and} \quad \Psi_s : \Sigma(X_A[s]) \to \Sigma(X_A; \geq s)$$

by

$$\Phi_s(L) := \phi_s(L \cap \widetilde{X}_A[s]) \quad \text{and} \quad \Psi_s(L') := \phi_s^{-1}(L') \cup \{s\}.$$

Note that $s \in L \in \Sigma(X_A)$ implies $L \subset T_s$ by Proposition 2.10. We can easily prove the following from the definitions.

Corollary 2.21. (1) *If $L \in \Sigma(X_A)$ and $L' \in \Sigma(X_A[s])$, then*

$$\dim \Phi_s(\, L\,) = \begin{cases} \dim L & \text{if } L \subset T_s \text{ and } s \notin L, \\ \dim L - 1 & \text{otherwise,} \end{cases}$$

and

$$\dim \Psi_s(L') = \dim L' + 1.$$

(2) *The two maps Φ_s and Ψ_s yield an isomorphism of posets between $\Sigma(X_A; \geq s)$ and $\Sigma(X_A[s])$ which shifts the dimension by* 1.

(3) *Suppose that $s \in L$. Then $\Phi_s(L) \wedge \Phi_s(L') = \Phi_s(L \wedge L')$ holds for any $L' \in \Sigma(X_A)$. In particular, if $s \in L$ and $s \notin L'$, then*

$$\dim(\Phi_s(L) \wedge \Phi_s(L')) = \dim(L \wedge L').$$

Using Corollaries 2.14 (3) and 2.21 (2), we have

$$|\Sigma^l(X_A)| = \frac{|\Sigma^0(X_A)| \cdot |\Sigma^{l-1}(X_A[s])|}{(q^{2l+2} - 1)/(q^2 - 1)}.$$

On the other hand, we have $|\Sigma^0(X_A)| = F(n, n+1)$ by Propositions 2.7 and 2.12. Combining these formulas, we can calculate $|\Sigma^l(X_A)|$ by induction on $\dim X_A$ and l. This result corresponds, via Proposition 2.12, to Segre's formula [11, n. 32] on the number of k-rational linear subspaces on a nonsingular Hermitian hypersurface. In particular, we obtain the following neat formula.

Corollary 2.22. *If $\dim X_A = 2m$, then $|\Sigma^m(X_A)| = \prod_{\nu=0}^{m}(q^{2\nu+1} + 1)$.*

3. Configuration of special linear subspaces

From now to the end of the paper, we assume that the dimension $n - 1$ of the p-quadric hypersurfaces is even. We put $n := 2m + 1$ where $m > 0$, and consider the configuration of m-dimensional linear subspaces contained in a nonsingular p-quadric hypersurface X_A of dimension $2m$. Recall that every linear subspace of dimension m contained in X_A is special with respect to X_A (cf. Corollary 2.17).

Suppose that X_A is nonsingular. For $H \in \Sigma^{m-1}(X_A)$, we put

$$B_A(H) := \{\, \Pi \in \Sigma^m(X_A) \;:\; \Pi > H \,\};$$

that is, the set $B_A(H)$ consists of $\Pi \in \Sigma^m(X_A)$ that contains H.

Proposition 3.1. (1) *The set $B_A(H)$ consists of $q + 1$ elements for every $H \in \Sigma^{m-1}(X_A)$.*

(2) *For any couple of $\Pi \in \Sigma^m(X_A)$ and $H \in \Sigma^{m-1}(X_A)$, there is a unique element $\beta_A(H, \Pi)$ of $B_A(H)$ such that, for any $\Pi' \in B_A(H)$, the following holds*:

$$\dim(\Pi' \wedge \Pi) = \begin{cases} \dim(H \wedge \Pi) & \text{if } \Pi' \neq \beta_A(H, \Pi), \\ \dim(H \wedge \Pi) + 1 & \text{if } \Pi' = \beta_A(H, \Pi). \end{cases} \tag{3.1}$$

(3) *For each $H \in \Sigma^{m-1}(X_A)$, there is a linear subspace P_H of $\mathbb{P}^{2m+1}$ with $\dim P_H = m + 1$ such that $P_H \cap X_A$ is the union of the $q + 1$ elements of $B_A(H)$.*

Proof. We proceed by induction on m. Suppose that $m = 1$. Then H is a special point s of X_A. Hence $X_A \cap T_s$ is a union of distinct $q+1$ lines passing through s by Proposition 2.20 (1), and the set $B_A(s)$ consists of these lines by Proposition 2.10. Therefore we can take T_s as P_s. Let Π be an arbitrary line on X_A. If Π is contained in T_s, then Π is an element of $B_A(s)$ and hence, with $\beta_A(s, \Pi) = \Pi$, the formula (3.1) holds. If Π is not contained in T_s, then the intersection $T_s \cap \Pi$ of a line and a plane consists of a single point t, which is not s because of Proposition 2.10. There exists exactly one line among $B_A(s)$ that passes through t. Then, with $\beta_A(s, \Pi)$ being this line, the formula (3.1) holds. Suppose that $m > 1$. We choose a special point s of X_A that is on H and make the projection $\widetilde{X}_A[s] \to X_A[s]$ defined in the previous section. We denote $X_A[s]$ by $X_{A'}$, where A' is a certain matrix in $GL(2m, \bar{k})$. Note that $\Phi_s(H)$ is an element of $\Sigma^{m-2}(X_{A'})$. By Corollary 2.21 (2), the map Φ_s induces a bijection

$$B_A(H) \cong B_{A'}(\Phi_s(H)). \tag{3.2}$$

Using the induction hypothesis, we obtain $|B_A(H)| = q+1$. For P_H, we can choose the closure of the pull-back of $P_{\Phi_s(H)} \subset \mathbb{P}T_s$ by the projection $T_s \setminus \{s\} \to \mathbb{P}T_s$ with the center s. Let Π be an arbitrary element of $\Sigma^m(X_A)$. If Π contains H, then we have $\Pi \in B_A(H)$, and (3.1) holds if we put $\beta_A(H, \Pi) = \Pi$. Suppose that Π does not contain H. We can choose the special point s from $H \setminus (H \cap \Pi)$ by virtue of Corollary 2.14 (3). Then we have $\Phi_s(\Pi) \in \Sigma^{m-1}(X_{A'})$ by Corollary 2.21 (1). Moreover we have

$$\dim(H \wedge \Pi) = \dim(\Phi_s(H) \wedge \Phi_s(\Pi)) \quad \text{and} \quad \dim(\Pi' \wedge \Pi) = \dim(\Phi_s(\Pi') \wedge \Phi_s(\Pi))$$

for any $\Pi' \in B_A(H)$ by Corollary 2.21 (3). The element $\beta_{A'}(\Phi_s(H), \Phi_s(\Pi))$ is given uniquely by the induction hypothesis. We set $\beta_A(H, \Pi)$ to be the unique element of $B_A(H)$ that corresponds to $\beta_{A'}(\Phi_s(H), \Phi_s(\Pi))$ by the bijection (3.2). Then (3.1) holds. *q.e.d.*

Corollary 3.2. *Let* Π_0, Π_1 *and* Π_2 *be mutually distinct elements of* $\Sigma^m(X_A)$ *such that* $\dim(\Pi_0 \wedge \Pi_1) = \dim(\Pi_0 \wedge \Pi_2) = m-1$. *Then*

$$\dim(\Pi_1 \wedge \Pi_2) = \begin{cases} m-2 & \textit{if } \Pi_0 \wedge \Pi_1 \neq \Pi_0 \wedge \Pi_2, \\ m-1 & \textit{if } \Pi_0 \wedge \Pi_1 = \Pi_0 \wedge \Pi_2. \end{cases}$$

Proof. It is obvious that $\dim(\Pi_1 \wedge \Pi_2) = m-1$ if $\Pi_0 \wedge \Pi_1 = \Pi_0 \wedge \Pi_2$. Suppose that $\Pi_0 \wedge \Pi_1 \neq \Pi_0 \wedge \Pi_2$. We put $H_1 := \Pi_0 \wedge \Pi_1$ and $H_2 := \Pi_0 \wedge \Pi_2$, both of which are hyperplanes of Π_0. Note that Π_0 and Π_1 are distinct elements of $B_A(H_1)$. Since $\Pi_0 \wedge \Pi_1 \neq \Pi_0 \wedge \Pi_2$, we have $\dim(H_1 \wedge \Pi_2) = \dim(H_1 \wedge H_2) = m-2$ and hence $\dim(\Pi_0 \wedge \Pi_2) = \dim(H_1 \wedge \Pi_2) + 1$. The uniqueness of $\beta_A(H_1, \Pi_2)$ implies that $\beta_A(H_1, \Pi_2) = \Pi_0$. Since $\Pi_1 \neq \Pi_0$, we have $\dim(\Pi_1 \wedge \Pi_2) = \dim(H_1 \wedge \Pi_2) = m-2$. *q.e.d.*

Recall that $h \in N^m(X_A)$ is the numerical equivalence class of the intersection of X_A with a general linear subspace of dimension $m+1$ in $\mathbb{P}^{2m+1}$. For $\Pi \in \Sigma^m(X_A)$, let $[\Pi] \in N^m(X_A)$ denote the numerical equivalence class of Π. From Proposition 3.1 (3), we obtain the following.

Corollary 3.3. *For any* $H \in \Sigma^{m-1}(X_A)$, *the numerical equivalence class* h *is equal to the sum of* $[\Pi]$, *where* Π *runs through the set* $B_A(H)$.

We choose homogeneous coordinates ${}^t(x_0, \dots, x_m, y_0, \dots, y_m)$ of $\mathbb{P}^{2m+1}$ defined over k, and put

$$M_0 := \{\, y_0 = \dots = y_m = 0 \,\} \quad \text{and} \quad M_\infty := \{\, x_0 = \dots = x_m = 0 \,\}.$$

We also put

$$J := \left(\begin{array}{c|c} O & I_{m+1} \\ \hline -I_{m+1} & O \end{array} \right).$$

Then the hypersurface X_J defined by the homogeneous equation

$$\sum_{i=0}^{m} (x_i y_i^q - x_i^q y_i) = 0$$

is a nonsingular Hermitian hypersurface containing M_0 and M_∞.

Remark 3.4. Using Lemma 2.2, we can prove the following. Suppose that a pair (Π_0, Π_∞) of elements of $\Sigma^m(X_A)$ such that $\Pi_0 \wedge \Pi_\infty = \emptyset$ is given, where A is an element of $GL(2m+2, \bar{k})$. Then there is a linear transformation $g \in GL(2m+2, \bar{k})$ such that $g(X_A) = X_J$, $g(\Pi_0) = M_0$ and $g(\Pi_\infty) = M_\infty$.

We define a subgroup $G_J^{0\infty}$ of $G_J = \{g \in GL(2m+2, \bar{k}) \;:\; \rho(g)(J) = J\}$ by

$$G_J^{0\infty} := \{\, g \in G_J \;:\; g(M_0) = M_0,\; g(M_\infty) = M_\infty \,\},$$

which acts on the triple (X_J, M_0, M_∞) projectively.

Proposition 3.5. *In terms of the coordinates ${}^t(x_0, \dots, x_m, y_0, \dots, y_m)$, the group $G_J^{0\infty}$ is expressed as follows*:

$$G_J^{0\infty} = \left\{ \left(\begin{array}{c|c} \gamma & O \\ \hline O & ({}^t\gamma^{-1})^{(q)} \end{array} \right) \;:\; \gamma \in GL(m+1, k) \right\}.$$

Proof. We decompose $g \in GL(2m+2, \bar{k})$ as follows:

$$g = \left(\begin{array}{c|c} g_{11} & g_{12} \\ \hline g_{21} & g_{22} \end{array} \right),$$

where g_{ij} are square matrices of size $m+1$. Then $g(M_0) = M_0$ and $g(M_\infty) = M_\infty$ hold if and only if $g_{12} = g_{21} = O$. Suppose that $g_{12} = g_{21} = O$. Then g is contained in G_J if and only if ${}^t g_{11} g_{22}^{(q)} = {}^t g_{22} g_{11}^{(q)} = I_{m+1}$ holds, which is equivalent to $g_{22} = ({}^t g_{11}^{-1})^{(q)}$ and $g_{11}^{(q^2)} = g_{11}$. The latter holds if and only if $g_{11} \in GL(m+1, k)$. *q.e.d.*

We fix an identification

$$G_J^{0\infty} \cong GL(m+1, k) \tag{3.3}$$

given by

$$\left(\begin{array}{c|c} \gamma & O \\ \hline O & ({}^t\gamma^{-1})^{(q)} \end{array} \right) \mapsto \gamma.$$

Then the action of $G_J^{0\infty}$ on the projective space M_0 factors through this isomorphism; that is, the group $G_J^{0\infty}$ acts on M_0 over k as the full linear transformation group.

Let a and b be integers such that $-1 \le a, b \le m$. We put

$$\Sigma^m(X_J)^{(a,b)} := \{\, \Pi \in \Sigma^m(X_J) \;:\; \dim(\Pi \wedge M_0) = a,\; \dim(\Pi \wedge M_\infty) = b \,\}.$$

Because $M_0 \wedge M_\infty = \emptyset$, this set $\Sigma^m(X_J)^{(a,b)}$ is empty when $a+b > m-1$. Suppose that $a+b \le m-1$. We define a linear subspace $M_1^{(a,b)}$ of dimension m in $\mathbb{P}^{2m+1}$ by the equations

$$x_\lambda = 0 \quad (a+1 \le \lambda \le a+b+1), \qquad y_\mu = 0 \quad (0 \le \mu \le a), \quad \text{and}$$
$$x_\nu - y_\nu = 0 \quad (a+b+2 \le \nu \le m).$$

For example, we have $M_1^{(-1,m)} = M_\infty$ and $M_1^{(m,-1)} = M_0$. The linear subspace $M_1^{(a,b)}$ is contained in $\Sigma^m(X_J)^{(a,b)}$, and hence $\Sigma^m(X_J)^{(a,b)}$ is non-empty if and only if $a+b \le m-1$. Note that $G_J^{0\infty}$ acts on the set $\Sigma^m(X_J)^{(a,b)}$.

Proposition 3.6. (1) *For each (a,b), the action of $G_J^{0\infty}$ on $\Sigma^m(X_J)^{(a,b)}$ is transitive.*

(2) *We put $c := m-1-a-b$. The number $|\Sigma^m(X_J)^{(a,b)}|$ is equal to*

$$|GL(m+1,k)| \,/\, (\,|GU(c,k)| \cdot |GL(a+1,k)| \cdot |GL(b+1,k)| \cdot q^{2n(a,b)}\,),$$

where $n(a,b) := (a+1)(b+1) + c(a+b+2)$. Here we understand that $|GU(0,k)| = |GL(0,k)| = 1$.

Proof. We decompose matrices $g \in GL(2m+2,\bar{k})$ and $A \in GL(2m+2,\bar{k})$ as follows:

$$g = \left[\begin{array}{c|c} \begin{array}{c|c|c} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \hline \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \hline \gamma_{31} & \gamma_{32} & \gamma_{33} \end{array} & g_{12} \\ \hline g_{21} & \begin{array}{c|c|c} \gamma'_{11} & \gamma'_{12} & \gamma'_{13} \\ \hline \gamma'_{21} & \gamma'_{22} & \gamma'_{23} \\ \hline \gamma'_{31} & \gamma'_{32} & \gamma'_{33} \end{array} \end{array}\right]$$

and

$$A = \left[\begin{array}{c|c} A_{11} & \begin{array}{c|c|c} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \hline \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \hline \alpha_{31} & \alpha_{32} & \alpha_{33} \end{array} \\ \hline \begin{array}{c|c|c} \alpha'_{11} & \alpha'_{12} & \alpha'_{13} \\ \hline \alpha'_{21} & \alpha'_{22} & \alpha'_{23} \\ \hline \alpha'_{31} & \alpha'_{32} & \alpha'_{33} \end{array} & A_{22} \end{array}\right].$$

Here g_{ij} and A_{ij} are square matrices of size $m+1$, and $\gamma_{\mu\nu}$, $\gamma'_{\mu\nu}$, $\alpha_{\mu\nu}$, and $\alpha'_{\mu\nu}$ are matrices of shape $s(\mu) \times s(\nu)$, where $s(1) := a+1$, $s(2) := b+1$ and $s(3) := c$. We define a subgroup $H^{(a,b)}$ of $GL(2m+2,\bar{k})$ and a closed subvariety $Z^{(a,b)}$ of $GL(2m+2,\bar{k})$ as follows:

$$H^{(a,b)} := \{\, g \in GL(2m+2,\bar{k}) \;:\; g(M_0) = M_0,\ g(M_\infty) = M_\infty,\ g(M_1^{(a,b)}) = M_1^{(a,b)} \,\},$$
$$Z^{(a,b)} := \{\, A \in GL(2m+2,\bar{k}) \;:\; M_0 \cup M_\infty \cup M_1^{(a,b)} \subset X_A \,\}.$$

By definition, the group $H^{(a,b)}$ acts on $Z^{(a,b)}$ by ρ. It is easy to see that

$g \in H^{(a,b)} \iff g_{12},\ g_{21},\ \gamma'_{12},\ \gamma_{21},\ \gamma'_{32},\ \gamma_{23},\ \gamma'_{13},\ \gamma_{31}$ are zero matrices, and $\gamma_{33} = \gamma'_{33}$,

$A \in Z^{(a,b)} \iff A_{11},\ A_{22},\ \alpha'_{21},\ \alpha_{12},\ \alpha'_{23},\ \alpha_{32},\ \alpha'_{31},\ \alpha_{13}$ and $\alpha_{33} + \alpha'_{33}$ are zero matrices.

This implies that both of $H^{(a,b)}$ and $Z^{(a,b)}$ are irreducible, and that $\dim H^{(a,b)}$ is equal to $\dim Z^{(a,b)}$. By Lemma 2.2, the action of $H^{(a,b)}(\bar{k})$ on the set $Z^{(a,b)}(\bar{k})$ is transitive. Let Π be an arbitrary element of $\Sigma^m(X_J)^{(a,b)}$. By looking at the action of the group $\{g \in GL(2m+2,\bar{k}) \ : \ g(M_0) = M_0,\ g(M_\infty) = M_\infty\}$ on the Grassmannian variety of m-dimensional linear subspaces in $\mathbb{P}^{2m+1}$, we see that there exists a linear transformation $g \in GL(2m+2,\bar{k})$ such that $g(M_0) = M_0$, $g(M_\infty) = M_\infty$ and $g(\Pi) = M_1^{(a,b)}$. Then we have $\rho(g)(J) \in Z^{(a,b)}(\bar{k})$. By the above argument, there exists an element $g' \in H^{(a,b)}(\bar{k})$ such that $\rho(g')\rho(g)(J) = J$. Then we have $g'g \in G_J^{0\infty}$ and $g'g(\Pi) = M_1^{(a,b)}$. Thus the first assertion is proved.

The stabilizer subgroup of $M_1^{(a,b)} \in \Sigma^m(X_J)^{(a,b)}$ in $G_J^{0\infty}$ is given by $H^{(a,b)} \cap G_J$. An element

$$g = \left(\begin{array}{c|c} \gamma & O \\ \hline O & \gamma' \end{array}\right) \quad \text{with} \quad \gamma = \left[\begin{array}{c|c|c} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \hline O & \gamma_{22} & O \\ \hline O & \gamma_{32} & \gamma_{33} \end{array}\right], \quad \gamma' = \left[\begin{array}{c|c|c} \gamma'_{11} & O & O \\ \hline \gamma'_{21} & \gamma'_{22} & \gamma'_{23} \\ \hline \gamma'_{31} & O & \gamma_{33} \end{array}\right]$$

of $H^{(a,b)}$ is contained in G_J if and only if γ is contained in $GL(m+1,k)$ and ${}^t\gamma'\gamma^{(q)}$ is the identity matrix by Proposition 3.5. If ${}^t\gamma'\gamma^{(q)} = I_{m+1}$, then γ_{33} must be an element of the group $GU(c,k)$. Hence the map $g \mapsto \gamma$ yields a homomorphism from $H^{(a,b)} \cap G_J$ to the group

$$H_J^{(a,b)} := \left\{ \left[\begin{array}{c|c|c} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \hline O & \gamma_{22} & O \\ \hline O & \gamma_{32} & \gamma_{33} \end{array}\right] \in GL(m+1,k) \quad ; \quad \gamma_{33} \in GU(c,k) \right\}.$$

We can check that the order of $H_J^{(a,b)}$ is equal to the denominator of the formula given in the assertion (2). The map $g \mapsto \gamma$ is an isomorphism between $H^{(a,b)} \cap G_J$ and $H_J^{(a,b)}$. Indeed, if $\gamma \in H_J^{(a,b)}$, then $\gamma' := ({}^t\gamma^{-1})^{(q)}$ satisfies $\gamma'_{12} = O$, $\gamma'_{32} = O$, $\gamma'_{13} = O$, and $\gamma_{33} = \gamma'_{33}$, and thus

$$\gamma \mapsto \left(\begin{array}{c|c} \gamma & O \\ \hline O & ({}^t\gamma^{-1})^{(q)} \end{array}\right)$$

yields the inverse homomorphism. Combining this with (3.3), we proved the assertion (2). *q.e.d.*

Because X_J is Hermitian, the set of $H \in \Sigma^{m-1}(X_J)$ satisfying $H \subset M_0$ is identified with the set $M_0^\vee(k)$ of k-rational points of the dual projective space of M_0 by Proposition 2.12. For $H \in M_0^\vee(k)$, we put

$$\Pi_H^+ := \beta_J(H, M_\infty).$$

Because $H \wedge M_\infty = \emptyset$, the definition means that Π_H^+ is the unique element of $B_J(H)$ such that $M_\infty \wedge \Pi_H^+ \neq \emptyset$. Thus $B_J(H)$ has two distinguished elements M_0 and Π_H^+. Therefore, for each $\Pi \in \Sigma^m(X_J)$, we can decompose $M_0^\vee(k)$ into the disjoint union of the following subsets:

$$\begin{aligned} C_0(\Pi) &:= \{\, H \in M_0^\vee(k) \ : \ \beta_J(H,\Pi) = M_0 \,\}, \\ C_\infty(\Pi) &:= \{\, H \in M_0^\vee(k) \ : \ \beta_J(H,\Pi) = \Pi_H^+ \,\}, \\ C_1(\Pi) &:= \{\, H \in M_0^\vee(k) \ : \ \beta_J(H,\Pi) \in B_J(H) \setminus \{M_0, \Pi_H^+\} \ \}. \end{aligned}$$

Table 3.1. $C_0(\Pi), C_\infty(\Pi)$ and $C_1(\Pi)$.

$\setminus\ \Pi$	M_0	M_∞	Π_H^+	Π_H^-
$C_0(\Pi)$	$M_0^\vee(k)$	$\emptyset$	$M_0^\vee(k)\setminus\{H\}$	$M_0^\vee(k)\setminus\{H\}$
$C_\infty(\Pi)$	$\emptyset$	$M_0^\vee(k)$	$\{H\}$	$\emptyset$
$C_1(\Pi)$	$\emptyset$	$\emptyset$	$\emptyset$	$\{H\}$

By the definition of $\beta_J(H,\Pi)$, we have

$$\begin{aligned} H\in C_0(\Pi) &\iff \dim(\Pi\wedge M_0)-1=\dim(\Pi\wedge H)=\dim(\Pi\wedge\Pi_H^+),\\ H\in C_\infty(\Pi) &\iff \dim(\Pi\wedge \Pi_H^+)-1=\dim(\Pi\wedge H)=\dim(\Pi\wedge M_0),\\ H\in C_1(\Pi) &\iff \dim(\Pi\wedge M_0)=\dim(\Pi\wedge\Pi_H^+)=\dim(\Pi\wedge H). \end{aligned} \tag{3.4}$$

For example, we have Table 3.1, where Π_H^- denotes an arbitrary element of $B_J(H)\setminus\{M_0,\Pi_H^+\}$.

Our aim is to describe the decomposition $M_0^\vee(k)=C_0(\Pi)\sqcup C_\infty(\Pi)\sqcup C_1(\Pi)$ for each $\Pi\in\Sigma^m(X_J)$. Let R be a k-rational linear subspace of M_0. We denote by $(M_0/R)^\vee$ the k-rational linear subspace of $M_0^\vee$ consisting of hyperplanes of M_0 containing R. Then we have

$$\dim R+\dim(M_0/R)^\vee+1=m.$$

When R is the empty set (that is, the linear subspace of dimension -1), we have $(M_0/\emptyset)^\vee=M_0^\vee$. (See Convention (4).)

Proposition 3.7. *Let Π be an element of $\Sigma^m(X_J)$, and R the intersection $\Pi\wedge M_0$. Then $C_\infty(\Pi)\sqcup C_1(\Pi)$ coincides with $(M_0/R)^\vee(k)$.*

Proof. We have $\Pi\wedge H=R\wedge H$ for any $H\in M_0^\vee(k)$. Hence $\dim(\Pi\wedge M_0)>\dim(\Pi\wedge H)$ holds if and only if $R\neq R\wedge H$; that is, if and only if $R\not\subset H$. *q.e.d.*

Definition 3.8. We define $\mathcal{P}$ to be the set of pairs (Γ_∞,Γ_1) of mutually disjoint subsets of $M_0^\vee(k)$. We have a map $\zeta:\Sigma^m(X_J)\to\mathcal{P}$ defined by

$$\zeta(\Pi):=(C_\infty(\Pi),C_1(\Pi)).$$

For integers a and b satisfying $-1\le a,b\le m$ and $a+b\le m-1$, we define $\mathcal{R}^{(a,b)}$ to be the set of couples (R,Y) such that

(i) R is a k-rational linear subspace of M_0 with $\dim R=a$, and
(ii) Y is a Hermitian hypersurface in $(M_0/R)^\vee$ with $\operatorname{rank}(Y)=m-1-a-b$.

Note that we can consider Y as a subvariety of $M_0^\vee$ because $(M_0/R)^\vee$ is a linear subspace of $M_0^\vee$. Let $\mathcal{R}$ be the disjoint union of the sets $\mathcal{R}^{(a,b)}$. We have a map $\xi:\mathcal{R}\to\mathcal{P}$ defined by

$$\xi(R,Y):=(\ Y(k),\ (M_0/R)^\vee(k)\setminus Y(k)\ ).$$

Recall that the group $G_J^{0\infty}$ acts on M_0 over k, and hence $G_J^{0\infty}$ acts on $\mathcal{P}$. By definition, the map ζ is equivariant under this action. The group $G_J^{0\infty}$ also acts on the set $\mathcal{R}^{(a,b)}$ in a natural way, and the map ξ is obviously equivariant under this action. Hence the subset $\xi(\mathcal{R}^{(a,b)})$ of $\mathcal{P}$ is stable under the action of $G_J^{0\infty}$.

Proposition 3.9. (1) *For each (a,b), the action of $G_J^{0\infty}$ on the set $\mathcal{R}^{(a,b)}$ is transitive.*

(2) *The number $|\mathcal{R}^{(a,b)}|$ is equal to*

$$h(m-a-1, m-1-a-b) \cdot |\operatorname{Grass}(\mathbb{P}^a, M_0)(k)|,$$

where h is the function given in Corollary 2.6 and $\operatorname{Grass}(\mathbb{P}^a, M_0)(k)$ is the set of k-rational linear subspaces of M_0 with dimension a.

Proof. The action of $G_J^{0\infty}$ on M_0 factors through the isomorphism (3.3). Hence $G_J^{0\infty}$ acts on the set $\operatorname{Grass}(\mathbb{P}^a, M_0)(k)$ transitively, and the stabilizer subgroup of a linear subspace $R \in \operatorname{Grass}(\mathbb{P}^a, M_0)(k)$ acts on the set of Hermitian hypersurfaces of rank $m-1-a-b$ in $(M_0/R)^\vee$ transitively by Proposition 1.2 (1). Thus the assertion (1) is proved. The assertion (2) is obvious. *q.e.d.*

Proposition 3.10. *For each (a,b), the map $\zeta : \Sigma^m(X_J) \to \mathcal{P}$ induces a surjective map from $\Sigma^m(X_J)^{(a,b)}$ to $\xi(\mathcal{R}^{(a,b)})$ such that each fiber consists of a single element if $a+b = m-1$, while it consists of $q-1$ elements if $a+b \neq m-1$.*

Proof. The map ξ is injective because of Proposition 2.8. Hence $|\xi(\mathcal{R}^{(a,b)})|$ is equal to $|\mathcal{R}^{(a,b)}|$. By Propositions 3.6 (2) and 3.9 (2), we have

$$|\Sigma^m(X_J)^{(a,b)}| \;/\; |\xi(\mathcal{R}^{(a,b)})| = \begin{cases} q-1 & \text{if } a+b \neq m-1, \\ 1 & \text{if } a+b = m-1. \end{cases}$$

Because ζ and ξ are equivariant under the action of $G_J^{0\infty}$, and because the actions are transitive both on $\Sigma^m(X_J)^{(a,b)}$ and $\xi(\mathcal{R}^{(a,b)})$ by Propositions 3.6 (1) and 3.9 (1), it is enough to show that an element of $\Sigma^m(X_J)^{(a,b)}$ is mapped to an element of $\xi(\mathcal{R}^{(a,b)})$ by ζ. Consider the element $M_1^{(a,b)} \in \Sigma^m(X_J)^{(a,b)}$. We put

$$R^a := M_1^{(a,b)} \cap M_0 = \{\, x_{a+1} = \cdots = x_m = y_0 = \cdots = y_m = 0 \,\}.$$

By Proposition 3.7, we have $C_\infty(M_1^{(a,b)}) \sqcup C_1(M_1^{(a,b)}) = (M_0/R^a)^\vee(k)$. We consider ${}^t(x_0, \dots, x_m)$ as homogeneous coordinates of M_0. Let $(\xi_0, \dots, \xi_m)$ be the homogeneous coordinates of $M_0^\vee$ dual to ${}^t(x_0, \dots, x_m)$. Then we have

$$(M_0/R^a)^\vee = \{\, \xi_0 = \cdots = \xi_a = 0 \,\}.$$

We fix an element H of $(M_0/R^a)^\vee(k)$. Let the defining equation of H in M_0 be

$$\alpha_{a+1}x_{a+1} + \cdots + \alpha_m x_m = 0,$$

where $\alpha_i \in k$ $(i = a+1, \dots, m)$. We will show that the linear subspace Π_H^+ is spanned by H and the point

$$\delta(H) := {}^t(\overbrace{0, \dots, 0,}^{m+a+2 \text{ times}}\ \alpha_{a+1}^q, \dots, \alpha_m^q) \in M_\infty.$$

Because Π_H^+ is the unique element of $\Sigma^m(X_J)$ that contains H and intersects M_∞, it is enough to show that the m-dimensional linear subspace spanned by H and $\delta(H)$ is contained in X_J. For a point $u := {}^t(u_0, \dots, u_m)$ of H, the line connecting $\delta(H)$ and u is given by

$$\{\, {}^t(\lambda u_0, \dots, \lambda u_m, 0, \dots, 0, \alpha_{a+1}^q, \dots, \alpha_m^q) \;:\; \lambda \in \mathbb{P}^1 \,\}.$$

Using $\alpha_{a+1}u_{a+1} + \cdots + \alpha_m u_m = 0$ and $\alpha_i^{q^2} = \alpha_i$, we can easily check that this line is contained in X_J. Thus the claim is proved.

By (3.4), the element H of $(M_0/R^a)^\vee(k)$ is contained in $C_\infty(M_1^{(a,b)})$ if and only if $\dim(\Pi_H^+ \wedge M_1^{(a,b)}) = a+1$ holds. Because R^a is contained in $\Pi_H^+ \wedge M_1^{(a,b)}$, this is equivalent to the existence of a point of $\Pi_H^+ \wedge M_1^{(a,b)}$ that is not contained in R^a. A point

$$^t(\lambda u_0, \dots, \lambda u_m, 0, \dots, 0, \alpha_{a+1}^q, \dots, \alpha_m^q)$$

of Π_H^+ is not contained in M_0 if and only if $\lambda \neq \infty$, and this point is contained in $M_1^{(a,b)}$ if and only if

$$\lambda u_{a+1} = \cdots = \lambda u_{a+b+1} = 0 \text{ and } \lambda u_j = \alpha_j^q \text{ for } j = a+b+2, \dots, m \tag{3.5}$$

hold. In order for a point $u \in H$ and $\lambda \in \mathbb{P}^1 \setminus \{\infty\}$ satisfying (3.5) to exist, it is necessary and sufficient that $\alpha_{a+b+2}^{q+1} + \cdots + \alpha_m^{q+1} = 0$ holds. Therefore $H \in C_\infty(M_1^{(a,b)})$ if and only if H is a k-rational point of the Hermitian hypersurface in $(M_0/R^a)^\vee$ defined by

$$\xi_{a+b+2}^{q+1} + \cdots + \xi_m^{q+1} = 0.$$

The rank of this Hermitian hypersurface is $m-1-a-b$. *q.e.d.*

Corollary 3.11. *Suppose that $q=2$. Then ζ induces a bijection from $\Sigma^m(X_J)^{(a,b)}$ to $\xi(\mathcal{R}^{(a,b)})$ for every (a,b).*

Corollary 3.12. *Suppose that Π is an element of $\Sigma^m(X_J)^{(a,b)}$. Then we have*

$$\begin{aligned} |C_0(\Pi)| &= |\mathbb{P}^m(k)| - |\mathbb{P}^{m-a-1}(k)|, \\ |C_\infty(\Pi)| &= F(m-a-1, m-1-a-b), \\ |C_1(\Pi)| &= |\mathbb{P}^{m-a-1}(k)| - F(m-a-1, m-1-a-b), \end{aligned}$$

where F is the function given in Proposition 2.7.

4. The signature and the discriminant of the lattice

In this section, we prove Theorem 1.1.

Let (X_J, M_0, M_∞) be the triple defined in the previous section. The intersection number of two elements Π, Π' of $\Sigma^m(X_J)$ on X_J is given by

$$\Pi.\Pi' = \theta(\nu) := \frac{1-(-q)^{\nu+1}}{1+q}, \quad \text{where } \nu := \dim(\Pi \wedge \Pi'). \tag{4.1}$$

(See, for example, Fulton [7, p. 102, Excess Intersection Formula].) Note that this formula holds even when Π and Π' are disjoint, because we have defined $\dim \emptyset$ to be -1. (See Convention (4).) For simplicity, we put

$$\Omega := \Sigma^m(X_J)^{(m-1,-1)} \quad \text{and} \quad \Xi := \Sigma^m(X_J)^{(m-1,0)}.$$

Recall that the set of $H \in \Sigma^{m-1}(X_J)$ lying on M_0 is identified with $M_0^\vee(k)$. We denote by

$$\tau : \Omega \sqcup \Xi \longrightarrow M_0^\vee(k)$$

the map $\Pi \mapsto \Pi \wedge M_0$. By Proposition 3.1, the fiber $\tau^{-1}(H)$ of τ over an element H of $M_0^\vee(k)$ coincides with the set $B_J(H) \setminus \{M_0\}$ consisting of q elements, and $\tau^{-1}(H) \cap \Xi$ consists of a single element $\Pi_H^+ = \beta_J(H, M_\infty)$. Thus we have $|\Omega| =$

$(q-1)\cdot|M_0^\vee(k)|$. On the other hand, the middle Betti number $b_{2m}(S_d^{2m})$ of a nonsingular hypersurface S_d^{2m} of dimension $2m$ and degree d is well-known:

$$b_{2m}(S_d^{2m}) = 2 + (\ (d-1)^{2m+2} - 1\ )\ /\ d.$$

Applying this formula to our case $d = q+1$, we obtain

$$|\Omega| = b_{2m}(X_J) - 2. \tag{4.2}$$

We put

$$\overline{\Omega} := \Omega \sqcup \{\eta, \omega\},$$

where η and ω are formal elements. We denote by E the real vector space $\mathbb{R}^{\overline{\Omega}}$ of $\mathbb{R}$-valued functions on $\overline{\Omega}$, and define a symmetric bilinear form $(\ ,\ )_E$ on E by

$$(f,g)_E := f(\omega)g(\omega) + (-1)^m\big[f(\eta)g(\eta) + \sum_{\Pi\in\Omega} f(\Pi)g(\Pi)\big].$$

Note that we have $\dim E = b_{2m}(X_J)$ by (4.2), and that the signature of E is $(1, b_{2m}(X_J)-1)$ when m is odd, while it is $(b_{2m}(X_J), 0)$ when m is even.

Let M' be an element of Ξ. We put $\Omega' := \Omega \sqcup \{M_0, M'\}$. We will show that there exists a map $v : \Omega' \to E$ with the following two properties:

(i) for any $\Pi, \Pi' \in \Omega'$, we have $\Pi.\Pi' = (v(\Pi), v(\Pi'))_E$, and

(ii) the determinant of the square matrix

$$\Gamma := (v(\Pi)(x)\ ;\ \Pi\in\Omega',\ x\in\overline{\Omega})$$

of size $b_{2m}(X_J)$ is a power of $\sqrt{q}$ up to sign.

First we assume that such a map v is constructed, and deduce Theorem 1.1. Because $\det\Gamma \neq 0$ and $|\Omega'| = \dim E$, the set of vectors $\{v(\Pi) : \Pi\in\Omega'\}$ constitutes a basis of E over $\mathbb{R}$. Combining this with the property (i), we see that v induces an isometry

$$NL^m(X_J)' \otimes_{\mathbb{Z}} \mathbb{R} \cong E,$$

where $NL^m(X_J)'$ is the sublattice of $NL^m(X_J)$ generated by the numerical equivalence classes $[\Pi]$ of $\Pi\in\Omega'$. It follows that the rank of $NL^m(X_J)'$ is equal to $b_{2m}(X_J)$, that the set of classes $\{[\Pi] : \Pi\in\Omega'\}$ becomes a $\mathbb{Z}$-basis of $NL^m(X_J)'$, and that the signature of $NL^m(X_J)'$ is the same with that of E. In particular, we recover the result of Tate [16] that X_J is supersingular, and hence $N^m(X_J)$ is of finite rank with $\mathrm{rank}(N^m(X_J)) = b_{2m}(X_J)$. By the property (i), the intersection matrix of $NL^m(X_J)'$ with respect to the basis $\{[\Pi] : \Pi\in\Omega'\}$ is given by ${}^t\Gamma\cdot\mathrm{Gram}(E)\cdot\Gamma$, where $\mathrm{Gram}(E)$ is the Gram matrix of E with respect to the standard basis. In particular, we have $\mathrm{disc}(NL^m(X_J)') = (\det\Gamma)^2$. By the property (ii), we see that $\mathrm{disc}(NL^m(X_J)')$ is a power of q. We know that the rank of $NL^m(X_J)$ is at most $\mathrm{rank}(N^m(X_J)) = b_{2m}(X_J)$. Thus $NL^m(X_J)'$ is of finite index in $NL^m(X_J)$. Therefore the signature of $NL^m(X_J)$ is equal to that of E, and the discriminant of $NL^m(X_J)$ divides that of $NL^m(X_J)'$. Thus Theorem 1.1 is proved.

Now we proceed to the construction of the map v. We will construct a map

$$v' : \Omega\sqcup\Xi\sqcup\{M_0, M_\infty\} \longrightarrow E$$

such that

$$\Pi.\Pi' = (v'(\Pi), v'(\Pi'))_E \tag{4.3}$$

Table 4.1. $\dim(\Pi \wedge \Pi')$.

<table>
<tr><th>$\Pi' \ \backslash \ \Pi$</th><th>M_0</th><th>M_∞</th><th>in Ω</th><th>in Ξ</th></tr>
<tr><td>M_0</td><td>m</td><td>-1</td><td>$m-1$</td><td>$m-1$</td></tr>
<tr><td>M_∞</td><td>-1</td><td>m</td><td>-1</td><td>0</td></tr>
<tr><td>in Ω</td><td>$m-1$</td><td>-1</td><td colspan="2" rowspan="2">$\begin{cases} m & \text{if } \Pi = \Pi' \\ m-1 & \text{if } \Pi \neq \Pi' \text{ and } \tau(\Pi) = \tau(\Pi') \\ m-2 & \text{if } \tau(\Pi) \neq \tau(\Pi') \end{cases}$</td></tr>
<tr><td>in Ξ</td><td>$m-1$</td><td>0</td></tr>
</table>

Table 4.2. The intersection number $A.B$.

<table>
<tr><th>$A \ \backslash \ B$</th><th>M_0</th><th>$M_\infty - M_0$</th><th>$\Pi - M_0 \ \ (\Pi \in \Omega)$</th><th>$\Pi - M_0 \ \ (\Pi \in \Xi)$</th></tr>
<tr><td>M_0</td><td>$\theta(m)$</td><td>$-\theta(m)$</td><td>$-(-q)^m$</td><td>$-(-q)^m$</td></tr>
<tr><td>$M_\infty - M_0$</td><td>$-\theta(m)$</td><td>$2\theta(m)$</td><td>$(-q)^m$</td><td>$1+(-q)^m$</td></tr>
<tr><td colspan="5">and, for $\Pi, \Pi' \in \Omega \sqcup \Xi$, we have $$(\Pi - M_0).(\Pi' - M_0) = \begin{cases} 2(-q)^m & \text{if } \Pi = \Pi' \\ (-q)^m & \text{if } \Pi \neq \Pi' \text{ and } \tau(\Pi) = \tau(\Pi') \\ -(q+1)(-q)^{m-1} & \text{if } \tau(\Pi) \neq \tau(\Pi') \end{cases}$$</td></tr>
</table>

holds for any $\Pi, \Pi' \in \Omega \sqcup \Xi \sqcup \{M_0, M_\infty\}$, and show that the restriction of v' to Ω' possesses the property (ii) for an arbitrary choice of $M' \in \Xi$.

By Corollary 3.2, the dimensions of $\Pi \wedge \Pi'$ are as in Table 4.1. Using the formula (4.1), we can calculate the intersection numbers $A.B$ as in Table 4.2, where A and B are M_0, $M_\infty - M_0$ or $\Pi - M_0$ ($\Pi \in \Omega \sqcup \Xi$). Here we have used the equalities

$$\theta(m) - \theta(m-1) = (-q)^m \quad \text{and} \quad \theta(m) - 2\,\theta(m-1) + \theta(m-2) = -(q+1)(-q)^{m-1}.$$

For simplicity, we put

$$\alpha_0 := \frac{1}{\sqrt{q+1}}, \quad \alpha_\infty := -\frac{\sqrt{q^{m+1}}}{\sqrt{q+1}}, \qquad \beta_1 := \frac{1}{\sqrt{q^{m+1}}}, \quad \beta_\infty := \frac{q^{m+1}+(-1)^m}{\sqrt{q+1}\sqrt{q^{m+1}}},$$

$$\gamma_\infty := \sqrt{q^{m-1}(q+1)}, \quad \gamma_1 := \frac{\sqrt{q^m}}{q-1}\left(1 - \frac{(-1)^m}{\sqrt{q}}\right), \quad \gamma_2 := \frac{\sqrt{q^m}}{q-1}\left((2-q) - \frac{(-1)^m}{\sqrt{q}}\right),$$

$$Q := (-1)^m \sqrt{q^{m-1}}.$$

We define the map v' by Table 4.3, which indicates the value of the function $v'(A)$ at $x \in \overline{\Omega}$. Here $v'(\Pi - M_0)$ means $v'(\Pi) - v'(M_0)$. Recall that, for a given $H \in M_0^\vee(k)$, there are exactly $q-1$ elements of Ω that are mapped to H by τ. Hence the values $(v'(A), v'(B))_E$ are calculated as in Table 4.4. We can easily check that Tables 4.2 and 4.4 coincide. Thus the map v' satisfies (4.3).

TABLE 4.3. $v'(A)(x)$.

$x \setminus A$	M_0	$M_\infty - M_0$	$\Pi - M_0$ $(\Pi \in \Omega)$	$\Pi - M_0$ $(\Pi \in \Xi)$
ω	α_0	0	0	0
η	α_∞	β_∞	γ_∞	γ_∞
in Ω	0	β_1	$\diamondsuit$	$\heartsuit$

where

$$\diamondsuit = \begin{cases} \gamma_2 & \text{if } x = \Pi \\ \gamma_1 & \text{if } x \neq \Pi \text{ and } \tau(x) = \tau(\Pi) \\ 0 & \text{if } \tau(x) \neq \tau(\Pi) \end{cases} \qquad \text{and} \qquad \heartsuit = \begin{cases} Q & \text{if } \tau(x) = \tau(\Pi) \\ 0 & \text{if } \tau(x) \neq \tau(\Pi) \end{cases}$$

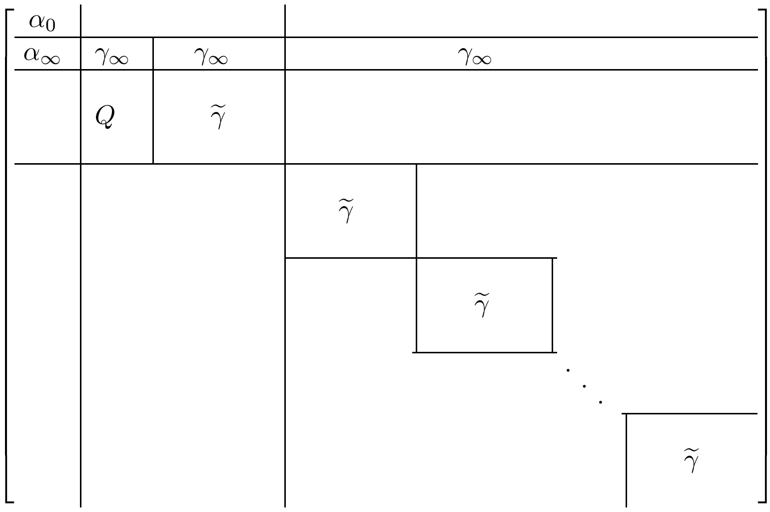
TABLE 4.4. $(v'(A), v'(B))_E$.

$A \setminus B$	M_0	$M_\infty - M_0$
M_0	$\alpha_0^2 + (-1)^m \alpha_\infty^2$	$(-1)^m \alpha_\infty \beta_\infty$
$M_\infty - M_0$	$(-1)^m \alpha_\infty \beta_\infty$	$(-1)^m (\beta_\infty^2 + \lvert\Omega\rvert\, \beta_1^2)$
$\Pi' - M_0$ $(\Pi' \in \Omega)$	$(-1)^m \alpha_\infty \gamma_\infty$	$(-1)^m \{\beta_\infty \gamma_\infty + \beta_1(\gamma_2 + (q-2)\gamma_1)\}$
$\Pi' - M_0$ $(\Pi' \in \Xi)$	$(-1)^m \alpha_\infty \gamma_\infty$	$(-1)^m \{\beta_\infty \gamma_\infty + (q-1)\beta_1 Q\}$

$A \setminus B$	$\Pi - M_0$ $(\Pi \in \Omega)$
$\Pi' - M_0$ $(\Pi' \in \Omega)$	$\begin{cases} (-1)^m \{\gamma_\infty^2 + \gamma_2^2 + (q-2)\gamma_1^2\} & \text{if } \Pi = \Pi' \\ (-1)^m \{\gamma_\infty^2 + (q-3)\gamma_1^2 + 2\gamma_1\gamma_2\} & \text{if } \Pi \neq \Pi' \text{ and } \tau(\Pi) = \tau(\Pi') \\ (-1)^m \gamma_\infty^2 & \text{if } \tau(\Pi) \neq \tau(\Pi') \end{cases}$
$\Pi' - M_0$ $(\Pi' \in \Xi)$	$\begin{cases} (-1)^m \{\gamma_\infty^2 + Q(\gamma_2 + (q-2)\gamma_1)\} & \text{if } \tau(\Pi) = \tau(\Pi') \\ (-1)^m \gamma_\infty^2 & \text{if } \tau(\Pi) \neq \tau(\Pi') \end{cases}$

$A \setminus B$	$\Pi - M_0$ $(\Pi \in \Xi)$
$\Pi' - M_0$ $(\Pi' \in \Xi)$	$\begin{cases} (-1)^m \{\gamma_\infty^2 + (q-1)Q^2\} & \text{if } \tau(\Pi) = \tau(\Pi') \\ (-1)^m \gamma_\infty^2 & \text{if } \tau(\Pi) \neq \tau(\Pi') \end{cases}$

FIGURE 4.1. The matrix Γ'.



,

We define a square matrix $\widetilde{\gamma} = (\gamma_{ij})$ of size $q-1$ by

$$\gamma_{ij} := \begin{cases} \gamma_2 & \text{if } i = j \\ \gamma_1 & \text{if } i \neq j. \end{cases}$$

We also put

$$\Omega'' := \{\, M_0, M' - M_0 \} \ \sqcup \ \{ \Pi - M_0 \ : \ \Pi \in \Omega \,\},$$

where M' is an arbitrary element of Ξ. By ordering elements of $\overline{\Omega}$ and Ω'' in a suitable way, the square matrix

$$\Gamma' := (v(A)(x) \ : \ A \in \Omega'',\ x \in \overline{\Omega})$$

of size $b_{2m}(X_J)$ is written in the form as in Figure 4.1, in which blank parts are zero matrices. Because we have

$$\det \widetilde{\gamma} = (\gamma_2 - \gamma_1)^{q-2} \cdot (\gamma_2 + (q-2)\gamma_1) = \pm \sqrt{q}^{mq-m-1}$$

and

$$\det \left(\begin{array}{c|c} \gamma_\infty & \gamma_\infty \\ \hline Q & \widetilde{\gamma} \end{array} \right) = \gamma_\infty \cdot (\gamma_2 - \gamma_1)^{q-2} \cdot (\gamma_2 + (q-2)\gamma_1 - (q-1)Q) = \pm \frac{1}{\alpha_0} \sqrt{q}^{mq},$$

it follows that $|\det \Gamma\,| = |\det \Gamma'\,|$ is a power of $\sqrt{q}$. *q.e.d.*

Because Ω' is contained in $\Omega \sqcup \Xi \sqcup \{M_0\}$, we have also proved the following.

Corollary 4.1. *The vector space* $NL^m(X_J) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is generated by the classes* $[\Pi]$, *where* Π *runs through the set* $\Omega \sqcup \Xi \sqcup \{M_0\}$.

The map v induces an isometry $\tilde{v}$ from $N^m(X_J) \otimes_{\mathbb{Z}} \mathbb{R}$ to E. Let us calculate the vector $\tilde{v}(h) \in E$, where $h \in N^m(X_J)$ is the numerical equivalence class of the intersection of X_J with a general linear subspace of $\mathbb{P}^{2m+1}$ with dimension $m+1$.

We choose H from $M_0^\vee(k)$. By Corollary 3.3, the vector $\tilde{v}(h)$ is the sum of $\tilde{v}([\Pi])$, where Π runs through the set $B_J(H)$. Because

$$(q+1)\alpha_\infty + q\gamma_\infty = 0 \quad \text{and} \quad \gamma_2 + (q-2)\gamma_1 + Q = 0,$$

we see that

$$\tilde{v}(h)(x) = \begin{cases} \alpha_0 & \text{if } x = \omega, \\ 0 & \text{if } x \in \overline{\Omega} \setminus \{\omega\}. \end{cases}$$

In particular, the primitive part $N^m_{\mathrm{prim}}(X_J) = (h)^\perp$ is mapped by $\tilde{v}$ to the subspace $\{f \in E \,:\, f(\omega) = 0\}$ of E, on which $(\ ,\ )_E$ multiplied by $(-1)^m$ is positive definite.

5. The structure of the lattice in the case $q = 2$

From now to the end of the paper, we put $p = q = 2$. In particular, the finite field k is $\mathbb{F}_4$, and X_J is projectively isomorphic to the cubic Fermat hypersurface.

Let (X_J, M_0, M_∞) be the triple defined in Section 2. We put $T := M_0^\vee(k)$, and let $\widetilde{L}^m$ be the lattice constructed from this T by the procedure described in the Introduction. Our aim in this section is to prove Theorem 1.4; that is, to construct an isomorphism $\mathcal{L}^m(X_J) \cong \widetilde{L}^m$ of lattices.

The lattice $\mathcal{L}^m(X_J)$ is generated by the numerical equivalence classes

$$[\![\Pi]\!] := [\Pi] - [M_0] \quad (\Pi \in \Sigma^m(X_J) \setminus \{M_0\}).$$

The symmetric bilinear form $(\ ,\ )_\mathcal{L}$ on $\mathcal{L}^m(X_J)$ is the intersection form multiplied by $(-1)^m$, and hence it is given by

$$([\![\Pi]\!], [\![\Pi']\!])_\mathcal{L} = (-1)^m(\theta(\dim(\Pi \wedge \Pi')) - \theta(\dim(\Pi \wedge M_0)) - \theta(\dim(\Pi' \wedge M_0)) + \theta(m)), \tag{5.1}$$

where θ is defined by (4.1). Recall that $\widetilde{T}$ is the set $T \sqcup \{\varphi\}$, where φ is a formal element, and that $\mathbb{Z}^{\widetilde{T}}$ is equipped with a $\mathbb{Q}$-valued bilinear symmetric form $(\ ,\ )_T$ given by (1.1). First we define a map $u : \Sigma^m(X_J) \to \mathbb{Z}^{\widetilde{T}}$ such that

$$([\![\Pi]\!], [\![\Pi']\!])_\mathcal{L} = (u(\Pi), u(\Pi'))_T \tag{5.2}$$

holds for any $\Pi, \Pi' \in \Sigma^m(X_J)$.

Suppose that $\dim(\Pi \wedge M_0) = a$. We define $u(\Pi) \in \mathbb{Z}^{\widetilde{T}}$ by

$$u(\Pi)(H) := \begin{cases} 0 & \text{if } H \in C_0(\Pi), \\ (-2)^{a+1} & \text{if } H \in C_\infty(\Pi), , \\ -(-2)^{a+1} & \text{if } H \in C_1(\Pi) \end{cases}$$

and

$$u(\Pi)(\varphi) := -(-2)^{a+1}\,\theta(m-a-1).$$

Then Table 3.1 implies that $u(M_0) = 0$. Because $q = 2$, the map $\tau : \Omega \sqcup \Xi \to T$ defined in Section 4 induces bijections $\Omega \cong T$ and $\Xi \cong T$ of sets. For $H \in T$, we denote by $\Pi_H^- \in \Omega$ and $\Pi_H^+ \in \Xi$ the unique elements such that $\tau(\Pi_H^-) = \tau(\Pi_H^+) = H$. Then $B_J(H)$ consists of the three elements M_0, Π_H^+ and Π_H^- for any $H \in T$. If $\Pi \in \Omega \sqcup \Xi$, then $\dim(\Pi \wedge M_0) = m - 1$ and hence $u(\Pi)(\varphi) = -(-2)^m$. Using Table 3.1, we can calculate $(u(\Pi), u(\Pi'))_T$ for $\Pi, \Pi' \in \Omega \sqcup \Xi$ as in Table 5.1. On the other hand, from Corollary 3.2, the dimensions $\dim(\Pi \wedge \Pi')$ are given by Table 5.2. We put

Table 5.1. $(u(\Pi^{\pm}_H), u(\Pi^{\pm}_{H'}))_T$.

$\Pi' \setminus \Pi$	Π^+_H	Π^-_H
$\Pi^+_{H'}$	$\begin{cases} 3\cdot 2^{m-1} & \text{if } H\ne H' \\ 4\cdot 2^{m-1} & \text{if } H=H' \end{cases}$	$\begin{cases} 3\cdot 2^{m-1} & \text{if } H\ne H' \\ 2\cdot 2^{m-1} & \text{if } H=H' \end{cases}$
$\Pi^-_{H'}$	$\begin{cases} 3\cdot 2^{m-1} & \text{if } H\ne H' \\ 2\cdot 2^{m-1} & \text{if } H=H' \end{cases}$	$\begin{cases} 3\cdot 2^{m-1} & \text{if } H\ne H' \\ 4\cdot 2^{m-1} & \text{if } H=H' \end{cases}$

Table 5.2. $\dim(\Pi^{\pm}_H \wedge \Pi^{\pm}_{H'})$.

$\Pi' \setminus \Pi$	Π^+_H	Π^-_H
$\Pi^+_{H'}$	$\begin{cases} m-2 & \text{if } H\ne H' \\ m & \text{if } H=H' \end{cases}$	$\begin{cases} m-2 & \text{if } H\ne H' \\ m-1 & \text{if } H=H' \end{cases}$
$\Pi^-_{H'}$	$\begin{cases} m-2 & \text{if } H\ne H' \\ m-1 & \text{if } H=H' \end{cases}$	$\begin{cases} m-2 & \text{if } H\ne H' \\ m & \text{if } H=H' \end{cases}$

$$\Theta_a(\nu) := (-1)^m(\theta(\nu) - \theta(a) - \theta(m-1) + \theta(m)).$$

If $\Pi, \Pi' \in \Omega \sqcup \Xi$, then $([\![\Pi]\!], [\![\Pi']\!])_{\mathcal{L}} = \Theta_{m-1}(\dim(\Pi \wedge \Pi'))$ by (5.1). It is easy to check that the equalities

$$\Theta_{m-1}(m-2) = 3\cdot 2^{m-1}, \quad \Theta_{m-1}(m-1) = 2\cdot 2^{m-1} \quad \text{and} \quad \Theta_{m-1}(m) = 4\cdot 2^{m-1}$$

hold. Hence (5.2) is satisfied when Π and Π' are elements of $\Omega \sqcup \Xi$. By Corollary 4.1, the vector space $\mathcal{L}^m(X_J) \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by the numerical equivalence classes $[\![\Pi]\!]$ of $\Pi \in \Omega \sqcup \Xi$. Hence the map $u|_{\Omega\sqcup\Xi} : \Omega \sqcup \Xi \to \mathbb{Z}^{\widetilde{T}}$ induces an isomorphism of $\mathbb{Q}$-lattices

$$\mathcal{L}^m(X_J) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Z}^{\widetilde{T}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Therefore, in order to check (5.2), it is enough to show that

$$([\![\Pi]\!], [\![\Pi^+_H]\!])_{\mathcal{L}} = (u(\Pi), u(\Pi^+_H))_T \quad \text{and} \quad ([\![\Pi]\!], [\![\Pi^-_H]\!])_{\mathcal{L}} = (u(\Pi), u(\Pi^-_H))_T \tag{5.3}$$

hold for any couple of $\Pi \in \Sigma^m(X_J) \setminus (\Omega \sqcup \Xi \sqcup \{M_0\})$ and $H \in T$. Note that the set $\Sigma^m(X_J) \setminus (\Omega \sqcup \Xi \sqcup \{M_0\})$ is the disjoint union of $\Sigma^m(X_J)^{(a,b)}$ with $a \le m-2$. Let Π be an element of $\Sigma^m(X_J)^{(a,b)}$ with $a \le m-2$. We put

$$\Phi_a := \frac{1}{2^{m+1}}(3\cdot u(\Pi)(\varphi)\cdot u(\Pi^{\pm}_H)(\varphi)) = 3\cdot(-1)^{m+1}\cdot(-2)^a\cdot\theta(m-a-1).$$

From the definition of u and Table 3.1, we can calculate $(u(\Pi), u(\Pi^{\pm}_H))_T$ as in Table 5.3. On the other hand, from the definitions of $\beta_J(H,\Pi)$ and $C_0(\Pi)$, $C_\infty(\Pi)$, $C_1(\Pi)$, we obtain Table 5.4. (See (3.4).) Because

$$([\![\Pi]\!], [\![\Pi^+_H]\!])_{\mathcal{L}} = \Theta_a(\dim(\Pi \wedge \Pi^+_H)), \qquad ([\![\Pi]\!], [\![\Pi^-_H]\!])_{\mathcal{L}} = \Theta_a(\dim(\Pi \wedge \Pi^-_H))$$

and

$$\Theta_a(a-1) = \Phi_a, \quad \Theta_a(a+1) = \Phi_a + (-1)^{m+1}(-2)^a \quad \text{and} \quad \Theta_a(a) = \Phi_a - (-1)^{m+1}(-2)^a,$$

we see that the map u satisfies (5.3) and hence (5.2).

TABLE 5.3. $(u(\Pi), u(\Pi_H^{\pm}))_T$.

	$(u(\Pi), u(\Pi_H^{+}))_T$	$(u(\Pi), u(\Pi_H^{-}))_T$
$H \in C_0(\Pi)$	Φ_a	Φ_a
$H \in C_\infty(\Pi)$	$\Phi_a + (-1)^{m+1}(-2)^a$	$\Phi_a - (-1)^{m+1}(-2)^a$
$H \in C_1(\Pi)$	$\Phi_a - (-1)^{m+1}(-2)^a$	$\Phi_a + (-1)^{m+1}(-2)^a$

TABLE 5.4. $\dim(\Pi \wedge X)$ for $X = H, M_0, \Pi_H^{\pm}$.

	$\dim(\Pi \wedge H)$	$\dim(\Pi \wedge M_0)$	$\dim(\Pi \wedge \Pi_H^{+})$	$\dim(\Pi \wedge \Pi_H^{-})$
$H \in C_0(\Pi)$	$a-1$	a	$a-1$	$a-1$
$H \in C_\infty(\Pi)$	a	a	$a+1$	a
$H \in C_1(\Pi)$	a	a	a	$a+1$

The map u induces an embedding $\tilde{u} : \mathcal{L}^m(X_J) \hookrightarrow \mathbb{Z}^{\widetilde{T}}$ of the lattice $\mathcal{L}^m(X_J)$ into $\mathbb{Z}^{\widetilde{T}}$. Next we show that the image of $\tilde{u}$ coincides with $\widetilde{L}^m$. First note that the vectors $u(\Pi_H^+) - u(\Pi_H^-)$ and $u(\Pi_H^+) + u(\Pi_H^-)$, where H runs through T, generate the kernel of the natural projection $\widetilde{\mathrm{pr}} : \mathbb{Z}^{\widetilde{T}} \to R^{\widetilde{T}}$, where $R = \mathbb{Z}/(2^{m+1})$. Hence we have $\operatorname{Ker} \widetilde{\mathrm{pr}} \subset \operatorname{Im} \tilde{u}$. Second, using Corollary 3.12, we can check that

$$\begin{aligned} -u(\Pi)(\varphi) + \sum_{H \in T} u(\Pi)(H) &= (-2)^{a+1}\theta(m-a-1) + (|C_\infty(\Pi)| - |C_1(\Pi)|) \cdot (-2)^{a+1} \\ &= (-2)^{m+1} \cdot (-1 + (-2)^{b+1}) \,/\, 3 \end{aligned}$$

is 0 modulo 2^{m+1} for any $\Pi \in \Sigma^m(X_J)$. This means that $\widetilde{\mathrm{pr}}(u(\Pi)) \in R^{\widetilde{T}}$ is the extension of its restriction $\widetilde{\mathrm{pr}}(u(\Pi))|_T \in R^T$ to T. Therefore, it is enough to show that the restriction $\widetilde{\mathrm{pr}}(\operatorname{Im} \tilde{u})|_T \subset R^T$ of the R-submodule $\widetilde{\mathrm{pr}}(\operatorname{Im} \tilde{u})$ of $R^{\widetilde{T}}$ to $T \subset \widetilde{T}$ coincides with the R-submodule $\overline{L}^m$ of R^T.

First we prepare several lemmas. Recall that $\overline{L}^m$ is generated by $\overline{V}_T$ and $2\,\overline{V}_S$, where S runs through the linear subspace $\mathcal{H}_m$ of $\mathbb{F}_2^T$ generated by the sets of k-rational points of Hermitian hypersurfaces in $M_0^\vee$. Here we identify $\mathbb{F}_2^T$ with the power set 2^T of T. (See Convention (6).) Let H_{m+1} be the vector space over $\mathbb{F}_2$ of Hermitian matrices of size $m+1$. Let $\xi = (\xi_0, \dots, \xi_m)$ be k-rational homogeneous coordinates of $M_0^\vee$. For $A \in H_{m+1}$, let $f_A(\xi, \xi)$ be the cubic homogeneous polynomial $\xi \cdot A \cdot {}^t\xi^{(2)}$, and let Y_A be the Hermitian hypersurface of $M_0^\vee$ defined by $f_A(\xi, \xi) = 0$.

Lemma 5.1. *If $A, A' \in H_{m+1}$, then the subset $Y_A(k) + Y_{A'}(k)$ of T coincides with $T + Y_{A+A'}(k) = T \setminus Y_{A+A'}(k)$. In particular, the map $A \mapsto T + Y_A(k)$ yields a homomorphism $\alpha : H_{m+1} \to \mathbb{F}_2^T$ of vector spaces over $\mathbb{F}_2$.*

Proof. If $a = (a_0, \dots, a_m)$ is a vector with $a_i \in k$, then $f_A(a, a)$ satisfies $f_A(a, a)^2 = f_A(a, a)$, because ${}^tA^{(2)} = A$ and $a^{(4)} = a$. Hence $f_A(a, a)$ is in $\mathbb{F}_2$. Then we have

$$\begin{aligned} a \notin Y_{A+A'}(k) &\iff f_A(a, a) + f_{A'}(a, a) = 1 \iff \\ &(f_A(a, a), f_{A'}(a, a)) = (1, 0) \;\text{ or }\; (0, 1) \iff a \in Y_A(k) + Y_{A'}(k). \end{aligned}$$

Thus the first assertion is proved. The second assertion is now obvious. *q.e.d.*

Lemma 5.2. *Suppose that S is an element of $\mathcal{H}_m$. Then one and only one of the following holds*;

(i) *there is a Hermitian hypersurface Y of $M_0^\vee$ such that $S = Y(k)$, or*
(ii) *there is a Hermitian hypersurface Y of $M_0^\vee$ such that $S = T + Y(k)$.*

Proof. First note that $\mathcal{H}_m$ is stable under the involution $S \mapsto T + S$ of the power set $\mathbb{F}_2^T$, because $T \in \mathcal{H}_m$. Let S be an element of $\mathcal{H}_m$. Then there are Hermitian matrices $A_1, \dots, A_l \in H_{m+1}$ such that $S = Y_{A_1}(k) + \cdots + Y_{A_l}(k)$. By Lemma 5.1, we have

$$(T + Y_{A_1}(k)) + \cdots + (T + Y_{A_l}(k)) = T + Y_{A_1+\cdots+A_l}(k).$$

Thus S is $Y_{A_1+\cdots+A_l}(k)$ if l is odd, while it is $T + Y_{A_1+\cdots+A_l}(k)$ if l is even. Assume that there were two Hermitian matrices $A, A' \in H_{m+1}$ such that $Y_A(k) = T + Y_{A'}(k)$. By Lemma 5.1, it would follow that $Y_{A+A'}(k) = \emptyset$. However any Hermitian hypersurface has at least one k-rational point. Hence we get a contradiction. *q.e.d.*

Lemma 5.3. *Let $A_1, \dots, A_d$ be a basis of H_{m+1} over $\mathbb{F}_2$, where $d = (m+1)^2$. Then T, $Y_{A_1}(k), \dots, Y_{A_d}(k)$ form a basis of $\mathcal{H}_m$.*

Proof. Because $\mathbb{F}_2^\times = \{1\}$, Proposition 2.8 implies that the homomorphism α defined in Lemma 5.1 is injective. The subspace $\mathcal{H}_m$ of $\mathbb{F}_2^T$ is generated by the image of α and T. Because there are no Hermitian matrices $A \in H_{m+1}$ such that $Y_A(k) = \emptyset$, we have $T \notin \operatorname{Im} \alpha$. *q.e.d.*

For a linear code $\mathcal{H} \subseteq \mathbb{F}_2^T$, its weight enumerator is defined as follows:

$$\operatorname{we}(\mathcal{H}) := \sum_{S \in \mathcal{H}} z^{|S|},$$

where z is a formal variable. By Lemma 5.2 combined with Propositions 2.7 and 2.8 and Corollary 2.6, we obtain the following formula.

Corollary 5.4.

$$\operatorname{we}(\mathcal{H}_m) = \sum_{r=0}^{m+1} h(m,r) \cdot (z^{F(m,r)} + z^{|T| - F(m,r)}).$$

Lemma 5.5. *Let $S_1, \dots, S_l$ be elements of $\mathcal{H}_m$. We put $W := S_1 \cap \cdots \cap S_l$. Then $2^l\, \overline{V}_W$ is contained in $\overline{L}^m$.*

Proof. We put $I := \{1, \dots, l\}$. For a positive integer λ with $\lambda \le l$, let $I^{[\lambda]}$ be the set of non-ordered λ-tuples of distinct elements of I. For $\mathbf{i} = \{i_1, \dots, i_\lambda\} \in I^{[\lambda]}$, we put

$$S_{\mathbf{i}} := S_{i_1} + \cdots + S_{i_\lambda}.$$

Note that $2\,\overline{V}_{S_{\mathbf{i}}}$ is an element of $\overline{L}^m$ for any λ and any $\mathbf{i} \in I^{[\lambda]}$. On the other hand, it is easy to see that

$$\sum_{\lambda=1}^{l} (-1)^{\lambda+1} \sum_{\mathbf{i} \in I^{[\lambda]}} \overline{V}_{S_{\mathbf{i}}} = 2^{l-1}\, \overline{V}_W.$$

Thus $2^l\, \overline{V}_W$ is an element of $\overline{L}^m$. *q.e.d.*

Table 6.1. MOG.

μ_1	μ_5	μ_9	μ_{13}	μ_{17}	μ_{21}
μ_2	μ_6	μ_{10}	μ_{14}	μ_{18}	μ_{22}
μ_3	μ_7	μ_{11}	μ_{15}	μ_{19}	μ_{23}
μ_4	μ_8	μ_{12}	μ_{16}	μ_{20}	μ_{24}

We are now ready to prove $\widetilde{\mathrm{pr}}\,(\operatorname{Im}\tilde{u})|_T = \overline{L}^m$. For simplicity, we put

$$\overline{\mathcal{L}}^m(X_J) := \widetilde{\mathrm{pr}}\,(\operatorname{Im}\tilde{u})|_T, \quad \text{and} \quad \bar{u}(\Pi) := \widetilde{\mathrm{pr}}\,(u(\Pi))|_T \in R^T.$$

Then $\overline{\mathcal{L}}^m(X_J)$ is the R-submodule of R^T generated by $\bar{u}(\Pi)$, where Π runs through $\Sigma^m(X_J)$. First we will show that $\overline{\mathcal{L}}^m(X_J)$ contains $\overline{L}^m$. By Table 3.1, we have

$$\bar{u}(M_\infty) = \overline{V}_T.$$

Let Y be a Hermitian hypersurface of rank r in $M_0^\vee$. By Corollary 3.11, there is an element $\Pi \in \Sigma^m(X_J)^{(-1,m-r)}$ such that $C_\infty(\Pi) = Y(k)$ and $C_1(\Pi) = T \setminus Y(k)$. Then we have

$$\bar{u}(M_\infty) + \bar{u}(\Pi) = 2\,\overline{V}_{Y(k)}, \quad \text{and} \quad \bar{u}(M_\infty) - \bar{u}(\Pi) = 2\,\overline{V}_{T+Y(k)}.$$

Using these identities and Lemma 5.2, we see that $\overline{\mathcal{L}}^m(X_J)$ contains all generators of $\overline{L}^m$. Next we show that $\overline{\mathcal{L}}^m(X_J)$ is contained in $\overline{L}^m$. It is enough to show that $\bar{u}(\Pi) \in \overline{L}^m$ for any $\Pi \in \Sigma^m(X_J)$. Note that a k-rational hyperplane K of $M_0^\vee$ is a Hermitian hypersurface of $M_0^\vee$ with rank 1, and hence $K(k) \in \mathcal{H}_m$. Let Π be an element of $\Sigma^m(X_J)^{(a,b)}$. By Corollary 3.11, there is a couple (R, Y) of a k-rational linear subspace R of M_0 with $\dim R = a$, and a Hermitian hypersurface Y in $(M_0/R)^\vee$ such that $C_\infty(\Pi) \sqcup C_1(\Pi) = (M_0/R)^\vee(k)$ and $C_\infty(\Pi) = Y(k)$. Then we have

$$\begin{aligned}\bar{u}(\Pi) &= (-2)^{a+1}\,\overline{V}_{C_\infty(\Pi)} - (-2)^{a+1}\,\overline{V}_{C_1(\Pi)} \\ &= -(-1)^{a+1} \cdot (2^{a+1}\,\overline{V}_{(M_0/R)^\vee(k)} - 2^{a+2}\,\overline{V}_{Y(k)}).\end{aligned}$$

On the other hand, there are k-rational hyperplanes $K_1, \ldots, K_{a+1}$ of $M_0^\vee$ and a Hermitian hypersurface $\tilde{Y}$ of $M_0^\vee$ such that $(M_0/R)^\vee = K_1 \cap \cdots \cap K_{a+1}$ and $Y = K_1 \cap \cdots \cap K_{a+1} \cap \tilde{Y}$. Lemma 5.5 implies that $2^{a+1}\,\overline{V}_{(M_0/R)^\vee(k)}$ and $2^{a+2}\,\overline{V}_{Y(k)}$ are elements of $\overline{L}^m$. Hence $\bar{u}(\Pi)$ is an element of $\overline{L}^m$.

Thus the proof of Theorem 1.4 is completed. *q.e.d.*

Corollary 5.6. *The lattice $\widetilde{L}^m$ is an even lattice with minimal norm at most 2^m.*

Proof. The generators $[\Pi] - [\Pi']$ of $\mathcal{L}^m(X_J)$ have even norms. When $\dim(\Pi \wedge \Pi') = m - 2$, the norm of $[\Pi] - [\Pi']$ is equal to 2^m. *q.e.d.*

6. Edge-Jónsson-McKay correspondence

In this section, we put $m = 2$ and prove Theorem 1.5.

First let us recall the construction of the Leech lattice Λ_{24} and the laminated lattice Λ_{22}. For details, see Conway and Sloane's book [2]. We label the positions of the Miracle Octad Generator (MOG) as in Table 6.1, and put $M := \{\mu_1, \ldots, \mu_{24}\}$. Let $\mathcal{C}_{24} \subset \mathbb{F}_2^M$ be the Golay code. (See [2, Chapter 11, Section 5] for the MOG and the Golay code.) A subset S of M is said to be a $\mathcal{C}$-set if S is an element of $\mathcal{C}_{24}$

Table 6.2. Definition of γ.

$(1:\omega:0)$ $(1:\omega:1)$	$(1:\bar{\omega}:0)$ $(1:\bar{\omega}:1)$	$(1:1:0)$ $(1:1:1)$	$(1:0:0)$ $(1:0:1)$	$(0:1:0)$ $(0:1:1)$	$(0:0:1)$ φ
$(1:\omega:\omega)$ $(1:\omega:\bar{\omega})$	$(1:\bar{\omega}:\omega)$ $(1:\bar{\omega}:\bar{\omega})$	$(1:1:\omega)$ $(1:1:\bar{\omega})$	$(1:0:\omega)$ $(1:0:\bar{\omega})$	$(0:1:\omega)$ $(0:1:\bar{\omega})$	φ φ

under the identification $\mathbb{F}_2^M = 2^M$. We equip $\mathbb{Z}^M$ with a positive definite symmetric bilinear form $(\ ,\ )_M$ defined by

$$(v, w)_M := \frac{1}{8} \sum_{\mu \in M} v(\mu) w(\mu). \tag{6.1}$$

We use [2, Chapter 10, Theorem 25] as a definition of the Leech lattice.

Definition 6.1. The Leech lattice Λ_{24} is the sublattice of $\mathbb{Z}^M$ consisting of vectors $x \in \mathbb{Z}^M$ satisfying the following:

(i) the coordinates $x(\mu_i)$ are all even or all odd,
(ii) for any $\alpha \in \mathbb{Z}/(4)$, the set $\{\mu_i \in M \,:\, x(\mu_i) \bmod 4 = \alpha\}$ is a $\mathcal{C}$-set, and
(iii) if $x(\mu_i)$ are even, then $\sum_{i=1}^{24} x(\mu_i) \bmod 8 = 0$ holds, while if $x(\mu_i)$ are odd, then $\sum_{i=1}^{24} x(\mu_i) \bmod 8 = 4$ holds.

For an abelian group A, we write by $(A^M)_{22}$ the submodule of A^M consisting of functions $a : M \to A$ satisfying $a(\mu_{22}) = a(\mu_{23}) = a(\mu_{24})$. We use [2, Chapter 6, Figure 6.2] as the definition of laminated lattices.

Definition 6.2. The laminated lattice Λ_{22} is defined to be $\Lambda_{24} \cap (\mathbb{Z}^M)_{22}$.

The minimal norm of Λ_{22} is 4 and the kissing number is 49896.

We will construct an isomorphism of lattices between $\widetilde{L}^2$ and Λ_{22}. In the present case $m = 2$, the set $\widetilde{T} = M_0^\vee(k) \sqcup \{\varphi\}$ consists of 22 elements. We define a map $\gamma : M \to \widetilde{T}$ by the MOG diagram given in Table 6.2, where $\omega \in k = \mathbb{F}_4$ is a root of the equation $x^2 + x + 1 = 0$ and $\bar{\omega} = \omega^2$. Then the natural homomorphism $\gamma^* : \mathbb{Z}^{\widetilde{T}} \to \mathbb{Z}^M$ induced by γ yields an isomorphism of $\mathbb{Q}$-lattices $\mathbb{Z}^{\widetilde{T}} \cong (\mathbb{Z}^M)_{22}$, where the symmetric bilinear forms on $\mathbb{Z}^{\widetilde{T}}$ and $\mathbb{Z}^M$ are defined by (1.1) and (6.1), respectively. It is enough to show that $\gamma^*(\widetilde{L}^2)$ coincides with Λ_{22}.

By Corollary 5.4, the weight distribution of the linear code $\mathcal{H}_2 \subset \mathbb{F}_2^T$ is

$$0^1 5^{21} 8^{210} 9^{280} 12^{280} 13^{210} 16^{21} 21^1. \tag{6.2}$$

In particular, we see that $|S| \bmod 4$ is either 0 or 1 for any $S \in \mathcal{H}_2$. By definition, the subspace $(\mathbb{F}_2^M)_{22} \subset \mathbb{F}_2^M$ consists of the subsets $S \subseteq T$ satisfying either $\{\mu_{22}, \mu_{23}, \mu_{24}\} \cap S = \emptyset$ or $\{\mu_{22}, \mu_{23}, \mu_{24}\} \subseteq S$. We define a map $\tilde{\gamma}^* : \mathcal{H}_2 \to (\mathbb{F}_2^M)_{22}$ as follows:

$$\tilde{\gamma}^*(S) := \begin{cases} \gamma^{-1}(S) & \text{if } |S| \bmod 4 = 0, \\ \gamma^{-1}(S) \sqcup \{\mu_{22}, \mu_{23}, \mu_{24}\} & \text{if } |S| \bmod 4 = 1. \end{cases}$$

We put

$$\mathcal{C}_{22} := \mathcal{C}_{24} \cap (\mathbb{F}_2^M)_{22}.$$

The map γ is defined in such a way that the following holds.

The map $\tilde{\gamma}^*$ induces an isomorphism $\mathcal{H}_2 \cong \mathcal{C}_{22}$ of vector spaces over $\mathbb{F}_2$. (6.3)

This claim is proved as follows. We see that $F(2, r) \bmod 4 = 1$ holds for $r = 0, \dots, 3$, where F is the function given in Proposition 2.7. Hence Lemma 5.2 implies that a code word S of $\mathcal{H}_2$ satisfies $|S| \bmod 4 = 0$ if and only if S is of the form $T + Y_A(k)$ for some Hermitian matrix A. Then Lemma 5.1 implies that, for $S_1, S_2 \in \mathcal{H}_2$, we have

$$|S_1 + S_2| \bmod 4 = 0 \quad \Longleftrightarrow \quad (\,|S_1| \bmod 4\ ,\ |S_2| \bmod 4\,) = (0, 0) \text{ or } (1, 1).$$

Hence the map $\tilde{\gamma}^*$ is a linear homomorphism over $\mathbb{F}_2$. We can write down a basis $S_1, \dots, S_{10}$ of $\mathcal{H}_2$ by Lemma 5.3. Using the method described in [2, Chapter 11, Section 5], we can check that $\tilde{\gamma}^*(S_i)$ is a $\mathcal{C}$-set for each S_i. Hence the image of $\tilde{\gamma}^*$ is in $\mathcal{C}_{22}$. It is obvious that $\tilde{\gamma}^*$ is injective. Hence it suffices to show that $\dim_{\mathbb{F}_2} \mathcal{C}_{22}$ is equal to $\dim_{\mathbb{F}_2} \mathcal{H}_2 = 10$. Let N be the subset $\{\mu_1, \dots, \mu_{21}\}$ of M, and $\mathrm{res}^M : (\mathbb{F}_2^M)_{22} \to \mathbb{F}_2^N$ the restriction homomorphism. The kernel of res^M intersects $\mathcal{C}_{22}$ only at the zero vector (that is, the empty set), because there are no non-empty $\mathcal{C}$-sets consisting of three or fewer elements. Therefore $\mathcal{C}_{22}$ and $\mathrm{res}^M(\mathcal{C}_{22})$ are isomorphic as vector spaces over $\mathbb{F}_2$. The weight distribution of the linear code $\mathrm{res}^M(\mathcal{C}_{22})$ can be read from [2, Chapter 10, Tables 10.1 and 10.2], and it coincides with (6.2). Hence we have $\dim_{\mathbb{F}_2} \mathcal{C}_{22} = \dim_{\mathbb{F}_2} \mathrm{res}^M(\mathcal{C}_{22}) = 10$. Thus (6.3) is proved.

Let us consider the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{Z}^{\widetilde{T}} & \xrightarrow[\gamma^*]{\sim} & (\mathbb{Z}^M)_{22} \\
{\scriptstyle \widetilde{\mathrm{pr}}^{\,T}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}^M} \\
R^{\widetilde{T}} & \xrightarrow[\gamma_R^*]{\sim} & (R^M)_{22},
\end{array}$$

where R is the finite ring $\mathbb{Z}/(8)$ and the vertical arrows are the natural projections. It is easy to see that $\mathrm{Ker}(\mathrm{pr}^M)$ is contained in Λ_{22}. Hence, in order to prove $\gamma^*(\widetilde{L}^2) = \Lambda_{22}$, it is enough to show that $\gamma_R^*((\overline{L}^2)\tilde{\ })$ coincides with $\mathrm{pr}^M(\Lambda_{22})$.

First we show that $\gamma_R^*((\overline{L}^2)\tilde{\ })$ is contained in $\mathrm{pr}^M(\Lambda_{22})$. The R-module $\gamma_R^*((\overline{L}^2)\tilde{\ })$ is generated by $\gamma_R^*((\overline{V}_T)\tilde{\ })$ and $\gamma_R^*(2\,(\overline{V}_S)\tilde{\ })$, where S runs through $\mathcal{H}_2$. The vector $\gamma_R^*((\overline{V}_T)\tilde{\ }) \in (R^M)_{22}$ is given by

$$\gamma_R^*((\overline{V}_T)\tilde{\ })(\mu_i) = \begin{cases} 1 & \text{if } i \le 21, \\ 5 & \text{if } 22 \le i \le 24, \end{cases}$$

which is an element of $\mathrm{pr}^M(\Lambda_{22})$. Suppose that $S \in \mathcal{H}_2$. The value of $2\,(\overline{V}_S)\tilde{\ }$ at the formal element φ is 0 if $|S| \bmod 4 = 0$, while it is 2 if $|S| \bmod 4 = 1$. Hence the set $\{\mu_i \in M\ :\ \gamma_R^*(2\,(\overline{V}_S)\tilde{\ })(\mu_i) \bmod 8 = 2\}$ coincides with $\tilde{\gamma}^*(S)$, which is an element of $\mathcal{C}_{22}$ by (6.3). Hence the vector $v'_S \in (\mathbb{Z}^M)_{22}$ defined by

$$v'_S(\mu_i) := \begin{cases} 0 & \text{if } \mu_i \notin \tilde{\gamma}^*(S), \\ 2 & \text{if } \mu_i \in \tilde{\gamma}^*(S) \end{cases}$$

is a vector of Λ_{22}, which is mapped to $\gamma_R^*(2\,(\overline{V}_S)\tilde{\ })$ by pr^M. Therefore $\gamma_R^*(2\,(\overline{V}_S)\tilde{\ })$ is also an element of $\mathrm{pr}^M(\Lambda_{22})$. Next we show that an arbitrary element $\bar{w}$ of $\mathrm{pr}^M(\Lambda_{22})$ is contained in $\gamma_R^*((\overline{L}^2)\tilde{\ })$. If $\bar{w}(\mu_i) \bmod 2 = 1$, then we replace $\bar{w}$ by

$\bar{w} - \gamma_R^*((\overline{V}_T)\tilde{\ })$ so that we can assume $\bar{w}(\mu_i) \mod 2 = 0$ for any $\mu_i \in M$. The set $\{\mu_i \in M \ :\ \bar{w}(\mu_i) \bmod 4 = 2\}$ is an element of $\mathcal{C}_{22}$. By (6.3), it coincides with $\tilde{\gamma}^*(S)$ for some $S \in \mathcal{H}_2$. Then the element $\bar{w} + \gamma_R^*(2\,(\overline{V}_S)\tilde{\ })$ is contained in the R-submodule

$$\{\ \bar{w}' \in \mathrm{pr}^M(\Lambda_{22}) \ :\ \bar{w}'(\mu_i) \bmod 4 = 0 \ \text{ for all } \ \mu_i \in M\ \}$$

of $(R^M)_{22}$. This submodule is generated by elements $\bar{w}_i$ $(i = 1, \dots, 21)$ defined by

$$\bar{w}_i(\mu_j) := \begin{cases} 4 \bmod 8 & \text{if } j = i \text{ or } j \in \{22, 23, 24\}, \\ 0 \bmod 8 & \text{otherwise.} \end{cases}$$

Hence it suffices to show that these $\bar{w}_i$ belong to $\gamma_R^*((\overline{L}^2)\tilde{\ })$. Let $p_i \in T$ be the k-rational point of $M_0^\vee$ such that $\gamma(\mu_i) = p_i$. There are k-rational lines l and l' on $M_0^\vee$ such that $l \cap l' = \{p_i\}$. Because $l(k)$, $l'(k)$ and $l(k) + l'(k)$ are elements of $\mathcal{H}_2$, and because $\bar{w}_i$ is written as

$$\bar{w}_i = \gamma_R^*(2\,(\overline{V}_{l(k)})\tilde{\ } + 2\,(\overline{V}_{l'(k)})\tilde{\ } - 2\,(\overline{V}_{l(k)+l'(k)})\tilde{\ }),$$

we have $\bar{w}_i \in \gamma_R^*((\overline{L}^2)\tilde{\ })$. *q.e.d.*

7. Discriminant, minimal norm and kissing number of $\widetilde{L}^3$

In this section, we prove Theorem 1.6.

First we will present a method for estimating $\mathrm{disc}(\widetilde{L}^m)$ from above. This method yields $\mathrm{disc}(\widetilde{L}^3) = 2^{16} \cdot 3$ stated in Theorem 1.6.

The discriminant of the $\mathbb{Q}$-lattice $\mathbb{Z}^{\widetilde{T}}$ with the symmetric bilinear form $(\ ,\ )_T$ given by (1.1) is $2^{-(m+1)(|T|+1)} \cdot 3$. The index of $\widetilde{L}^m$ in $\mathbb{Z}^{\widetilde{T}}$ is

$$|R^{\widetilde{T}}/(\overline{L}^m)\tilde{\ }| = 2^{(m+1)(|T|+1)}/\,|\overline{L}^m|,$$

because $(\overline{L}^m)\tilde{\ }$ and $\overline{L}^m$ are isomorphic as R-modules. Hence we have

$$\mathrm{disc}(\widetilde{L}^m) = \mathrm{disc}(\mathbb{Z}^{\widetilde{T}}) \cdot [\mathbb{Z}^{\widetilde{T}} : \widetilde{L}^m]^2 = 3 \cdot 2^{(m+1)(|T|+1)}/\,|\overline{L}^m|^2. \tag{7.1}$$

For a non-negative integer n with $n \le m+1$, let ϕ^n be the natural projection from $R^T = (\mathbb{Z}/(2^{m+1}))^T$ to $(\mathbb{Z}/(2^n))^T$. We put

$$(\overline{L}^m)^n := \mathrm{Ker}\,\phi^n \cap \overline{L}^m.$$

Then we obtain a decreasing filtration

$$\overline{L}^m = (\overline{L}^m)^0 \supseteq (\overline{L}^m)^1 \supseteq (\overline{L}^m)^2 \supseteq \cdots \supseteq (\overline{L}^m)^m \supseteq (\overline{L}^m)^{m+1} = 0$$

of $\overline{L}^m$. There is a natural identification of $\mathrm{Ker}\,\phi^n/\,\mathrm{Ker}\,\phi^{n+1}$ with $\mathbb{F}_2^T$. Using this identification, we define a linear code $\mathcal{C}^{\,m}(n) \subseteq \mathbb{F}_2^T$ by

$$\mathcal{C}^{\,m}(n) := (\overline{L}^m)^n/(\overline{L}^m)^{n+1}.$$

Then we have

$$|\,\overline{L}^m\,| = \prod_{n=0}^{m} |\,\mathcal{C}^{\,m}(n)|. \tag{7.2}$$

An element α of R is uniquely written in the form

$$\alpha = r_0(\alpha) + 2\,r_1(\alpha) + 4\,r_2(\alpha) + \cdots + 2^m\,r_m(\alpha),$$

TABLE 7.1. $\dim_{\mathbb{F}_2} \mathcal{H}_m(n)$.

$m \setminus n$	0	1	2	3	4	5
3	1	17	61	85		
4	1	26	146	296	341	
5	1	37	302	882	1289	1365

where $r_i(\alpha) \in \{0,1\}$. Hence, for any vector $\bar{v} \in R^T$, there is a unique sequence $U_0(\bar{v}), U_1(\bar{v}), \dots, U_m(\bar{v})$ of subsets of T such that

$$\bar{v} = \overline{V}_{U_0(\bar{v})} + 2\,\overline{V}_{U_1(\bar{v})} + 4\,\overline{V}_{U_2(\bar{v})} + \cdots + 2^m\,\overline{V}_{U_m(\bar{v})}. \tag{7.3}$$

Note that a vector $\bar{v}$ of $\overline{L}^m$ is contained in $(\overline{L}^m)^n$ if and only if $U_0(\bar{v}) = \cdots = U_{n-1}(\bar{v}) = \emptyset$, and that, if $\bar{v} \in (\overline{L}^m)^n$, then the image of $\bar{v}$ in $\mathcal{C}^m(n)$ by the natural projection $(\overline{L}^m)^n \to \mathcal{C}^m(n)$ is $U_n(\bar{v})$. It is easy to see from the definition of $\overline{L}^m$ that $\mathcal{C}^m(0) = \{\emptyset, T\}$. For a positive integer l, let $\mathcal{H}_m(l) \subseteq \mathbb{F}_2^T$ be the linear code generated by the subsets

$$S_1 \cap \cdots \cap S_l \qquad (S_1, \dots, S_l \in \mathcal{H}_m)$$

of T. We have $\mathcal{H}_m(1) = \mathcal{H}_m$. We put $\mathcal{H}_m(0) := \mathcal{C}^m(0) = \{\emptyset, T\}$.

Lemma 7.1. *If $\bar{v} \in \overline{L}^m$, then $U_i(\bar{v})$ is a member of $\mathcal{H}_m(2^{i-1})$ for each i.*

Proof. A vector $\bar{v}$ of $\overline{L}^m$ is written in the form

$$\bar{v} = \overline{V}_{U_0(\bar{v})} + 2\,\overline{V}_{S_1} + \cdots + 2\,\overline{V}_{S_N}, \tag{7.4}$$

where $U_0(\bar{v}) \in \mathcal{C}^m(0) = \{\emptyset, T\}$ and $S_1, \dots, S_N \in \mathcal{H}_m$. Using the formula

$$2^l\,\overline{V}_S + 2^l\,\overline{V}_{S'} = 2^l\,\overline{V}_{S+S'} + 2^{l+1}\,\overline{V}_{S\cap S'}$$

and

$$(S + S') \cap S'' = (S \cap S'') + (S' \cap S'') \tag{7.5}$$

recursively, we can rewrite (7.4) into the form (7.3), where $U_i(\bar{v}) \in \mathcal{H}_m(2^{i-1})$. *q.e.d.*

Lemma 7.2. *For a positive integer n with $n \le m$, we have $\mathcal{H}_m(n) \subseteq \mathcal{C}^m(n) \subseteq \mathcal{H}_m(2^{n-1})$. In particular, the subcode $\mathcal{H}_m(1)$ coincides with $\mathcal{C}^m(1)$ and the subcode $\mathcal{H}_m(2)$ coincides with $\mathcal{C}^m(2)$. Moreover, we have $\mathcal{H}_m(m) = \mathcal{C}^m(m) = \mathbb{F}_2^T$.*

Proof. The inclusion $\mathcal{H}_m(n) \subseteq \mathcal{C}^m(n)$ follows from Lemma 5.5. The inclusion $\mathcal{C}^m(n) \subseteq \mathcal{H}_m(2^{n-1})$ follows from Lemma 7.1. If K is a k-rational hyperplane of $M_0^\vee$, then the subset $K(k)$ of T is a member of $\mathcal{H}_m$. Any k-rational point of $M_0^\vee$ is expressed as the intersection of k-rational hyperplanes $K_1, \dots, K_m$ of $M_0^\vee$. Hence $\{H\}$ is a code word of $\mathcal{H}_m(m)$ for any $H \in T$, which implies that $\mathcal{H}_m(m) = \mathbb{F}_2^T$. *q.e.d.*

By Lemma 5.3, we can write down a basis of $\mathcal{H}_m(1)$, which consists of N subsets $B_1, \dots, B_N$ of T, where $N := (m+1)^2 + 1$. Suppose that a basis $B'_1, \dots, B'_M$ of $\mathcal{H}_m(n)$ is given. Because of the formula (7.5), we see that $\mathcal{H}_m(n+1)$ is generated by the subsets $B_i \cap B'_j$, where $i = 1, \dots, N$ and $j = 1, \dots, M$. Hence we can pick up a basis of $\mathcal{H}_m(n+1)$ from these NM subsets using a simple linear algebra over $\mathbb{F}_2$. By this inductive method, we obtain Table 7.1 of $\dim_{\mathbb{F}_2} \mathcal{H}_m(n)$ for $m \le 5$. Combining Table 7.1 with (7.1), (7.2) and Lemma 7.2, we obtain the following:

$$\begin{aligned}
\operatorname{disc}(\widetilde{L}^3) &= 2^{16}\cdot 3,\\
\operatorname{disc}(\widetilde{L}^4) &= 2^{\nu(4)}\cdot 3 \quad\text{with}\quad \nu(4)\le 90,\\
\operatorname{disc}(\widetilde{L}^5) &= 2^{\nu(5)}\cdot 3 \quad\text{with}\quad \nu(5)\le 444.
\end{aligned}$$

Next we consider the minimal norm and the kissing number of $\widetilde{L}^3$. We put $m = 3$ and $R = \mathbb{Z}/(16)$. Let $\operatorname{ws}(\mathcal{H}_3(n))$ be the set $\{|S| : S \in \mathcal{H}_3(n)\}$ of weights of the code words of $\mathcal{H}_3(n)$. By Corollary 5.4, the weight enumerator $\operatorname{we}(\mathcal{H}_3(1))$ and hence the weight set $\operatorname{ws}(\mathcal{H}_3(1))$ are easily calculated:

$$\begin{aligned}
\operatorname{we}(\mathcal{H}_3(1)) =& z^{85} + 85\, z^{64} + 3570\, z^{53} + 23800\, z^{48} + 38080\, z^{45} +\\
&+ 38080\, z^{40} + 23800\, z^{37} + 3570\, z^{32} + 85\, z^{21} + 1,\\
\operatorname{ws}(\mathcal{H}_3(1)) =& \{0, 21, 32, 37, 40, 45, 48, 53, 64, 85\}.
\end{aligned}$$

We can calculate $\operatorname{ws}(\mathcal{H}_3(2))$ as follows. Let $\mathbb{F}_2^T$ be equipped with a non-degenerate symmetric bilinear form $(S, S') := |S \cap S'| \bmod 2$. The orthogonal complement $\mathcal{H}_3(2)^\perp$ of $\mathcal{H}_3(2)$ in $\mathbb{F}_2^T$ is of dimension $85 - 61 = 24$. Because we have calculated a basis of $\mathcal{H}_3(2)$, it is easy to obtain a basis of $\mathcal{H}_3(2)^\perp$. The weight enumerator of $\mathcal{H}_3(2)^\perp$ can be calculated by brute strength using a computer, because it has relatively small dimension. The weight enumerator of $\mathcal{H}_3(2)$ is related to that of $\mathcal{H}_3(2)^\perp$ via the MacWilliams formula. (See, for example, van Lint [17, p. 39, Theorem 3.5.3].) The result is as follows:

$$\begin{gathered}
\operatorname{we}(\mathcal{H}_3(2)) := z^{85} + 357\, z^{80} + 17850\, z^{77} + 23800\, z^{76} + 45696\, z^{75} + 1142400\, z^{74} +\\
\cdots\\
19007943360\, z^{17} + 4668633585\, z^{16} + 1074247680\, z^{15} + 229785600\, z^{14} + 44666650\, z^{13} +\\
8020600\, z^{12} + 1142400\, z^{11} + 45696\, z^{10} + 23800\, z^{9} + 17850\, z^{8} + 357\, z^{5} + 1.
\end{gathered}$$

In particular, we have

$$\operatorname{ws}(\mathcal{H}_3(2)) = \{0, 1, 2, \dots, 85\} \setminus \{1, 2, 3, 4, 6, 7, 78, 79, 81, 82, 83, 84\}.$$

The group $G_J^{0\infty} \cong GL(4, k)$ acts on $M_0^\vee$ projectively over k, and hence on the codes $\mathcal{H}_3(1)$ and $\mathcal{H}_3(2)$. Recall that every code word of $\mathcal{H}_3 \cong \mathcal{H}_3(1)$ is either the set of k-rational points of a Hermitian surface in $M_0^\vee \cong \mathbb{P}^3$ or its complement. (Lemma 5.2.) By Corollary 2.6, the action is transitive on the set of code words of a given weight in $\mathcal{H}_3(1)$. The upper half of Table 7.2 shows the number of Hermitian surfaces S of each rank and the weight of corresponding code words $S(k)$ and $M_0^\vee(k) \setminus S(k)$.

Note that the number 357 of the code words of weight 5 in $\mathcal{H}_3(2)$ is equal to the number of k-rational lines in $M_0^\vee$. Because any k-rational line ℓ can be expressed as an intersection of two k-rational planes and $\ell(k)$ has five points, a code word of weight 5 is a member of $\mathcal{H}_3(2)$ if and only if it is the set of k-rational points of a k-rational line. The lower half of Table 7.2 shows the number of k-rational lines ℓ that intersect at distinct points of a given number with a Hermitian surface S of a fixed rank.

Suppose that a non-zero vector v of $\widetilde{L}^3$ contained in the kernel of the natural projection $\widetilde{\mathrm{pr}} : \widetilde{L}^3 \to \widetilde{R^T}$ is given. Then we have $(v, v)_T \ge 16$, and hence v cannot be a minimal non-zero vector.

TABLE 7.2. Number of Hermitian surfaces and lines.

rank of S	0	1	2	3	4
$\lvert S(k)\rvert$	85	21	53	37	45
$\lvert M_0^{\vee}(k) \setminus S(k)\rvert$	0	64	32	48	40
number of S	1	85	3570	23800	38080
$\lvert S(k) \cap \ell(k)\rvert$	number of ℓ				
0	0	0	0	0	0
1	0	336	40	156	90
2	0	0	0	0	0
3	0	0	256	192	240
4	0	0	0	0	0
5	357	21	61	9	27

For an element α of R, let $\langle \alpha \rangle$ be the unique integer such that $-8 < \langle \alpha \rangle \le 8$ and $\langle \alpha \rangle \bmod 16 = \alpha$. For an element $\bar{w} \in \overline{L}^3$, let $\langle \bar{w}^{\sim} \rangle \in \widetilde{L}^3$ be the vector defined by

$$\langle \bar{w}^{\sim} \rangle(H) := \langle \bar{w}(H) \rangle \quad \text{for} \quad H \in T \quad \text{and} \qquad \langle \bar{w}^{\sim} \rangle(\varphi) := \Big\langle \sum_{H \in T} \bar{w}(H) \Big\rangle.$$

It is obvious that $\langle \bar{w}^{\sim} \rangle$ is one of the vectors of minimal norm in $\widetilde{\mathrm{pr}}^{-1}(\bar{w}^{\sim})$. (If the number of $H \in \widetilde{T}$ such that $\bar{w}(H) = 8$ is ν, then the set of the vectors of minimal norm in $\widetilde{\mathrm{pr}}^{-1}(\bar{w}^{\sim})$ consists of 2^{ν} elements.) The vector $\bar{w}$ is written in one of the following forms:

$$\begin{aligned} \bar{w} &= \overline{V}_T + 2\,\overline{V}_{U_1(\bar{w})} + 4\,\overline{V}_{U_2(\bar{w})} + 8\,\overline{V}_{U_3(\bar{w})} \quad \text{or} \\ \bar{w} &= 2\,\overline{V}_{U_1(\bar{w})} + 4\,\overline{V}_{U_2(\bar{w})} + 8\,\overline{V}_{U_3(\bar{w})}, \end{aligned}$$

where $U_i(\bar{w}) \in \mathcal{H}_3(2^{i-1})$ for $i = 1, 2, 3$ are the code words given by Lemma 7.1. We say that $\bar{w}$ is *odd* or *even* according to whether $\bar{w}$ is written in the first form or in the second form. Because $\overline{V}_T \in \overline{L}^3$, if $\bar{w} \in \overline{L}^3$ is odd, then the even vector

$$\bar{w} + \overline{V}_T = 2\,\overline{V}_{T+U_1(\bar{w})} + 4\,\overline{V}_{U_1(\bar{w})+U_2(\bar{w})} + 8\,\overline{V}_{U_3(\bar{w})+U_1(\bar{w})\cap U_2(\bar{w})}$$

is also contained in $\overline{L}^3$, while if $\bar{w} \in \overline{L}^3$ is even, then the even vector

$$\bar{w} + 2\,\overline{V}_T = 2\,\overline{V}_{T+U_1(\bar{w})} + 4\,\overline{V}_{U_1(\bar{w})+U_2(\bar{w})} + 8\,\overline{V}_{U_3(\bar{w})+U_1(\bar{w})\cap U_2(\bar{w})}$$

is also contained in $\overline{L}^3$. We define the function $\delta_{\bar{w}} : R \to \mathbb{Z}_{\ge 0}$ by

$$\delta_{\bar{w}}(\alpha) := |\{ H \in T \;:\; \bar{w}(H) = \alpha \}|.$$

For any function $\delta : R \to \mathbb{Z}_{\ge 0}$, we put

$$N(\delta) := \frac{1}{16} \Big(\sum_{\alpha \in R} \langle \alpha \rangle^2 \delta(\alpha) + 3 \Big\langle \sum_{\alpha \in R} \alpha\, \delta(\alpha) \Big\rangle^2 \Big).$$

Then the norm of $\langle \bar{w}^{\sim} \rangle \in \widetilde{L}^3$ is equal to $N(\delta_{\bar{w}})$.

Suppose that a function $\delta : R \to \mathbb{Z}_{\ge 0}$ is equal to $\delta_{\bar{w}}$ for some non-zero element $\bar{w} \in \overline{L}^3$. Then δ must satisfy the following conditions.

(1) The sum $\sum_{\alpha\in R}\delta(\alpha)$ is equal to 85, and $\delta(0) < 85$. Moreover, we have $\delta(\alpha) = 0$ for all even α if $\bar{w}$ is odd, while $\delta(\alpha) = 0$ for all odd α if $\bar{w}$ is even.

(2) The integer $|U_1(\bar{w})|$ appears in $\mathrm{ws}(\mathcal{H}_3(1))$, where

$$|U_1(\bar{w})| = \begin{cases} \delta(3)+\delta(7)+\delta(11)+\delta(15) & \text{if } \bar{w} \text{ is odd,} \\ \delta(2)+\delta(6)+\delta(10)+\delta(14) & \text{if } \bar{w} \text{ is even.} \end{cases}$$

(3) The integer $|U_2(\bar{w})|$ appears in $\mathrm{ws}(\mathcal{H}_3(2))$, where

$$|U_2(\bar{w})| = \begin{cases} \delta(5)+\delta(7)+\delta(13)+\delta(15) & \text{if } \bar{w} \text{ is odd,} \\ \delta(4)+\delta(6)+\delta(12)+\delta(14) & \text{if } \bar{w} \text{ is even.} \end{cases}$$

(4) The following integer appears in $\mathrm{ws}(\mathcal{H}_3(2))$:

$$\begin{cases} |U_2(\bar{w}+\overline{V}_T)| = |U_1(\bar{w})+U_2(\bar{w})| = \delta(3)+\delta(5)+\delta(11)+\delta(13) & \text{if } \bar{w} \text{ is odd,} \\ |U_2(\bar{w}+2\,\overline{V}_T)| = |U_1(\bar{w})+U_2(\bar{w})| = \delta(2)+\delta(4)+\delta(10)+\delta(12) & \text{if } \bar{w} \text{ is even.} \end{cases}$$

(5) The norm $N(\delta) = (\langle \bar{w}\tilde{\ }\rangle, \langle \bar{w}\tilde{\ }\rangle)_T$ is an even integer.

We list up all the functions $\delta : R \to \mathbb{Z}_{\geq 0}$ that satisfy the conditions (1)-(5) and $N(\delta) \leq 8$. There are no such functions δ with $N(\delta) \leq 4$. The list of δ with $N(\delta) = 6$ or $N(\delta) = 8$ is given in Table 7.3, where the function δ is expressed by the concatenation of $\alpha^{\delta(\alpha)}$ ($\alpha \in R$) with $\delta(\alpha) > 0$. The third column of Table 7.3 indicates the number $\nu(\delta)$ of the vectors $w \in \widetilde{L}^3$ such that $\delta_{\bar{w}} = \delta$ and $(w,w)_T = N(\delta)$, where $\bar{w}$ is the restriction of $\widetilde{\mathrm{pr}}\,(w) \in R^{\widetilde{T}}$ to T.

In order to calculate $\nu(\delta)$, we need the following lemma. Let A be a subset of $M_0^\vee(k)$. We define $\mathcal{H}_3(2, A)$ to be the linear subcode of $\mathcal{H}_3(2)$ consisting of all the code words of $\mathcal{H}_3(2)$ that are contained in A.

Lemma 7.3. (1) *Let P be a k-rational plane of $M_0^\vee$, and p and q two distinct points of $M_0^\vee(k) \setminus P(k)$. Then we have*

$$\mathcal{H}_3(2, P(k) \sqcup \{p, q\}) = \mathcal{H}_3(2, P(k)).$$

The weight enumerator of $\mathcal{H}_3(2, P(k))$ is

$$z^{21} + 21\,z^{16} + 210\,z^{13} + 280\,z^{12} + 280\,z^{9} + 210\,z^{8} + 21\,z^{5} + 1.$$

(2) *Let A be a code word of $\mathcal{H}_3(1)$ with weight* 32; *that is, A is $M_0^\vee(k) \setminus S(k)$ where S is a Hermitian surface of rank* 2. *Then the weight enumerator of $\mathcal{H}_3(2, A)$ is*

$$z^{32} + 140\,z^{24} + 3520\,z^{20} + 9062\,z^{16} + 3520\,z^{12} + 140\,z^{8} + 1.$$

Proof. The action of the group $G_J^{0\infty} \cong GL(4, k)$ on $M_0^\vee$ is transitive on the set of k-rational planes of $M_0^\vee$, and the stabilizer subgroup of a k-rational plane P_0 acts 2-transitively on $M_0^\vee(k) \setminus P_0(k)$. Hence it is enough to check the assertion (1) only for one choice of P_0 and p, q. Since we have already obtained a basis of $\mathcal{H}_3(2)$, this checking can be easily done by a computer. In fact, we see that $\mathcal{H}_3(2, P(k))$ is isomorphic to $\mathcal{H}_2(1)$. A similar argument can be applied to the assertion (2). *q.e.d.*

We will demonstrate how to calculate $\nu(\delta)$ on several examples.

Example 1. $\delta = 2^{11}14^{10}$ (No. 0). If $\delta = \delta_{\bar{w}}$, then $|U_1(\bar{w})| = 21$ and hence $U_1(\bar{w})$ should coincide with $P(k)$ for some k-rational plane P. Then $U_2(\bar{w})$ is a code

Table 7.3. Kissing numbers.

Norm 6

No.	δ	$\nu(\delta)$
0	$2^{11}14^{10}$	0
1	$2^{10}14^{11}$	0

Norm 8: even

No.	δ	$\nu(\delta)$
2	$0^{80}14^{32}$	3570
3	$0^{80}12^{5}$	357
4	$0^{77}12^{8}$	17850
5	$0^{83}8^{2}$	14280
6	$0^{64}6^{1}14^{20}$	1785
7	$0^{80}4^{1}12^{4}$	1785
8	$0^{80}4^{2}12^{3}$	3570
9	$0^{77}4^{2}12^{6}$	499800
10	$0^{80}4^{3}12^{2}$	3570
11	$0^{80}4^{4}12^{1}$	1785
12	$0^{77}4^{4}12^{4}$	1249500
13	$0^{80}4^{5}$	357
14	$0^{77}4^{6}12^{2}$	499800
15	$0^{77}4^{8}$	17850
16	$0^{62}2^{3}4^{1}12^{1}14^{18}$	0
17	$0^{64}2^{4}10^{1}14^{16}$	8925
18	$0^{64}2^{5}6^{1}14^{15}$	28560
19	$0^{62}2^{6}4^{1}12^{1}14^{15}$	0
20	$0^{64}2^{7}10^{1}14^{13}$	142800
21	$0^{62}2^{7}4^{1}12^{1}14^{14}$	0
22	$0^{53}2^{8}14^{24}$	499800
23	$0^{62}2^{8}12^{2}14^{13}$	0
24	$0^{64}2^{8}10^{1}14^{12}$	214200
25	$0^{64}2^{8}6^{1}14^{12}$	232050
26	$0^{62}2^{8}4^{2}14^{13}$	0

No.	δ	$\nu(\delta)$
27	$0^{62}2^{9}12^{2}14^{12}$	0
28	$0^{64}2^{9}6^{1}14^{11}$	285600
29	$0^{62}2^{9}4^{2}14^{12}$	0
30	$0^{62}2^{10}4^{1}12^{1}14^{11}$	0
31	$0^{64}2^{11}10^{1}14^{9}$	285600
32	$0^{62}2^{11}4^{1}12^{1}14^{10}$	0
33	$0^{53}2^{12}14^{20}$	12566400
34	$0^{62}2^{12}12^{2}14^{9}$	0
35	$0^{64}2^{12}10^{1}14^{8}$	232050
36	$0^{64}2^{12}6^{1}14^{8}$	214200
37	$0^{62}2^{12}4^{2}14^{9}$	0
38	$0^{62}2^{13}12^{2}14^{8}$	0
39	$0^{64}2^{13}6^{1}14^{7}$	142800
40	$0^{62}2^{13}4^{2}14^{8}$	0
41	$0^{62}2^{14}4^{1}12^{1}14^{7}$	0
42	$0^{64}2^{15}10^{1}14^{5}$	28560
43	$0^{62}2^{15}4^{1}12^{1}14^{6}$	0
44	$0^{53}2^{16}14^{16}$	32351340
45	$0^{64}2^{16}6^{1}14^{4}$	8925
46	$0^{62}2^{18}4^{1}12^{1}14^{3}$	0
47	$0^{53}2^{20}14^{12}$	12566400
48	$0^{64}2^{20}10^{1}$	1785
49	$0^{53}2^{24}14^{8}$	499800
50	$0^{53}2^{32}$	3570

Norm 8: odd

No.	δ	$\nu(\delta)$
51	$3^{5}15^{80}$	357
52	$1^{16}13^{5}15^{64}$	1785
53	$1^{18}3^{2}13^{3}15^{62}$	0
54	$1^{20}3^{4}13^{1}15^{60}$	28560
55	$1^{28}3^{1}13^{4}15^{52}$	142800
56	$1^{30}3^{3}13^{2}15^{50}$	913920
57	$1^{32}13^{5}15^{48}$	214200
58	$1^{32}3^{5}15^{48}$	217770
59	$1^{34}3^{2}13^{3}15^{46}$	4569600
60	$1^{36}3^{1}13^{4}15^{44}$	3427200
61	$1^{36}3^{4}13^{1}15^{44}$	3712800
62	$1^{38}3^{3}13^{2}15^{42}$	9139200
63	$1^{40}13^{5}15^{40}$	1028160

No.	δ	$\nu(\delta)$
64	$1^{40}3^{5}15^{40}$	1028160
65	$1^{42}3^{2}13^{3}15^{38}$	9139200
66	$1^{44}3^{1}13^{4}15^{36}$	3712800
67	$1^{44}3^{4}13^{1}15^{36}$	3427200
68	$1^{46}3^{3}13^{2}15^{34}$	4569600
69	$1^{48}13^{5}15^{32}$	217770
70	$1^{48}3^{5}15^{32}$	214200
71	$1^{50}3^{2}13^{3}15^{30}$	913920
72	$1^{52}3^{4}13^{1}15^{28}$	142800
73	$1^{60}3^{1}13^{4}15^{20}$	28560
74	$1^{62}3^{3}13^{2}15^{18}$	0
75	$1^{64}3^{5}15^{16}$	1785
76	$1^{80}13^{5}$	357

word of $\mathcal{H}_3(2, P(k))$ with weight 10. However there are no such code words by Lemma 7.3, and hence

$$\nu(2^{11}14^{10}) = 0.$$

Example 2. $\delta = 0^{83}8^2$ (No. 5). The code $\mathcal{H}_3(3)$ coincides with $\mathbb{F}_2^T$. Hence there are $\binom{85}{2}$ elements $\bar{w}$ of $\overline{L}^3$ such that $\delta = \delta_{\bar{w}}$. For each such $\bar{w}$, there are four vectors $w \in \widetilde{L}^3$ such that $\widetilde{\mathrm{pr}}\,(w)|_T = \bar{w}$ and $(w, w)_T = 8$, because we can lift $8 \in R$ to either $8 \in \mathbb{Z}$ or $-8 \in \mathbb{Z}$. Hence we have

$$\nu(0^{83}8^2) = \binom{85}{2} \cdot 4 = 14280.$$

Example 3. $\delta = 0^{77}4^4 12^4$ (No. 12). Suppose that $\delta = \delta_{\bar{w}}$. There is only one $w \in \widetilde{L}^3$ such that $\widetilde{\mathrm{pr}}\,(w)|_T = \bar{w}$ and $(w, w)_T = 8$. In such a situation, we say that the lift of $\bar{w}$ is unique. The code word $U_2(\bar{w})$ is one of 17850 code words of weight 8 in $\mathcal{H}_3(2)$ and, for each such code word, there are $\binom{8}{4}$ choices of $U_3(\bar{w}) \subset U_2(\bar{w})$. Hence

$$\nu(0^{77}4^4 12^4) = 17850 \cdot \binom{8}{4} = 1249500.$$

Example 4. $\delta = 0^{62}2^3 4^1 12^1 14^{18}$ (No. 16). Suppose that $\delta = \delta_{\bar{w}}$. The lift of $\bar{w}$ is unique. Since $|U_1(\bar{w})| = 21$, $U_1(\bar{w})$ is equal to $P(k)$ for some k-rational plane P. The code word $U_2(\bar{w})$ of weight 20 must satisfy $|U_2(\bar{w}) \setminus P(k)| = 2$. There are no such code words by Lemma 7.3. Hence

$$\nu(0^{62}2^3 4^1 12^1 14^{18}) = 0.$$

Example 5. $\delta = 0^{53}2^{16}14^{16}$ (No. 44). Suppose that $\delta = \delta_{\bar{w}}$. The lift of $\bar{w}$ is unique. The code word $U_1(\bar{w})$ is one of 3570 code words of weight 32 in $\mathcal{H}_3(1)$. The code word $U_2(\bar{w})$ is an element of $\mathcal{H}_3(2, U_1(\bar{w}))$ with weight 16. There are 9062 such code words by Lemma 7.3. The code word $U_3(\bar{w})$ must be equal to $U_2(\bar{w})$. Hence

$$\nu(0^{53}2^{16}14^{16}) = 3570 \cdot 9062 = 32351340.$$

Example 6. $\delta = 1^{38}3^3 13^2 15^{42}$ (No. 62). Suppose that $\delta = \delta_{\bar{w}}$. The lift of $\bar{w}$ is unique. We put $\bar{v} = \bar{w} + \overline{V}_T$. Then $\delta_{\bar{v}} = 2^{38}4^3 14^2$. The code word $U_1(\bar{v})$ is one of 38080 code words of weight 40 in $\mathcal{H}_3(1)$. The code word $U_2(\bar{v})$ is an element of $\mathcal{H}_3(2)$ with weight 5, and hence there is a k-rational line ℓ such that $U_2(\bar{v}) = \ell(k)$. This line ℓ must intersect with $U_1(\bar{v})$ at distinct two points. There are 240 such lines by Table 7.2. The code word $U_3(\bar{v})$ must be equal to $U_1(\bar{v}) \cap U_2(\bar{v})$. Hence

$$\nu(1^{38}3^3 13^2 15^{42}) = 38080 \cdot 240 = 9139200.$$

Now we see that there are no non-zero vectors $w \in \widetilde{L}^3$ with $(w, w)_T < 8$, and the kissing number of $\widetilde{L}^3$ is

$$\sum \nu(\delta) = 109421928.$$

q.e.d.

8. Concluding remarks

Let X be the cubic Fermat hypersurface of dimension $2m$ in characteristic 2. The lattice $\mathcal{L}^m(X)$ has geometrically natural generators $[\Pi]-[\Pi']$ $(\Pi, \Pi' \in \Sigma^m(X))$. The norm of these generators is at least 2^m, and the minimal value 2^m is attained if and only if $\dim(\Pi \wedge \Pi') = m - 2$. Using induction on m, we see that, for each

$A \in \Sigma^{m-2}(X)$, there are $27 \cdot 16 = 432$ ordered pairs Π, Π' such that $\Pi \wedge \Pi' = A$. Hence there are $27 \cdot 16 \cdot |\Sigma^{m-2}(X)|$ ordered pairs Π, Π' such that

$$([\Pi] - [\Pi'], [\Pi] - [\Pi'])_{\mathcal{L}} = 2^m.$$

For $m = 1, 2, 3$, the number $|\Sigma^{m-2}(X)|$ is equal to 1, 693 and 1519749, respectively. On the other hand, for each vector $v = [\Pi_1] - [\Pi'_1]$ with $(v, v)_{\mathcal{L}} = 2^m$, there are at least six ordered pairs Π_i, Π'_i $(i = 1, \ldots, 6)$ such that $v = [\Pi_i] - [\Pi'_i]$ holds. Indeed, there are five elements $\Xi_2, \ldots, \Xi_6$ of $\Sigma^m(X)$ such that $\dim(\Pi_1 \wedge \Xi_i) = \dim(\Pi'_1 \wedge \Xi_i) = m - 1$. Let Π'_i be the unique element of $\Sigma^m(X)$ that is distinct from Π_1 and Ξ_i and contains $\Pi_1 \wedge \Xi_i$, and Π_i the unique element of $\Sigma^m(X)$ that is distinct from Π'_1 and Ξ_i and contains $\Pi'_1 \wedge \Xi_i$. Both of $[\Pi_1] + [\Pi'_i] + [\Xi_i]$ and $[\Pi'_1] + [\Pi_i] + [\Xi_i]$ are equal to the numerical equivalence class h by Corollary 3.3. Hence we have $v = [\Pi_i] - [\Pi'_i]$ for $i = 1, \ldots, 6$.

We have obtained the following.

Observation. For $m = 1, 2, 3$, the minimal norm of $\mathcal{L}^m(X)$ is 2^m and the kissing number is equal to

$$(27 \cdot 16 \cdot |\Sigma^{m-2}(X)|)/6 = 72 \cdot |\Sigma^{m-2}(X)|.$$

We may therefore expect that the same formulae continue to be valid for $m \geq 4$.

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