THE AUTOMORPHISM GROUP OF A SUPERSINGULAR $K3$ SURFACE WITH ARTIN INVARIANT 1 IN CHARACTERISTIC 3

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Abstract. We present a finite set of generators of the automorphism group of a supersingular $K3$ surface with Artin invariant 1 in characteristic 3.

1. Introduction

To determine the automorphism group $\mathrm{Aut}(Y)$ of a given $K3$ surface Y is an important problem. In this paper, we present a set of generators of the automorphism group of a supersingular $K3$ surface X in characteristic 3 with Artin invariant 1. Our method is computational, and relies heavily on computer-aided calculation. It gives us generators in explicit form, and it can be easily applied to many other $K3$ surfaces by modifying computer programs.

A $K3$ surface defined over an algebraically closed field k is said to be *supersingular* (in the sense of Shioda) if its Picard number is 22. Supersingular $K3$ surfaces exist only when k is of positive characteristic. Let Y be a supersingular $K3$ surface in characteristic $p > 0$, and let S_Y denote its Néron-Severi lattice. Artin [3] showed that the discriminant group of S_Y is a p-elementary abelian group of rank 2σ, where σ is an integer such that $1 \le \sigma \le 10$. This integer σ is called the *Artin invariant* of Y. Ogus [18, 19] proved that a supersingular $K3$ surface with Artin invariant 1 in characteristic p is unique up to isomorphisms (see also [21]).

It is known that the Fermat quartic surface

$$X := \{w^4 + x^4 + y^4 + z^4 = 0\} \subset \mathbb{P}^3$$

defined over an algebraically closed field k of characteristic 3 is a supersingular $K3$ surface with Artin invariant 1 (see [33]). Let

$$h_0 := [\mathcal{O}_X(1)] \in S_X$$

denote the class of the hyperplane section of X. The projective automorphism group $\mathrm{Aut}(X, h_0)$ of $X \subset \mathbb{P}^3$ is equal to the finite subgroup $\mathrm{PGU}_4(\mathbb{F}_9)$ of $\mathrm{PGL}_4(k)$ with order $13,063,680$.

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Let (w, x, y) be the affine coordinates of $\mathbb{P}^3$ with $z = 1$, and let F_{1j} and F_{2j} be polynomials of (w, x, y) with coefficients in

$$\mathbb{F}_9 = \mathbb{F}_3(i) = \{0, \pm 1, \pm i, \pm(1+i), \pm(1-i)\}, \quad \text{where } i := \sqrt{-1},$$

given in Table 1.1.

Proposition 1.1. *For $i = 1$ and 2, the rational map*

$$(w, x, y) \mapsto [F_{i0} : F_{i1} : F_{i2}] \in \mathbb{P}^2$$

induces a morphism $\phi_i : X \to \mathbb{P}^2$ of degree 2.

We denote by

$$X \xrightarrow{\psi_i} Y_i \xrightarrow{\pi_i} \mathbb{P}^2$$

the Stein factorization of $\phi_i : X \to \mathbb{P}^2$, and let $B_i \subset \mathbb{P}^2$ be the branch curve of the finite morphism $\pi_i : Y_i \to \mathbb{P}^2$ of degree 2. Note that Y_i is a normal $K3$ surface, and hence Y_i has only rational double points as its singularities (see [1, 2]). Let $[x_0 : x_1 : x_2]$ be the homogeneous coordinates of $\mathbb{P}^2$.

Proposition 1.2. (1) *The ADE-type of the singularities of Y_1 is $6A_1 + 4A_2$. The branch curve B_1 is defined by $f_1 = 0$, where*

$$\begin{aligned} f_1 :=\, & {x_0}^6 + {x_0}^5 x_1 - {x_0}^3 {x_1}^3 - x_0 {x_1}^5 - {x_0}^4 {x_2}^2 \\ & + x_0 {x_1}^3 {x_2}^2 + {x_1}^4 {x_2}^2 + {x_0}^2 {x_2}^4 + {x_1}^2 {x_2}^4 + {x_2}^6. \end{aligned}$$

(2) *The ADE-type of the singularities of Y_2 is $A_1 + A_2 + 2A_3 + 2A_4$. The branch curve B_2 is defined by $f_2 = 0$, where*

$$\begin{aligned} f_2 :=\, & {x_0}^5 x_1 + {x_0}^2 {x_1}^4 - {x_0}^4 {x_2}^2 + x_0 {x_1}^3 {x_2}^2 \\ & + {x_1}^4 {x_2}^2 - {x_0}^2 {x_2}^4 - x_0 x_1 {x_2}^4 - {x_1}^2 {x_2}^4 - {x_2}^6. \end{aligned}$$

Our main result is as follows:

Theorem 1.3. *Let $g_i \in \mathrm{Aut}(X)$ denote the involution induced from the deck-transformation of $\pi_i : Y_i \to \mathbb{P}^2$. Then $\mathrm{Aut}(X)$ is generated by $\mathrm{Aut}(X, h_0) = \mathrm{PGU}_4(\mathbb{F}_9)$ and g_1, g_2.*

See Theorem 7.1 for a more explicit description of the involutions g_1 and g_2.

Let $\mathcal{P}_{S_X}$ denote the connected component of $\{x \in S_X \otimes \mathbb{R} \mid x^2 > 0\}$ that contains h_0. Following Borcherds [4], we prove Theorem 1.3 by calculating a closed chamber D_{S0} in the cone $\mathcal{P}_{S_X}$ with the following properties (see Section 6):

1. (1) The chamber D_{S0} is invariant under the action of $\mathrm{Aut}(X, h_0)$.
2. (2) For any nef class $v \in S_X$, there exists $\gamma \in \mathrm{Aut}(X)$ such that $v^\gamma \in D_{S0}$.
3. (3) For nef classes v, v' in the interior of D_{S0}, there exists $\gamma \in \mathrm{Aut}(X)$ such that $v' = v^\gamma$ if and only if there exists $\tau \in \mathrm{Aut}(X, h_0)$ such that $v' = v^\tau$.

$$
\begin{aligned}
F_{10} = {} & (1+i) + (1+i)\,w + (1-i)\,x - y - (1-i)\,wx - x^2 + i\,wy \\
& + i\,xy - i\,y^2 + (1+i)\,w^3 - i\,w^2x + (1+i)\,wx^2 - i\,x^3 + w^2y \\
& + (1+i)\,wxy + (1+i)\,x^2y - (1-i)\,wy^2 - (1+i)\,xy^2 + i\,y^3 \\
F_{11} = {} & (1-i) - (1+i)\,x - (1-i)\,y - (1-i)\,w^2 - (1-i)\,wx - (1-i)\,x^2 \\
& - (1+i)\,wy - xy - (1+i)\,y^2 - w^3 + (1-i)\,w^2x + wx^2 - i\,x^3 \\
& - (1+i)\,w^2y - (1+i)\,wxy + x^2y - i\,wy^2 - xy^2 + (1-i)\,y^3 \\
F_{12} = {} & (1+i)\,w - i\,x - y - w^2 - wx - i\,x^2 - i\,xy + i\,y^2 + i\,w^3 \\
& - (1+i)\,wx^2 + i\,x^3 - i\,w^2y - wxy + (1-i)\,wy^2 + (1+i)\,y^3
\end{aligned}
$$

$$
\begin{aligned}
F_{20} = {} & -1 - i\,w + (1+i)\,x - y - (1+i)\,w^2 - wx - (1-i)\,x^2 - i\,wy + (1+i)\,xy \\
& - (1-i)\,w^3 + w^2x - wx^2 + x^3 - w^2y + (1-i)\,wxy + x^2y + (1-i)\,wy^2 \\
& + (1-i)\,xy^2 + (1+i)\,y^3 - w^3x - i\,w^2x^2 - wx^3 + w^3y - (1+i)\,w^2xy \\
& - (1-i)\,wxy^2 + x^2y^2 - (1-i)\,wy^3 - (1+i)\,xy^3 - y^4 + (1-i)\,w^3x^2 - i\,x^5 \\
& + (1-i)\,w^3xy + (1+i)\,wx^3y - i\,w^3y^2 + (1+i)\,w^2xy^2 - (1+i)\,wx^2y^2 \\
& + i\,x^3y^2 - w^2y^3 - (1+i)\,wxy^3 - (1-i)\,x^2y^3 + i\,wy^4 + (1-i)\,xy^4 + (1+i)\,y^5 \\
F_{21} = {} & -(1-i) + i\,w + (1-i)\,y - (1+i)\,w^2 + wx + (1+i)\,x^2 + (1+i)\,wy - (1+i)\,xy \\
& - i\,y^2 - w^3 + i\,w^2x + (1+i)\,wx^2 - x^3 - (1+i)\,w^2y - (1-i)\,wxy - (1-i)\,x^2y \\
& - i\,wy^2 - (1+i)\,xy^2 + y^3 - (1-i)\,w^3x - wx^3 + (1-i)\,x^4 + (1-i)\,w^3y + i\,w^2xy \\
& + (1-i)\,wx^2y - i\,x^3y + (1-i)\,w^2y^2 + (1-i)\,wxy^2 - (1+i)\,x^2y^2 + (1-i)\,wy^3 \\
& - i\,xy^3 + i\,y^4 + w^3x^2 + w^2x^3 + (1-i)\,wx^4 - i\,x^5 - i\,w^3xy + w^2x^2y + (1+i)\,wx^3y \\
& + x^4y + w^3y^2 - w^2xy^2 - wx^2y^2 + i\,w^2y^3 + (1+i)\,wxy^3 - i\,wy^4 - i\,xy^4 + y^5 \\
F_{22} = {} & (1-i) - (1+i)\,w - (1+i)\,x - (1-i)\,y + i\,w^2 - (1+i)\,wx - (1-i)\,x^2 + i\,wy \\
& - (1+i)\,xy - w^3 - i\,w^2x - wx^2 + x^3 - (1-i)\,w^2y + wxy + x^2y + (1+i)\,wy^2 \\
& - (1+i)\,xy^2 - y^3 + i\,w^3x - (1-i)\,w^2x^2 - wx^3 - (1+i)\,x^4 + i\,w^3y + w^2xy \\
& + (1-i)\,wx^2y - (1-i)\,w^2y^2 + (1+i)\,wxy^2 + i\,wy^3 + xy^3 + (1-i)\,y^4 - i\,w^3x^2 \\
& - (1+i)\,wx^4 + x^5 - (1-i)\,w^3xy - i\,w^2x^2y + (1+i)\,wx^3y + (1-i)\,x^4y - w^3y^2 \\
& - (1+i)\,w^2xy^2 + i\,wx^2y^2 + i\,x^3y^2 - wxy^3 - (1-i)\,x^2y^3 - wy^4 - xy^4 - y^5
\end{aligned}
$$

TABLE 1.1. Polynomials F_{1j} and F_{2j}

This chamber D_{S0} is bounded by $112+648+5184$ hyperplanes in $\mathcal{P}_{S_X}$. See Proposition 4.5 for the explicit description of these walls. Using D_{S0} and these walls, we can also present a finite set of generators of $\mathrm{O}^+(S_X)$ (see Theorem 8.2).

Vinberg [35] determined the automorphism groups of two complex $K3$ surfaces with Picard number 20 by investigating the orthogonal groups of their Néron-Severi lattices and the associated hyperbolic geometry.

Let L denote an even unimodular lattice of rank 26 with signature $(1, 25)$, which is unique up to isomorphisms by Eichler's theorem. Conway [6] determined the fundamental domain in a positive cone of $L\otimes\mathbb{R}$ under the action of the subgroup of

$\mathrm{O}^+(L)$ generated by the reflections with respect to the vectors of square norm -2. Borcherds [4] applied Conway theory to the investigation of the orthogonal groups of even hyperbolic lattices S primitively embedded in L. Then the first author [15] determined the automorphism group of a generic Jacobian Kummer surface by embedding its Néron-Severi lattice into L and using Conway theory. Keum and the first author [14] applied this method to the Kummer surface of the product of two elliptic curves, Dolgachev and Keum [11] applied it to quartic Hessian surfaces, and Dolgachev and the first author [10] applied it to the supersingular $K3$ surface in characteristic 2 with Artin invariant 1.

Recently, configurations of smooth rational curves on our supersingular $K3$ surface X was studied in [13] with respect to an embedding of S_X into L, and elliptic fibrations on X was classified in [25] by embedding S_X into L.

The new idea introduced in this paper is that, in order to find automorphisms of X necessary to generate $\mathrm{Aut}(X)$, we search for polarizations of degree 2 whose classes are located on the walls of the chamber decomposition of the cone $\mathcal{P}_{S_X}$. The computational tools used in this paper have been developed by the second author for the study [30] of various double plane models of a supersingular $K3$ surface in characteristic 5 with Artin invariant 1. The computational data for this paper is available from the second author's webpage [31].

In [27] and [29], the second author showed that every supersingular $K3$ surface in any characteristic with arbitrary Artin invariant is birational to a double cover of the projective plane. In [28], [32] and [20, 30], projective models of supersingular $K3$ surfaces in characteristic 2, 3 and 5 were investigated, respectively.

This paper is organized as follows. In Section 2, we give a review of the theory of Conway and Borcherds, and investigate chamber decomposition induced on a positive cone of a primitive hyperbolic sublattice S of L. In Section 3, we give explicitly a basis of the Néron-Severi lattice S_X of X, and describe a method to compute the action of $\mathrm{Aut}(X, h_0)$ on S_X. The fact that S_X is generated by the classes of lines in X enables us to calculate projective models of X explicitly. In Section 4, we embed S_X into L, and study the obtained chamber decomposition in detail. In particular, we investigate the walls of the chamber D_{S0} that contains the class h_0. In Section 5, we prove Propositions 1.1 and 1.2, and show that the involutions g_1 and g_2 map h_0 to its mirror images into walls of the chamber D_{S0}. Then we can prove Theorem 1.3 in Section 6. In Section 7, we give another description of the involutions g_i. In the last section, we give a set of generators of $\mathrm{O}^+(S_X)$.

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2. Leech roots

2.1. **Terminologies and notation.** We fix some terminologies and notation about lattices. A *lattice* M is a free $\mathbb{Z}$-module of finite rank with a non-degenerate symmetric bilinear form

$$(\ ,\ )_M : M \times M \to \mathbb{Z}.$$

A submodule N of M is said to be *primitive* if M/N is torsion free. For a submodule N of M, we denote by $N^\perp \subset M$ the submodule defined by

$$N^\perp := \{\ u \in M \ \mid\ (u, v)_M = 0 \ \text{ for all }\ v \in N\ \},$$

which is primitive by definition. We denote by $\mathrm{O}(M)$ the orthogonal group of M. Throughout this paper, we let $\mathrm{O}(M)$ act on M from *right*. Suppose that M is of rank r. We say that M is *hyperbolic* (resp. *negative-definite*) if the signature of the symmetric bilinear form $(\ ,\ )_M$ on $M \otimes \mathbb{R}$ is $(1, r-1)$ (resp. $(0, r)$). We define the *dual lattice* $M^\vee$ of M by

$$M^\vee := \{\ u \in M \otimes \mathbb{Q} \ \mid\ (u, v)_M \in \mathbb{Z} \ \text{ for all }\ v \in M\ \}.$$

Then M is contained in $M^\vee$ as a submodule of finite index. The finite abelian group $M^\vee/M$ is called the *discriminant group* of M. We say that M is *unimodular* if $M = M^\vee$.

A lattice M is said to be *even* if $(v, v)_M \in 2\mathbb{Z}$ holds for any $v \in M$. The discriminant group $M^\vee/M$ of an even lattice M is naturally equipped with the quadratic form

$$q_M : M^\vee/M \to \mathbb{Q}/2\mathbb{Z}$$

defined by $q_M(u \bmod M) := (u, u)_M \bmod 2\mathbb{Z}$. We call q_M the *discriminant form* of M. The automorphism group of q_M is denoted by $\mathrm{O}(q_M)$. There exists a natural homomorphism $\mathrm{O}(M) \to \mathrm{O}(q_M)$.

Suppose that M is hyperbolic. Then the open subset

$$\{\ x \in M \otimes \mathbb{R} \ \mid\ (x, x)_M > 0\ \}$$

of $M \otimes \mathbb{R}$ has two connected components. A *positive cone* of M is one of them. We fix a positive cone $\mathcal{P}$. The *autochronous orthogonal group* $\mathrm{O}^+(M)$ of M is the group of isometries of M that preserve $\mathcal{P}$. Then $\mathrm{O}^+(M)$ is a subgroup of $\mathrm{O}(M)$ with index 2. Note that $\mathrm{O}^+(M)$ acts on $\mathcal{P}$. For a nonzero vector $u \in M \otimes \mathbb{R}$, we denote by $(u)^\perp_M$ the hyperplane of $M \otimes \mathbb{R}$ defined by

$$(u)^\perp_M := \{\ x \in M \otimes \mathbb{R} \ \mid\ (x, u)_M = 0\ \}.$$

Let $\mathcal{R}$ be a set of non-zero vectors of $M \otimes \mathbb{R}$, and let

$$\mathcal{H} := \{\ (u)^\perp_M \ \mid\ u \in \mathcal{R}\ \}$$

be the family of hyperplanes defined by $\mathcal{R}$. Suppose that $\mathcal{H}$ is locally finite in $\mathcal{P}$. Then the closure in $\mathcal{P}$ of each connected component of

$$\mathcal{P} \setminus \left(\mathcal{P} \cap \bigcup_{u \in \mathcal{R}} (u)^{\perp}_{M} \right)$$

is called an $\mathcal{R}$-*chamber*. Let D be an $\mathcal{R}$-chamber. We denote by D° the interior of D. We say that a hyperplane $(u)^{\perp}_{M} \in \mathcal{H}$ *bounds* D, or that $(u)^{\perp}_{M}$ is a *wall* of D, if $(u)^{\perp}_{M} \cap D$ contains a non-empty open subset of $(u)^{\perp}_{M}$. We denote the set of walls of D by

$$\mathcal{W}(D) := \{ \, (u)^{\perp}_{M} \in \mathcal{H} \mid (u)^{\perp}_{M} \text{ bounds } D \, \}.$$

Suppose that $\mathcal{R}$ is invariant under $u \mapsto -u$. We choose a point $p \in D^{\circ}$, and put

$$\widetilde{\mathcal{W}}(D) := \{ \, u \in \mathcal{R} \mid (u)^{\perp}_{M} \text{ bounds } D \text{ and } (u, p)_M > 0 \, \},$$

which is independent of the choice of p. It is obvious that D is equal to

$$\{ \, x \in \mathcal{P} \mid (x, u)_M \geq 0 \ \text{ for all } \ u \in \widetilde{\mathcal{W}}(D) \, \}.$$

2.2. **Conway theory.** We review the theory of Conway [6]. Let L be an even unimodular hyperbolic lattice of rank 26, which is unique up to isomorphisms by Eichler's theorem (see, for example, [5, Chapter 11, Theorem 1.4]). We choose and fix a positive cone $\mathcal{P}_L$ once and for all. A vector $r \in L$ is called a *root* if the reflection $s_r : L \otimes \mathbb{R} \to L \otimes \mathbb{R}$ defined by

$$x \mapsto x - \frac{2(x, r)_L}{(r, r)_L} \cdot r$$

preserves L and $\mathcal{P}_L$, or equivalently, if $(r, r)_L = -2$. We denote by $\mathcal{R}_L$ the set of roots of L, which is invariant under $r \mapsto -r$. Let $W(L)$ denote the subgroup of $\mathrm{O}^+(L)$ generated by the reflections s_r associated with all the roots $r \in \mathcal{R}_L$. Then $W(L)$ is a normal subgroup of $\mathrm{O}^+(L)$. The family of hyperplanes

$$\mathcal{H}_L := \{ \, (r)^{\perp}_{L} \mid r \in \mathcal{R}_L \, \}$$

is locally finite in $\mathcal{P}_L$. Hence we can consider $\mathcal{R}_L$-chambers. By definition, each $\mathcal{R}_L$-chamber is a fundamental domain of the action of $W(L)$ on $\mathcal{P}_L$.

A non-zero primitive vector $w \in L$ is called a *Weyl vector* if $(w, w)_L = 0$, w is contained in the closure of $\mathcal{P}_L$ in $L \otimes \mathbb{R}$, and the negative-definite even unimodular lattice $\langle w \rangle^{\perp} / \langle w \rangle$ of rank 24 has no vectors of square norm -2. Let $w \in L$ be a Weyl vector. We put

$$LR(w) := \{ \, r \in \mathcal{R}_L \mid (w, r)_L = 1 \, \}.$$

A root in $LR(w)$ is called a *Leech root with respect to* w.

Suppose that w is a non-zero primitive vector of norm 0 contained in the closure of $\mathcal{P}_L$. Then there exists a vector $w' \in L$ such that $(w, w')_L = 1$ and $(w', w')_L = 0$. Let $U \subset L$ denote the hyperbolic sublattice of rank 2 generated by w and w'. By

Niemeier's classification [16] of even definite unimodular lattices of rank 24 (see also [9, Chapter 18]), we see that the condition that $\langle w\rangle^{\perp}/\langle w\rangle$ have no vectors of square norm -2 is equivalent to the condition that the orthogonal complement $U^{\perp}$ of U in L be isomorphic to the (negative-definite) Leech lattice Λ. From this fact, we can deduce the following:

Proposition 2.1. *The group* $\mathrm{O}^+(L)$ *acts on the set of Weyl vectors transitively.*

Proposition 2.2. *Suppose that* w *is a Weyl vector and that* $w' \in L$ *satisfies* $(w, w')_L = 1$ *and* $(w', w')_L = 0$*. Via an isomorphism* $\rho : \Lambda \xrightarrow{\sim} U^{\perp}$*, the map*

$$\lambda \;\mapsto\; -\frac{2 + (\lambda, \lambda)_\Lambda}{2} w + w' + \rho(\lambda)$$

induces a bijection from the Leech lattice Λ *to the set* $LR(w)$*.*

Using Vinberg's algorithm [34] and the result on the covering radius of the Leech lattice [8], Conway [6] proved the following:

Theorem 2.3. *Let* $w \in L$ *be a Weyl vector. Then*

$$D_L(w) := \{\, x \in \mathcal{P}_L \;\mid\; (x, r)_L \ge 0 \text{ for all } r \in LR(w) \,\}$$

is an $\mathcal{R}_L$*-chamber, and* $\widetilde{\mathcal{W}}(D_L(w))$ *is equal to* $LR(w)$*; that is,* $(r)^{\perp}_L$ *bounds* $D_L(w)$ *for any* $r \in LR(w)$*. The map* $w \mapsto D_L(w)$ *is a bijection from the set of Weyl vectors to the set of* $\mathcal{R}_L$*-chambers.*

Remark 2.4. Using Proposition 2.2, Conway [6] also showed that the automorphism group $\mathrm{Aut}(D_L(w)) \subset \mathrm{O}^+(L)$ of an $\mathcal{R}_L$-chamber $D_L(w)$ is isomorphic to the group $\cdot\infty$ of *affine* automorphisms of the Leech lattice Λ. Hence $\mathrm{O}^+(L)$ is isomorphic to the split extension of $\cdot\infty$ by $W(L)$.

2.3. **Restriction of $\mathcal{R}_L$-chambers to a primitive sublattice.** Let S be an even hyperbolic lattice of rank $r < 26$ primitively embedded in L. Following Borcherds [4], we explain how the Leech roots of L induce a chamber decomposition on the positive cone

$$\mathcal{P}_S := \mathcal{P}_L \cap (S \otimes \mathbb{R})$$

of $S \otimes \mathbb{R}$.

The orthogonal complement $T := S^{\perp}$ of S in L is negative-definite of rank $26-r$, and we have

$$S \oplus T \;\subset\; L \;\subset\; S^{\vee} \oplus T^{\vee}$$

with $[L : S \oplus T] = [S^{\vee} \oplus T^{\vee} : L]$. The projections $L \otimes \mathbb{R} \to S \otimes \mathbb{R}$ and $L \otimes \mathbb{R} \to T \otimes \mathbb{R}$ are denoted by

$$x \mapsto x_S \quad\text{and}\quad x \mapsto x_T,$$

respectively. Note that, if $v \in L$, then $v_S \in S^{\vee}$ and $v_T \in T^{\vee}$.

Let $r \in L$ be a root. Then the hyperplane $(r)^{\perp}_{L}$ contains $S \otimes \mathbb{R}$ if and only if $r_S = 0$, or equivalently, $r \in T$. Since T is negative-definite, the set

$$\mathcal{R}_T := \{\ v \in T \ \mid\ (v, v)_T = -2\ \}$$

is finite, and therefore there exist only finite number of hyperplanes $(r)^{\perp}_{L}$ that contain $S \otimes \mathbb{R}$. Suppose that $r_S \neq 0$. If $(r_S, r_S)_S \geq 0$, then either $\mathcal{P}_S$ is entirely contained in the interior of the halfspace

$$\{\ x \in L \otimes \mathbb{R} \ \mid\ (x, r)_L \geq 0\ \}$$

or is disjoint from this halfspace. Hence the hyperplane

$$(r_S)^{\perp}_{S} = (r)^{\perp}_{L} \cap (S \otimes \mathbb{R})$$

of $S \otimes \mathbb{R}$ intersects $\mathcal{P}_S$ if and only if $(r_S, r_S)_S < 0$. We put

$$\begin{aligned} \mathcal{R}_S &:= \{\ r_S \ \mid\ r \in \mathcal{R}_L \ \text{ and } \ (r_S, r_S)_S < 0\ \} \\ &= \{\ r_S \ \mid\ r \in \mathcal{R}_L \ \text{ and } \ (r_S)^{\perp}_{S} \cap \mathcal{P}_S \neq \emptyset\ \}. \end{aligned}$$

Then the associated family of hyperplanes

$$\mathcal{H}_S := \{\ (r_S)^{\perp}_{S} \ \mid\ r_S \in \mathcal{R}_S\ \}$$

is locally finite in $\mathcal{P}_S$, and hence we can consider $\mathcal{R}_S$-chambers in $\mathcal{P}_S$. Note that $\mathcal{R}_S$ is invariant under $r_S \mapsto -r_S$. We investigate the relation between $\mathcal{R}_S$-chambers and $\mathcal{R}_L$-chambers.

If $D_S \subset \mathcal{P}_S$ is an $\mathcal{R}_S$-chamber, then there exists an $\mathcal{R}_L$-chamber $D_L(w) \subset \mathcal{P}_L$ such that $D_S = D_L(w) \cap (S \otimes \mathbb{R})$ holds.

For a given $\mathcal{R}_S$-chamber D_S, the set of $\mathcal{R}_L$-chambers $D_L(w)$ satisfying $D_S = D_L(w) \cap (S \otimes \mathbb{R})$ is in one-to-one correspondence with the set of connected components of

$$(T \otimes \mathbb{R}) \setminus \bigcup_{r \in \mathcal{R}_T} (r)^{\perp}_{T}.$$

Conversely, suppose that an $\mathcal{R}_L$-chamber $D_L(w)$ is given.

Definition 2.5. We say that $D_L(w)$ is S-*nondegenerate* if $D_L(w) \cap (S \otimes \mathbb{R})$ is an $\mathcal{R}_S$-chamber.

By definition, $D_L(w)$ is S-nondegenerate if and only if w satisfies the following two conditions:

(i) There exists $v \in \mathcal{P}_S$ such that $(v, r)_L \geq 0$ holds for any $r \in LR(w)$.

(ii) There exists $v' \in \mathcal{P}_S$ such that $(v', r)_L > 0$ holds for any $r \in LR(w)$ with $(r_S, r_S)_S < 0$.

If $D_S = D_L(w) \cap (S \otimes \mathbb{R})$ is an $\mathcal{R}_S$-chamber, then $\widetilde{\mathcal{W}}(D_S)$ is contained in the image of the set

$$LR(w, S) := \{\ r \in LR(w) \ \mid\ r_S \in \mathcal{R}_S\ \} = \{\ r \in LR(w) \ \mid\ (r_S, r_S)_S < 0\ \}$$

by the projection $L \to S^\vee$. The following proposition shows that D_S is bounded by a finite number of walls if $w_T \neq 0$, and its proof indicates an effective procedure to calculate $LR(w, S)$. (See [30, Section 3] for the details of the necessary algorithms.)

Proposition 2.6. *Let $w \in L$ be a Weyl vector such that $w_T \neq 0$. Then $LR(w, S)$ is a finite set.*

Proof. Since T is negative-definite and $w_T \neq 0$, we have

$$(w_S, w_S)_S = -(w_T, w_T)_T > 0.$$

Suppose that $r \in LR(w)$. Then we have

$$(w_S, r_S)_S + (w_T, r_T)_T = 1, \quad (r_S, r_S)_S + (r_T, r_T)_T = -2.$$

We have $(r_S, r_S)_S < 0$ if and only if $(r_T, r_T)_T > -2$. Since T is negative-definite, the set

$$V_T := \{\ v \in T^\vee \ \mid\ (v, v)_T > -2\ \}$$

is finite. For $v \in V_T$, we put

$$a_v := 1 - (w_T, v)_T, \quad n_v := -2 - (v, v)_T \quad \text{and} \quad A := \{\ (a_v, n_v)\ \mid\ v \in V_T\ \}.$$

For each $(a, n) \in A$, we put

$$V_S(a, n) := \{\ u \in S^\vee \ \mid\ (w_S, u)_S = a,\ (u, u)_S = n\ \}.$$

Since S is hyperbolic and $(w_S, w_S)_S > 0$, the set $V_S(a, n)$ is finite, because $(\ ,\ )_S$ induces on the affine hyperplane

$$\{\ x \in S \otimes \mathbb{R} \ \mid\ (x, w_S)_S = a\ \}$$

of $S \otimes \mathbb{R}$ an inhomogeneous quadratic function whose quadratic part is negative-definite. Then the set $LR(w, S)$ is equal to

$$L \cap \{\ u + v \ \mid\ v \in V_T,\ u \in V_S(a_v, n_v)\ \},$$

where the intersection is taken in $S^\vee \oplus T^\vee$. □

The notion of $\mathcal{R}_S$-chamber is useful in the study on $\mathrm{O}^+(S)$ because of the following:

Proposition 2.7. *Suppose that the natural homomorphism $\mathrm{O}(T) \to \mathrm{O}(q_T)$ is surjective. Then the action of $\mathrm{O}^+(S)$ preserves $\mathcal{R}_S$. In particular, for an $\mathcal{R}_S$-chamber D_S and an isometry $\gamma \in \mathrm{O}^+(S)$, the image D_S^γ of D_S by γ is also an $\mathcal{R}_S$-chamber. Moreover, if the interior of D_S^γ has a common point with D_S, then $D_S^\gamma = D_S$ holds and γ preserves $\widetilde{\mathcal{W}}(D_S)$.*

Proof. By the assumption $\mathrm{O}(T) \twoheadrightarrow \mathrm{O}(q_T)$, every element $\gamma \in \mathrm{O}^+(S)$ lifts to an element $\tilde{\gamma} \in \mathrm{O}(L)$ that satisfies $\tilde{\gamma}(S) = S$ and $\tilde{\gamma}|_S = \gamma$ (see [17, Proposition 1.6.1]). Since $\tilde{\gamma}$ preserves $\mathcal{R}_L$ and γ preserves $\mathcal{P}_S$, γ preserves $\mathcal{R}_S$. □

3. A basis of the Néron-Severi lattice of X

Recall that $X \subset \mathbb{P}^3$ is the Fermat quartic surface in characteristic 3. From now on, we put

$$S := S_X,$$

which is an even hyperbolic lattice of rank 22 such that $S^\vee/S \cong (\mathbb{Z}/3\mathbb{Z})^2$. We use the affine coordinates w, x, y of $\mathbb{P}^3$ with $z = 1$.

Note that X is the Hermitian surface over $\mathbb{F}_9$ (see [12, Chapter 23]). Hence the number of lines contained in X is 112 (see [24, n. 32] or [26, Corollary 2.22]). Since the indices of these lines are important throughout this paper, we present defining equations of these lines in Table 3.1. (Note that $\ell_i \subset X$ implies that ℓ_i is not contained in the plane $z = 0$ at infinity.) From these 112 lines, we choose the following:

$$\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7, \ell_9, \ell_{10}, \ell_{11}, \ell_{17}, \ell_{18}, \ell_{19}, \ell_{21}, \ell_{22}, \ell_{23}, \ell_{25}, \ell_{26}, \ell_{27}, \ell_{33}, \ell_{35}, \ell_{49}. \tag{3.1}$$

The intersection matrix N of these 22 lines is given in Table 3.2. Since $\det N = -9$, the classes $[\ell_i] \in S$ of the lines ℓ_i in (3.1) form a basis of S. Throughout this paper, we fix this basis, and write elements of $S \otimes \mathbb{R}$ as row vectors

$$[x_1, \dots, x_{22}]_S.$$

When we use its dual basis, we write

$$[\xi_1, \dots, \xi_{22}]_S^\vee.$$

Since the hyperplane $w + (1 + i) = 0$ cuts out from X the divisor $\ell_1 + \ell_2 + \ell_3 + \ell_4$, the class $h_0 = [\mathcal{O}_X(1)] \in S$ of the hyperplane section is equal to

$$\begin{aligned} h_0 &= [1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]_S \\ &= [1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]_S^\vee. \end{aligned}$$

As a positive cone $\mathcal{P}_S$ of S, we choose the connected component containing h_0.

From the intersection numbers of the 112 lines, we can calculate their classes $[\ell_i] \in S$.

Remark 3.1. Since these 112 lines are all defined over $\mathbb{F}_9$, every class $v \in S$ is represented by a divisor defined over $\mathbb{F}_9$. More generally, Schütt [23] showed that a supersingular $K3$ surface with Artin invariant 1 in characteristic p has a projective model defined over $\mathbb{F}_{p^2}$, and its Néron-Severi lattice is generated by the classes of divisors defined over $\mathbb{F}_{p^2}$.

Proposition 3.2. *We have*

$$h_0 = \frac{1}{28} \sum_{i=1}^{112} [\ell_i].$$

ℓ_1	$:=$	$\{w+(1+i) = x+(1+i)y=0\}$	ℓ_2	$:=$	$\{w+(1+i) = x+(1-i)y=0\}$
ℓ_3	$:=$	$\{w+(1+i) = x-(1-i)y=0\}$	ℓ_4	$:=$	$\{w+(1+i) = x-(1+i)y=0\}$
ℓ_5	$:=$	$\{w+(1-i) = x+(1+i)y=0\}$	ℓ_6	$:=$	$\{w+(1-i) = x+(1-i)y=0\}$
ℓ_7	$:=$	$\{w+(1-i) = x-(1-i)y=0\}$	ℓ_8	$:=$	$\{w+(1-i) = x-(1+i)y=0\}$
ℓ_9	$:=$	$\{w-(1-i) = x+(1+i)y=0\}$	ℓ_{10}	$:=$	$\{w-(1-i) = x+(1-i)y=0\}$
ℓ_{11}	$:=$	$\{w-(1-i) = x-(1-i)y=0\}$	ℓ_{12}	$:=$	$\{w-(1-i) = x-(1+i)y=0\}$
ℓ_{13}	$:=$	$\{w-(1+i) = x+(1+i)y=0\}$	ℓ_{14}	$:=$	$\{w-(1+i) = x+(1-i)y=0\}$
ℓ_{15}	$:=$	$\{w-(1+i) = x-(1-i)y=0\}$	ℓ_{16}	$:=$	$\{w-(1+i) = x-(1+i)y=0\}$
ℓ_{17}	$:=$	$\{w+iy+i = x+iy-i=0\}$	ℓ_{18}	$:=$	$\{w+iy+i = x-iy+i=0\}$
ℓ_{19}	$:=$	$\{w+iy+i = x+y-1=0\}$	ℓ_{20}	$:=$	$\{w+iy+i = x-y+1=0\}$
ℓ_{21}	$:=$	$\{w+iy-i = x+iy+i=0\}$	ℓ_{22}	$:=$	$\{w+iy-i = x-iy-i=0\}$
ℓ_{23}	$:=$	$\{w+iy-i = x+y+1=0\}$	ℓ_{24}	$:=$	$\{w+iy-i = x-y-1=0\}$
ℓ_{25}	$:=$	$\{w+iy+1 = x+iy-1=0\}$	ℓ_{26}	$:=$	$\{w+iy+1 = x-iy+1=0\}$
ℓ_{27}	$:=$	$\{w+iy+1 = x+y+i=0\}$	ℓ_{28}	$:=$	$\{w+iy+1 = x-y-i=0\}$
ℓ_{29}	$:=$	$\{w+iy-1 = x+iy+1=0\}$	ℓ_{30}	$:=$	$\{w+iy-1 = x-iy-1=0\}$
ℓ_{31}	$:=$	$\{w+iy-1 = x+y-i=0\}$	ℓ_{32}	$:=$	$\{w+iy-1 = x-y+i=0\}$
ℓ_{33}	$:=$	$\{w-iy+i = x+iy+i=0\}$	ℓ_{34}	$:=$	$\{w-iy+i = x-iy-i=0\}$
ℓ_{35}	$:=$	$\{w-iy+i = x+y+1=0\}$	ℓ_{36}	$:=$	$\{w-iy+i = x-y-1=0\}$
ℓ_{37}	$:=$	$\{w-iy-i = x+iy-i=0\}$	ℓ_{38}	$:=$	$\{w-iy-i = x-iy+i=0\}$
ℓ_{39}	$:=$	$\{w-iy-i = x+y-1=0\}$	ℓ_{40}	$:=$	$\{w-iy-i = x-y+1=0\}$
ℓ_{41}	$:=$	$\{w-iy+1 = x+iy+1=0\}$	ℓ_{42}	$:=$	$\{w-iy+1 = x-iy-1=0\}$
ℓ_{43}	$:=$	$\{w-iy+1 = x+y-i=0\}$	ℓ_{44}	$:=$	$\{w-iy+1 = x-y+i=0\}$
ℓ_{45}	$:=$	$\{w-iy-1 = x+iy-1=0\}$	ℓ_{46}	$:=$	$\{w-iy-1 = x-iy+1=0\}$
ℓ_{47}	$:=$	$\{w-iy-1 = x+y+i=0\}$	ℓ_{48}	$:=$	$\{w-iy-1 = x-y-i=0\}$
ℓ_{49}	$:=$	$\{w+y+i = x+iy+1=0\}$	ℓ_{50}	$:=$	$\{w+y+i = x-iy-1=0\}$
ℓ_{51}	$:=$	$\{w+y+i = x+y-i=0\}$	ℓ_{52}	$:=$	$\{w+y+i = x-y+i=0\}$
ℓ_{53}	$:=$	$\{w+y-i = x+iy-1=0\}$	ℓ_{54}	$:=$	$\{w+y-i = x-iy+1=0\}$
ℓ_{55}	$:=$	$\{w+y-i = x+y+i=0\}$	ℓ_{56}	$:=$	$\{w+y-i = x-y-i=0\}$
ℓ_{57}	$:=$	$\{w+y+1 = x+iy-i=0\}$	ℓ_{58}	$:=$	$\{w+y+1 = x-iy+i=0\}$
ℓ_{59}	$:=$	$\{w+y+1 = x+y-1=0\}$	ℓ_{60}	$:=$	$\{w+y+1 = x-y+1=0\}$
ℓ_{61}	$:=$	$\{w+y-1 = x+iy+i=0\}$	ℓ_{62}	$:=$	$\{w+y-1 = x-iy-i=0\}$
ℓ_{63}	$:=$	$\{w+y-1 = x+y+1=0\}$	ℓ_{64}	$:=$	$\{w+y-1 = x-y-1=0\}$
ℓ_{65}	$:=$	$\{w+(1+i)y = x+(1+i)=0\}$	ℓ_{66}	$:=$	$\{w+(1+i)y = x+(1-i)=0\}$
ℓ_{67}	$:=$	$\{w+(1+i)y = x-(1-i)=0\}$	ℓ_{68}	$:=$	$\{w+(1+i)y = x-(1+i)=0\}$
ℓ_{69}	$:=$	$\{w+(1-i)y = x+(1+i)=0\}$	ℓ_{70}	$:=$	$\{w+(1-i)y = x+(1-i)=0\}$
ℓ_{71}	$:=$	$\{w+(1-i)y = x-(1-i)=0\}$	ℓ_{72}	$:=$	$\{w+(1-i)y = x-(1+i)=0\}$
ℓ_{73}	$:=$	$\{w-y+i = x+iy-1=0\}$	ℓ_{74}	$:=$	$\{w-y+i = x-iy+1=0\}$
ℓ_{75}	$:=$	$\{w-y+i = x+y+i=0\}$	ℓ_{76}	$:=$	$\{w-y+i = x-y-i=0\}$
ℓ_{77}	$:=$	$\{w-y-i = x+iy+1=0\}$	ℓ_{78}	$:=$	$\{w-y-i = x-iy-1=0\}$
ℓ_{79}	$:=$	$\{w-y-i = x+y-i=0\}$	ℓ_{80}	$:=$	$\{w-y-i = x-y+i=0\}$
ℓ_{81}	$:=$	$\{w-y+1 = x+iy+i=0\}$	ℓ_{82}	$:=$	$\{w-y+1 = x-iy-i=0\}$
ℓ_{83}	$:=$	$\{w-y+1 = x+y+1=0\}$	ℓ_{84}	$:=$	$\{w-y+1 = x-y-1=0\}$
ℓ_{85}	$:=$	$\{w-y-1 = x+iy-i=0\}$	ℓ_{86}	$:=$	$\{w-y-1 = x-iy+i=0\}$
ℓ_{87}	$:=$	$\{w-y-1 = x+y-1=0\}$	ℓ_{88}	$:=$	$\{w-y-1 = x-y+1=0\}$
ℓ_{89}	$:=$	$\{w-(1-i)y = x+(1+i)=0\}$	ℓ_{90}	$:=$	$\{w-(1-i)y = x+(1-i)=0\}$
ℓ_{91}	$:=$	$\{w-(1-i)y = x-(1-i)=0\}$	ℓ_{92}	$:=$	$\{w-(1-i)y = x-(1+i)=0\}$
ℓ_{93}	$:=$	$\{w-(1+i)y = x+(1+i)=0\}$	ℓ_{94}	$:=$	$\{w-(1+i)y = x+(1-i)=0\}$
ℓ_{95}	$:=$	$\{w-(1+i)y = x-(1-i)=0\}$	ℓ_{96}	$:=$	$\{w-(1+i)y = x-(1+i)=0\}$
ℓ_{97}	$:=$	$\{w+(1+i)x = y+(1+i)=0\}$	ℓ_{98}	$:=$	$\{w+(1+i)x = y+(1-i)=0\}$
ℓ_{99}	$:=$	$\{w+(1+i)x = y-(1-i)=0\}$	ℓ_{100}	$:=$	$\{w+(1+i)x = y-(1+i)=0\}$
ℓ_{101}	$:=$	$\{w+(1-i)x = y+(1+i)=0\}$	ℓ_{102}	$:=$	$\{w+(1-i)x = y+(1-i)=0\}$
ℓ_{103}	$:=$	$\{w+(1-i)x = y-(1-i)=0\}$	ℓ_{104}	$:=$	$\{w+(1-i)x = y-(1+i)=0\}$
ℓ_{105}	$:=$	$\{w-(1-i)x = y+(1+i)=0\}$	ℓ_{106}	$:=$	$\{w-(1-i)x = y+(1-i)=0\}$
ℓ_{107}	$:=$	$\{w-(1-i)x = y-(1-i)=0\}$	ℓ_{108}	$:=$	$\{w-(1-i)x = y-(1+i)=0\}$
ℓ_{109}	$:=$	$\{w-(1+i)x = y+(1+i)=0\}$	ℓ_{110}	$:=$	$\{w-(1+i)x = y+(1-i)=0\}$
ℓ_{111}	$:=$	$\{w-(1+i)x = y-(1-i)=0\}$	ℓ_{112}	$:=$	$\{w-(1+i)x = y-(1+i)=0\}$

TABLE 3.1. Lines on X

$$
\begin{bmatrix}
-2 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & -2 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & -2 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & -2 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & -2 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -2 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -2
\end{bmatrix}
$$

Table 3.2. Gram matrix N of S

Proof. The number of $\mathbb{F}_9$-rational points on X is 280. For each $\mathbb{F}_9$-rational point P of X, the tangent plane $T_{X,P} \subset \mathbb{P}^3$ to X at P cuts out a union of four lines from X. Since each line contains ten $\mathbb{F}_9$-rational points, we have $280\, h_0 = 10 \sum [\ell_i]$. □

As before, we let $\mathrm{O}(S)$ act on S from *right*, so that

$$\mathrm{O}(S) = \{\, T \in \mathrm{GL}_{22}(\mathbb{Z}) \;\mid\; T\, N\, {}^tT = N \,\}.$$

We also let the projective automorphism group $\mathrm{Aut}(X, h_0) = \mathrm{PGU}_4(\mathbb{F}_9)$ act on X from *right*. For each $\tau \in \mathrm{PGU}_4(\mathbb{F}_9)$, we can calculate its action τ_* on S by looking at the permutation of the 112 lines induced by τ.

Example 3.3. Consider the projective automorphism

$$\tau \quad : \quad [w:x:y:z] \mapsto [w:x:y:z] \begin{bmatrix} i & 0 & i & -1+i \\ 1 & 1-i & -1 & 0 \\ 1 & i & i & -i \\ 1 & -1 & -i & -1 \end{bmatrix}$$

of X. Then the images ℓ_i^τ of the lines ℓ_i in (3.1) are

$$\begin{array}{l} \ell_1^\tau = \ell_{60},\ \ \ell_2^\tau = \ell_{31},\ \ \ell_3^\tau = \ell_{105},\ \ \ell_4^\tau = \ell_{95},\ \ \ell_5^\tau = \ell_{92},\ \ \ell_6^\tau = \ell_{30}, \\ \ell_7^\tau = \ell_{76},\ \ \ell_9^\tau = \ell_{110},\ \ \ell_{10}^\tau = \ell_{29},\ \ \ell_{11}^\tau = \ell_{6},\ \ \ell_{17}^\tau = \ell_{20},\ \ \ell_{18}^\tau = \ell_{96}, \\ \ell_{19}^\tau = \ell_{102},\ \ \ell_{21}^\tau = \ell_{13},\ \ \ell_{22}^\tau = \ell_{87},\ \ \ell_{23}^\tau = \ell_{91},\ \ \ell_{25}^\tau = \ell_{108},\ \ \ell_{26}^\tau = \ell_{10}, \\ \ell_{27}^\tau = \ell_{57},\ \ \ell_{33}^\tau = \ell_{52},\ \ \ell_{35}^\tau = \ell_{51},\ \ \ell_{49}^\tau = \ell_{59}. \end{array}$$

Therefore the action τ_* on S is given by $v \mapsto vT_\tau$, where T_τ is the matrix whose row vectors are

$$\begin{array}{rcl}
[\ell_{60}] & = & [1,0,0,0,0,0,-1,0,0,0,0,0,0,-1,0,0,0,0,1,0,0,1]_S, \\
[\ell_{31}] & = & [1,1,1,1,0,0,0,0,0,0,0,0,-1,0,0,-1,0,0,-1,0,0,0]_S, \\
[\ell_{105}] & = & [2,2,2,3,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,-1,0,0,0,-1,0]_S, \\
[\ell_{95}] & = & [-3,-2,-2,-3,1,1,2,1,1,1,0,0,1,1,0,1,1,0,0,0,1,-1]_S, \\
[\ell_{92}] & = & [-1,0,0,0,0,1,0,0,1,0,0,1,0,0,1,0,0,1,0,-1,-1,-1]_S, \\
[\ell_{30}] & = & [1,1,1,1,0,0,0,0,0,0,0,-1,0,0,-1,0,0,-1,0,0,0,0]_S, \\
[\ell_{76}] & = & [0,-1,-1,-1,0,-1,0,1,0,1,0,0,0,0,0,0,0,0,0,1,1,1]_S, \\
[\ell_{110}] & = & [-1,0,0,-1,0,0,1,0,0,0,0,-1,0,1,0,0,1,0,0,0,1,0]_S, \\
[\ell_{29}] & = & [1,1,1,1,0,0,0,0,0,0,-1,0,0,-1,0,0,-1,0,0,0,0,0]_S, \\
[\ell_{6}] & = & [0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]_S, \\
[\ell_{20}] & = & [1,1,1,1,0,0,0,0,0,0,-1,-1,-1,0,0,0,0,0,0,0,0,0]_S, \\
[\ell_{96}] & = & [4,2,3,4,-2,-3,-2,-1,-2,0,1,0,-1,0,-1,-1,-1,-1,-1,2,0,1]_S, \\
[\ell_{102}] & = & [-1,-1,-1,-1,1,1,1,0,0,0,0,0,1,0,0,1,0,0,1,-1,0,0]_S, \\
[\ell_{13}] & = & [0,1,1,1,-1,0,0,-1,0,0,0,0,0,0,0,0,0,0,0,0,0,0]_S, \\
[\ell_{87}] & = & [-3,-2,-3,-3,2,2,1,1,1,0,0,1,1,0,1,1,1,1,1,-1,0,-1]_S, \\
[\ell_{91}] & = & [4,2,3,3,-1,-2,-1,0,-1,0,0,-1,-1,-1,-1,-1,-1,-1,-1,1,0,1]_S, \\
[\ell_{108}] & = & [-2,-2,-2,-3,1,0,1,1,0,1,1,0,1,1,0,1,1,0,0,0,1,0]_S, \\
[\ell_{10}] & = & [0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0]_S, \\
[\ell_{57}] & = & [1,2,1,2,-1,0,-1,-1,0,-1,0,1,0,0,0,-1,0,1,0,0,-1,-1]_S, \\
[\ell_{52}] & = & [-1,0,-1,-1,1,1,1,1,1,0,0,0,1,0,0,0,0,0,0,-1,0,-1]_S, \\
[\ell_{51}] & = & [1,1,1,2,0,0,-1,-1,0,-1,0,1,0,0,0,0,0,0,0,0,-1,-1]_S, \\
[\ell_{59}] & = & [2,1,2,2,-1,-2,0,0,-1,1,0,-1,-1,0,-1,0,-1,-1,-1,1,1,1]_S.
\end{array}$$

We put the representation

$$\tau \mapsto T_\tau \tag{3.2}$$

of $\mathrm{Aut}(X, h_0) = \mathrm{PGU}_4(\mathbb{F}_9)$ to $\mathrm{O}^+(S)$ in the computer memory. It turns out to be faithful. On the other hand, $\mathrm{Aut}(X, h_0)$ is just the stabilizer subgroup in $\mathrm{Aut}(X)$ of $h_0 \in S$. Therefore we confirm the following fact ([21, Section 8, Proposition 3]):

Proposition 3.4. *The action of* $\mathrm{Aut}(X)$ *on* S *is faithful.*

From now on, we regard $\mathrm{Aut}(X)$ as a subgroup of $\mathrm{O}^+(S)$, and write $v \mapsto v^\gamma$ instead of $v \mapsto v^{\gamma_*}$ for the action γ_* of $\gamma \in \mathrm{Aut}(X)$ on S.

4. Embedding of S into L

Next we embed the Néron-Severi lattice S of X into the even unimodular hyperbolic lattice of rank 26, and calculate the walls of an $\mathcal{R}_S$-chamber.

Let T be the negative-definite root lattice of type $2A_2$. We fix a basis of T in such a way that the Gram matrix is equal to

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

When we use this basis, we write elements of $T \otimes \mathbb{R}$ as $[y_1, y_2, y_3, y_4]_T$, while when we use its dual basis, we write as $[\eta_1, \eta_2, \eta_3, \eta_4]^\vee_T$. Elements of $(S \oplus T) \otimes \mathbb{R}$ are written as

$$[x_1, \dots, x_{22} \mid y_1, \dots, y_4]$$

using the bases of S and T, or as

$$[\xi_1, \dots, \xi_{22} \mid \eta_1, \dots, \eta_4]^\vee$$

using the dual bases of $S^\vee$ and $T^\vee$.

Consider the following vectors of $S^\vee \oplus T^\vee$:

$$\begin{aligned} a_1 &:= \frac{1}{3}[2,2,0,0,0,1,2,2,1,1,2,2,1,1,2,0,0,1,1,0,0,0 \mid 1,2,0,0], \\ a_2 &:= \frac{1}{3}[2,0,2,0,2,1,1,0,2,1,2,1,0,2,2,1,1,0,1,0,0,0 \mid 0,0,1,2]. \end{aligned}$$

We define $\alpha_1, \alpha_2 \in (S \oplus T)^\vee/(S \oplus T)$ by

$$\alpha_1 := a_1 \bmod (S \oplus T), \quad \alpha_2 := a_2 \bmod (S \oplus T).$$

Then α_1 and α_2 are linearly independent in $(S \oplus T)^\vee/(S \oplus T) \cong \mathbb{F}_3^4$. Since

$$q_{S\oplus T}(\alpha_1) = q_{S\oplus T}(\alpha_2) = q_{S\oplus T}(\alpha_1 + \alpha_2) = 0,$$

the vectors α_1 and α_2 generate a maximal isotropic subgroup of $q_{S\oplus T}$. Therefore, by [17, Proposition 1.4.1], the submodule

$$L := (S \oplus T) + \langle a_1 \rangle + \langle a_2 \rangle$$

of $S^\vee \oplus T^\vee$ is an even unimodular overlattice of $S \oplus T$ into which S and T are primitively embedded.

By construction, L is hyperbolic of rank 26. We choose $\mathcal{P}_L$ to be the connected component that contains $\mathcal{P}_S$. Then, by means of the roots of L, we obtain a decomposition of $\mathcal{P}_S$ into $\mathcal{R}_S$-chambers.

The order of $\mathrm{O}(T)$ is 288, while the order of $\mathrm{O}(q_T)$ is 8. It is easy to check that the natural homomorphism $\mathrm{O}(T) \to \mathrm{O}(q_T)$ is surjective. Therefore we obtain the following from Proposition 2.7:

Proposition 4.1. *The action of* $\mathrm{O}^+(S)$ *on* $S \otimes \mathbb{R}$ *preserves* $\mathcal{R}_S$.

We put

$$\begin{aligned} w_0 &:= [1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0 \mid -1,-1,-1,-1] \\ &= [1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \mid 1,1,1,1]^\vee. \end{aligned}$$

Note that the projection $w_{0S} \in S^\vee$ of w_0 to $S^\vee$ is equal to h_0.

Since $(w_0, w_0)_L = 0$ and $(w_0, h_0)_L > 0$, we see that w_0 is on the boundary of the closure of $\mathcal{P}_L$ in $L \otimes \mathbb{R}$.

Proposition 4.2. *The vector* w_0 *is a Weyl vector, and the* $\mathcal{R}_L$*-chamber* $D_L(w_0)$ *is* S*-nondegenerate. The* $\mathcal{R}_S$*-chamber*

$$D_{S0} := D_L(w_0) \cap (S \otimes \mathbb{R})$$

contains $w_{S0} = h_0$ *in its interior.*

Proof. The only non-trivial part of the first assertion is that $\langle w_0 \rangle^\perp / \langle w_0 \rangle$ has no vectors of square norm -2. We put

$$w_0' := [7,6,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,7 \mid 7,5,7,7]^\vee.$$

Then we have $(w_0', w_0')_L = 0$ and $(w_0, w_0')_L = 1$. Let $U \subset L$ be the sublattice generated by w_0 and w_0'. Calculating a basis $\lambda_1, \dots, \lambda_{24}$ of $U^\perp \subset L$, we obtain a Gram matrix of $U^\perp$, which is negative-definite of determinant 1. By the algorithm described in [30, Section 3.1], we verify that there are no vectors of square norm -2 in $U^\perp$.

We show that w_0 satisfies the conditions (i) and (ii) given after Definition 2.5. By Proposition 2.2, in order to verify the condition (i), it is enough to show that the function $Q : U^\perp \to \mathbb{Z}$ given by

$$Q(\lambda) := (h_0, -\frac{2 + (\lambda, \lambda)_L}{2} w_0 + w_0' + \lambda)_L$$

does not take negative values. Using the basis $\lambda_1, \dots, \lambda_{24}$ of $U^\perp$, we can write Q as an inhomogeneous quadratic function of 24 variables. Its quadratic part turns out to be positive-definite. By the algorithm described in [30, Section 3.1], we verify that there exist no vectors $\lambda \in U^\perp$ such that $Q(\lambda) < 0$. Next we show that $w_{0S} = h_0 \in \mathcal{P}_S$ has the property required for v' in the condition (ii), and hence h_0 is contained in the interior of D_{S0}. Note that $w_{0T} = [-1, -1, -1, -1]_T$ is non-zero. Hence we can calculate

$$LR(w_0, S) = \{\, r \in LR(w_0) \mid (r_S, r_S)_S < 0 \,\}$$

by the method described in the proof of Proposition 2.6. Then we can easily show that h_0 satisfies $(h_0, r)_L > 0$ for any $r \in LR(w_0, S)$. □

Remark 4.3. There exist exactly four vectors $\lambda \in U^\perp$ such that $Q(\lambda) = 0$. They correspond to the Leech roots $r \in LR(w_0)$ such that $r = r_T$.

From the surjectivity of $\mathrm{O}(T) \to \mathrm{O}(q_T)$ and Proposition 2.7, we obtain the following:

Corollary 4.4. *The action of* $\mathrm{Aut}(X, h_0)$ *on* $S \otimes \mathbb{R}$ *preserves* D_{S0} *and* $\widetilde{\mathcal{W}}(D_{S0})$.

Proposition 4.5. *The maps* $r \mapsto r_S$ *and* $r_S \mapsto (r_S)^\perp_S$ *induce bijections*

$$LR(w_0, S) \;\cong\; \widetilde{\mathcal{W}}(D_{S0}) \;\cong\; \mathcal{W}(D_{S0}).$$

The action of $\mathrm{Aut}(X, h_0)$ *decomposes* $\widetilde{\mathcal{W}}(D_{S0})$ *into the three orbits*

$$\widetilde{W}_{112} := \widetilde{\mathcal{W}}(D_{S0})_{[1,-2]}, \quad \widetilde{W}_{648} := \widetilde{\mathcal{W}}(D_{S0})_{[2,-4/3]} \quad \text{and} \quad \widetilde{W}_{5184} := \widetilde{\mathcal{W}}(D_{S0})_{[3,-2/3]}$$

of cardinalities 112*,* 648 *and* 5184*, respectively, where*

$$\widetilde{\mathcal{W}}(D_{S0})_{[a,n]} := \{\, r_S \in \widetilde{\mathcal{W}}(D_{S0}) \mid (r_S, h_0)_S = a,\ (r_S, r_S)_S = n \,\}.$$

The set $\widetilde{W}_{112}$ *coincides with the set of the classes* $[\ell_i]$ *of lines contained in* X*:*

$$\widetilde{W}_{112} = \{[\ell_1], [\ell_2], \dots, [\ell_{112}]\}.$$

The sets $\widetilde{W}_{648}$ *and* $\widetilde{W}_{5184}$ *are the orbits of*

$$\begin{aligned} b_1 &:= \frac{1}{3}[-1,0,-1,0,2,1,1,0,2,1,-1,1,0,-1,-1,1,1,0,1,0,0,0]_S \;\in\; \widetilde{W}_{648}, \quad \textit{and} \\ b_2 &:= \frac{1}{3}[0,1,-1,0,2,0,2,1,1,0,0,-1,2,1,0,1,1,-1,0,0,0,0]_S \;\in\; \widetilde{W}_{5184}, \end{aligned}$$

by the action of $\mathrm{Aut}(X, h_0)$*, respectively.*

Proof. We have calculated the finite set $LR(w_0, S)$ in the proof of Proposition 4.2. We have also stored the classes $[\ell_i]$ of the 112 lines and the action of $\mathrm{Aut}(X, h_0)$ on S in the computer memory. Thus the assertions of Proposition 4.5 are verified by a direct computation, except for the fact that, for any $r \in LR(w_0, S)$, the hyperplane $(r_S)^{\perp}_S$ actually bounds D_{S0}. This is proved by showing that the point

$$p := h_0 - \frac{(h_0, r_S)_S}{(r_S, r_S)_S} r_S$$

on $(r_S)^{\perp}_S$ satisfies $(p, r')_L > 0$ for any $r' \in LR(w_0, S) \setminus \{r\}$. □

Since Proposition 3.2 implies that the interior point h_0 of D_{S0} is determined by $\widetilde{W}_{112}$ and since $\mathrm{O}(T) \to \mathrm{O}(q_T)$ is surjective, we obtain the following from Proposition 2.7:

Corollary 4.6. *For $\gamma \in \mathrm{O}^+(S)$, the following are equivalent:* (i) *the interior of D_{S0}^{γ} has a common point with D_{S0},* (ii) $D_{S0}^{\gamma} = D_{S0}$, (iii) $\widetilde{W}_{112}^{\gamma} = \widetilde{W}_{112}$, (iv) $h_0^{\gamma} = h_0$, *and* (v) $h_0^{\gamma} \in D_{S0}$.

In particular, we obtain the following:

Corollary 4.7. *If $\gamma \in \mathrm{Aut}(X)$ satisfies $h_0^{\gamma} \in D_{S0}$, then γ is in $\mathrm{Aut}(X, h_0)$.*

5. The automorphisms g_1 and g_2

In order to find automorphisms $\gamma \in \mathrm{Aut}(X)$ such that $h_0^{\gamma} \notin D_{S0}$, we search for polarizations of degree 2 that are located on the walls $(b_1)^{\perp}_S$ and $(b_2)^{\perp}_S$.

We fix terminologies and notation. For a vector $v \in S$, we denote by $\mathcal{L}_v \to X$ a line bundle defined over $\mathbb{F}_9$ whose class is v (see Remark 3.1). We say that a vector $h \in S$ is a *polarization* of degree d if $(h, h)_S = d$ and the complete linear system $|\mathcal{L}_h|$ is nonempty and has no fixed components. If h is a polarization, then $|\mathcal{L}_h|$ has no base-points by [22, Corollary 3.2] and hence defines a morphism

$$\Phi_h : X \to \mathbb{P}^N,$$

where $N = \dim |\mathcal{L}_h|$.

A polynomial in $\mathbb{F}_9[w, x, y]$ is said to be *of normal form* if its degree with respect to w is ≤ 3. For each polynomial $G \in \mathbb{F}_9[w, x, y]$, there exists a unique polynomial $\overline{G}$ of normal form such that

$$G \equiv \overline{G} \mod (w^4 + x^4 + y^4 + 1).$$

We say that $\overline{G}$ is the *normal form* of G. For any $d \in \mathbb{Z}$, the vector space $H^0(X, \mathcal{L}_{dh_0})$ over $\mathbb{F}_9$ is naturally identified with the vector subspace

$$\Gamma(d) := \{\ G \in \mathbb{F}_9[w, x, y] \ \mid\ G \text{ is of normal form with total degree} \le d\ \}$$

of $\mathbb{F}_9[w, x, y]$. For an ideal J of $\mathbb{F}_9[w, x, y]$, we put

$$\Gamma(d, J) := \Gamma(d) \cap J.$$

A basis of $\Gamma(d, J)$ is easily obtained by a Gröbner basis of J. Let ℓ_i be a line contained in X. We denote by $I_i \subset \mathbb{F}_9[w,x,y]$ the affine defining ideal of ℓ_i in $\mathbb{P}^3$ (see Table 3.1), and put

$$I_i^{(\nu)} := I_i^{\nu} + (w^4 + x^4 + y^4 + 1) \quad \subset \quad \mathbb{F}_9[w,x,y]$$

for nonnegative integers ν. Suppose that $v \in S$ is written as

$$v = d\,h_0 - \sum_{i=1}^{112} a_i[\ell_i], \tag{5.1}$$

where a_i are nonnegative integers. Then there exists a natural isomorphism

$$H^0(X, \mathcal{L}_v) \cong \Gamma(d, \bigcap_{i=1}^{112} I_i^{(a_i)})$$

with the property that, for

$$v' = d' h_0 - \sum_{i=0}^{112} a'_i[\ell_i] \quad \in \quad S$$

with $a'_i \in \mathbb{Z}_{\geq 0}$, the multiplication homomorphism

$$H^0(X, \mathcal{L}_v) \times H^0(X, \mathcal{L}_{v'}) \to H^0(X, \mathcal{L}_{v+v'})$$

is identified with

$$\Gamma(d, \bigcap I_i^{(a_i)}) \times \Gamma(d', \bigcap I_i^{(a'_i)}) \to \Gamma(d+d', \bigcap I_i^{(a_i+a'_i)})$$

given by $(\overline{G}, \overline{G'}) \mapsto \overline{GG'}$.

Proposition 1.1 in Introduction is an immediate consequence of the following:

Proposition 5.1. *Consider the vectors*

$$\begin{aligned} m_1 &:= [-1,0,-1,-1,2,2,1,1,2,0,-1,1,1,-1,0,1,0,0,1,-1,0,0]_S \quad \textit{and} \\ m_2 &:= [2,2,1,2,1,-1,1,1,1,1,0,-1,0,0,0,0,0,-1,-1,1,0,1]_S \end{aligned}$$

of S. Then each m_i is a polarization of degree 2. If we choose a basis of the vector space $H^0(X, \mathcal{L}_{m_i})$ appropriately, the morphism $\Phi_{m_i} : X \to \mathbb{P}^2$ associated with $|\mathcal{L}_{m_i}|$ coincides with the morphism $\phi_i : X \to \mathbb{P}^2$ given in the statement of Proposition 1.1.

Proof. We have $(m_i, m_i)_S = 2$. By the method described in [30, Section 4.1], we see that m_i is a polarization; namely, we verify that the sets

$$\begin{aligned} &\{\, v \in S \mid (v,v)_S = -2,\ (v, m_i)_S < 0,\ (v, h_0)_S > 0 \,\} \quad \text{and} \\ &\{\, v \in S \mid (v,v)_S = 0,\ (v, m_i)_S = 1 \,\} \end{aligned}$$

are both empty. Since

$$m_1 = 3h_0 - ([\ell_{21}] + [\ell_{22}] + [\ell_{50}] + [\ell_{63}] + [\ell_{65}] + [\ell_{88}]) \quad \text{and} \tag{5.2}$$

$$\begin{aligned} m_2 = 5h_0 - (&[\ell_1] + [\ell_3] + [\ell_6] + [\ell_{18}] + \\ &+[\ell_{35}] + [\ell_{74}] + [\ell_{90}] + [\ell_{92}] + [\ell_{110}] + [\ell_{111}]), \end{aligned} \tag{5.3}$$

ℓ_{37}	$\mapsto$	$[1:1-i:1-i]$	(A_1-point)
ℓ_{23}	$\mapsto$	$[1:1+i:-(1+i)]$	(A_1-point)
ℓ_{62}	$\mapsto$	$[1:-(1+i):0]$	(A_1-point)
ℓ_{102}	$\mapsto$	$[1:-(1-i):0]$	(A_1-point)
ℓ_{68}	$\mapsto$	$[1:1+i:1+i]$	(A_1-point)
ℓ_{112}	$\mapsto$	$[1:1-i:-(1-i)]$	(A_1-point)
ℓ_{49}, ℓ_{29}	$\mapsto$	$[1:1:-i]$	(A_2-point)
ℓ_{73}, ℓ_{60}	$\mapsto$	$[1:1:i]$	(A_2-point)
ℓ_{18}, ℓ_{10}	$\mapsto$	$[0:1:-1]$	(A_2-point)
ℓ_{16}, ℓ_{99}	$\mapsto$	$[0:1:1]$	(A_2-point)

TABLE 5.1. Lines contracted by $\phi_1 : X \to \mathbb{P}^2$

the vector spaces $H^0(X, \mathcal{L}_{m_1})$ and $H^0(X, \mathcal{L}_{m_2})$ are identified with the subspaces

$$\begin{aligned} \Gamma_1 &:= \Gamma(3, I_{21} \cap I_{22} \cap I_{50} \cap I_{63} \cap I_{65} \cap I_{88}) \quad \text{and} \\ \Gamma_2 &:= \Gamma(5, I_1 \cap I_3 \cap I_6 \cap I_{18} \cap I_{35} \cap I_{74} \cap I_{90} \cap I_{92} \cap I_{110} \cap I_{111}) \end{aligned}$$

of $\mathbb{F}_9[w, x, y]$, respectively. We calculate a basis of Γ_i by means of Gröbner bases of the ideals I_i. The set $\{F_{i0}, F_{i1}, F_{i2}\}$ of polynomials in Table 1.1 is just a basis of Γ_i thus calculated. □

Remark 5.2. The polarizations m_1 and m_2 in Proposition 5.1 are located on the hyperplanes $(b_1)^{\perp}_S$ and $(b_2)^{\perp}_S$ bounding D_{S0}, respectively, where $b_1 \in \widetilde{W}_{648}$ and $b_2 \in \widetilde{W}_{5184}$ are given in Proposition 4.5.

Proof of Proposition 1.2. The set $\mathrm{Exc}(\phi_i)$ of the classes of (-2)-curves contracted by $\phi_i : X \to \mathbb{P}^2$ is calculated by the method described in [30, Section 4.2]. We first calculate the set

$$R_i^+ := \{ \, v \in S \mid (v,v)_S = -2, \ (v, m_i)_S = 0, \ (v, h_0)_S > 0 \, \}.$$

It turns out that every element of R_i^+ is written as a linear combination with coefficients in $\mathbb{Z}_{\geq 0}$ of elements $l \in R_i^+$ such that $(l, h_0)_S = 1$. Hence we have

$$\mathrm{Exc}(\phi_i) = \{ \, l \in R_i^+ \mid (l, h_0)_S = 1 \, \}.$$

The ADE-type of the root system $\mathrm{Exc}(\phi_i)$ is equal to $6A_1 + 4A_2$ for $i = 1$ and $A_1 + A_2 + 2A_3 + 2A_4$ for $i = 2$. Thus the assertion on the ADE-type of the singularities of Y_i is proved. Moreover we have proved that all (-2)-curves contracted by $\phi_i : X \to \mathbb{P}^2$ are lines. See Tables 5.1 and 5.2, in which the lines $\ell_{k_1}, \dots, \ell_{k_r}$ contracted by ϕ_i to a singular point P of type A_r are indicated in such an order that $(\ell_{k_\nu}, \ell_{k_{\nu+1}})_S = 1$ holds for $\nu = 1, \dots, r-1$.

The defining equation $f_i = 0$ of the branch curve $B_i \subset \mathbb{P}^2$ is calculated by the method given in [30, Section 5]. We calculate a basis of the vector space

ℓ_{43}	$\mapsto$	$[0:1:0]$	(A_1-point)
ℓ_{76}, ℓ_{94}	$\mapsto$	$[1:-1:0]$	(A_2-point)
$\ell_{22}, \ell_{49}, \ell_{20}$	$\mapsto$	$[1:-1:1]$	(A_3-point)
$\ell_{7}, \ell_{5}, \ell_{103}$	$\mapsto$	$[1:-1:-1]$	(A_3-point)
$\ell_{10}, \ell_{2}, \ell_{4}, \ell_{91}$	$\mapsto$	$[1:0:1]$	(A_4-point)
$\ell_{33}, \ell_{36}, \ell_{72}, \ell_{83}$	$\mapsto$	$[1:0:-1]$	(A_4-point)

TABLE 5.2. Lines contracted by $\phi_2 : X \to \mathbb{P}^2$

$H^0(X, \mathcal{L}_{3m_i})$ of dimension 11 using (5.2), (5.3) and Gröbner bases of $I_i^{(3)}$. Note that the ten normal forms $M_{i,1}, \dots, M_{i,10}$ of the cubic monomials of F_{i0}, F_{i1}, F_{i2} are contained in $H^0(X, \mathcal{L}_{3m_i})$. We choose a polynomial $G_i \in H^0(X, \mathcal{L}_{3m_i})$ that is not contained in the linear span of $M_{i,1}, \dots, M_{i,10}$. In the vector space $H^0(X, \mathcal{L}_{6m_i})$ of dimension 38, the 39 normal forms of the monomials of $G_i, F_{i0}, F_{i1}, F_{i2}$ of weighted degree 6 with weight $\deg G_i = 3$ and $\deg F_{ij} = 1$ have a non-trivial linear relation. Note that this linear relation is quadratic with respect to G_i. Completing the square and re-choosing G_i appropriately, we confirm that

$$\overline{G_i^2 + f_i(F_{i0}, F_{i1}, F_{i2})} = 0$$

holds. Hence Y_i is defined by $y^2 + f_i(x_0, x_1, x_2) = 0$. □

Remark 5.3. In order to obtain a defining equation of B_i with coefficients in $\mathbb{F}_3$, we have to choose the basis F_{i0}, F_{i1}, F_{i2} of $\Gamma_i = H^0(X, \mathcal{L}_{m_i})$ carefully. See [30, Section 6.10] for the method.

Remark 5.4. The polynomial

$$G_1 = G_{1(0)}(x,y) + G_{1(1)}(x,y)\,w + G_{1(2)}(x,y)\,w^2 + G_{1(3)}(x,y)\,w^3$$

is given in Table 5.3. The polynomial G_2 is too large to be presented in the paper (see [31]).

Proposition 5.5. *Let g_1 and g_2 be the involutions of X defined in Theorem 1.3. Then the action g_{i*} on S is given by $v \mapsto vA_i$, where A_i is the matrix given in Tables 5.4 and 5.5.*

Proof. Recall that $\mathrm{Exc}(\phi_i)$ is the set of the classes of (-2)-curves contracted by $\phi_i : X \to \mathbb{P}^2$. Suppose that $\gamma_1, \dots, \gamma_r \in \mathrm{Exc}(\phi_i)$ are the classes of (-2)-curves that are contracted to a singular point $P \in \mathrm{Sing}(B_i)$ of type A_r. We index them in such a way that $(\gamma_\nu, \gamma_{\nu+1})_S = 1$ holds for $\nu = 1, \dots, r-1$. Then g_{i*} interchanges γ_i and γ_{r+1-i}. Let $V(P) \subset S \otimes \mathbb{Q}$ denote the linear span of the invariant vectors $\gamma_i + \gamma_{r+1-i}$. Then the eigenspace of g_{i*} on $S \otimes \mathbb{Q}$ with eigenvalue 1 is equal to

$$\langle m_i \rangle \oplus \bigoplus_{P \in \mathrm{Sing}(B_i)} V(P),$$

$$
\begin{aligned}
G_{1(0)} = {} & -(1-i) + (1+i)\,x + (1+i)\,y + i\,x^2 - (1+i)\,xy - (1+i)\,y^2 - xy^2 \\
& + (1+i)\,y^3 - (1-i)\,x^4 - (1+i)\,x^3y - xy^3 - (1-i)\,y^4 - (1-i)\,x^5 - x^3y^2 \\
& - i\,x^2y^3 - (1-i)\,xy^4 + (1+i)\,y^5 - (1-i)\,x^6 + x^5y + i\,x^4y^2 - (1-i)\,x^3y^3 \\
& + (1+i)\,x^2y^4 - i\,xy^5 + (1-i)\,y^6 + (1+i)\,x^7 + x^4y^3 + (1+i)\,x^3y^4 \\
& + i\,xy^6 - i\,y^7 + i\,x^8 + (1+i)\,x^7y + i\,x^6y^2 - i\,x^5y^3 + (1-i)\,x^4y^4 + x^2y^6 \\
& + i\,xy^7 - i\,y^8 - (1+i)\,x^9 - i\,x^8y - (1-i)\,x^7y^2 - (1+i)\,x^6y^3 + i\,x^5y^4 \\
& + (1-i)\,x^4y^5 - (1+i)\,x^3y^6 - (1-i)\,x^2y^7 - (1-i)\,xy^8 - (1+i)\,y^9 \\
G_{1(1)} = {} & (1-i) + (1-i)\,x - (1+i)\,x^2 - xy + i\,y^2 + x^3 - (1-i)\,x^2y - xy^2 \\
& + (1+i)\,x^4 + (1-i)\,xy^3 - (1-i)\,y^4 - x^5 + x^4y + xy^4 - (1+i)\,y^5 \\
& + (1+i)\,x^5y - (1+i)\,xy^5 + y^6 - i\,x^7 - x^6y + x^5y^2 - (1-i)\,x^4y^3 \\
& + (1-i)\,x^2y^5 + (1-i)\,xy^6 - (1-i)\,y^7 - i\,x^8 + (1+i)\,x^7y - i\,x^6y^2 \\
& - i\,x^5y^3 - (1-i)\,x^4y^4 + (1+i)\,x^3y^5 - (1-i)\,x^2y^6 + (1-i)\,xy^7 + (1-i)\,y^8 \\
G_{1(2)} = {} & (1-i) - (1+i)\,x - (1+i)\,xy + y^2 - (1-i)\,x^3 - (1+i)\,x^2y - i\,xy^2 \\
& - (1+i)\,y^3 + x^4 - (1+i)\,x^3y + xy^3 - y^4 - x^5 - x^4y - xy^4 - y^5 \\
& - i\,x^6 + x^4y^2 + i\,x^3y^3 + (1+i)\,xy^5 - y^6 - (1-i)\,x^7 - i\,x^6y - i\,x^5y^2 \\
& - i\,x^4y^3 - (1+i)\,x^3y^4 - (1+i)\,x^2y^5 + (1+i)\,xy^6 - (1+i)\,y^7 \\
G_{1(3)} = {} & (1+i)\,x - (1-i)\,y - (1-i)\,x^2 - (1-i)\,xy + (1+i)\,y^2 - x^3 - x^2y \\
& - xy^2 + y^3 - (1-i)\,x^4 - i\,x^3y + (1-i)\,xy^3 - i\,y^4 + i\,x^5 + x^4y \\
& + (1+i)\,x^3y^2 + (1+i)\,x^2y^3 + (1+i)\,xy^4 + (1+i)\,y^5 - x^6 - (1-i)\,x^5y \\
& + (1+i)\,x^4y^2 + i\,x^3y^3 - (1-i)\,x^2y^4 - (1+i)\,xy^5 + (1+i)\,y^6
\end{aligned}
$$

TABLE 5.3. Polynomial G_1

and the eigenspace with eigenvalue -1 is its orthogonal complement. □

Using the matrix representations A_i of g_{i*}, we verify the following facts:

(1) The eigenspace of g_{i*} with eigenvalue 1 is contained in $(b_i)^{\perp}_{S}$. In particular, we have $b_i^{g_i} = -b_i$.
(2) The vector $h_0^{g_i}$ is equal to the image of h_0 by the reflection into the wall $(b_i)^{\perp}_{S}$, that is $h_0^{g_1} = h_0 + 3\,b_1$ and $h_0^{g_2} = h_0 + 9\,b_2$ hold.

Since $\mathrm{Aut}(X, h_0)$ acts on each of $\widetilde{W}_{648}$ and $\widetilde{W}_{5184}$ transitively, we obtain the following:

Corollary 5.6. *For any $r_S \in \widetilde{W}_{648} \cup \widetilde{W}_{5184}$, there exists $\tau \in \mathrm{Aut}(X, h_0)$ such that*

$$h_0^{g_i\tau} = h_0 + c_i\, r_S$$

holds, where $i = 1$ and $c_1 = 3$ if $r_S \in \widetilde{W}_{648}$ while $i = 2$ and $c_2 = 9$ if $r_S \in \widetilde{W}_{5184}$.

6. Proof of Theorem 1.3

We denote by

$$G := \langle \mathrm{Aut}(X, h_0), g_1, g_2 \rangle$$

$$
\begin{bmatrix}
-1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 1 & 1 \\
-2 & -1 & -2 & -2 & 2 & 1 & 1 & 1 & 2 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \\
-1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 2 & 1 & -1 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 3 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-2 & -1 & -2 & -2 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 3 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

TABLE 5.4. The matrix A_1

the subgroup of $\mathrm{Aut}(X)$ generated by $\mathrm{Aut}(X, h_0)$, g_1 and g_2. Note that the action of $\mathrm{Aut}(X)$ on S preserves the set of nef classes.

Theorem 6.1. *If $v \in S$ is nef, there exists $\gamma \in G$ such that $v^\gamma \in D_{S0}$.*

Proof. Let $\gamma \in G$ be an element such that $(v^\gamma, h_0)_S$ attains

$$\min\{(v^{\gamma'}, h_0)_S \mid \gamma' \in G\}.$$

We show that $(v^\gamma, r_S)_S \geq 0$ holds for any $r_S \in \widetilde{\mathcal{W}}(D_{S0})$. If $r_S \in \widetilde{W}_{112}$, then $r_S = [\ell_i]$ for some line $\ell_i \subset X$, and hence $(v^\gamma, r_S)_S \geq 0$ holds because v^γ is nef. Suppose that $r_S \in \widetilde{W}_{648} \cup \widetilde{W}_{5184}$. By Corollary 5.6, there exists $\tau \in \mathrm{Aut}(X, h_0)$ such that $h_0^{g_i \tau} = h_0 + c_i r_S$ holds, where $i = 1$ and $c_1 = 3$ if $r_S \in \widetilde{W}_{648}$ while $i = 2$

$$
\begin{bmatrix}
1 & 1 & -1 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 0 & -1 & 2 & 1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 4 & 0 & 4 & 2 & 2 & 0 & 0 & -2 & 4 & 2 & 0 & 2 & 2 & -2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -2 & 0 & 4 & 0 & 4 & 2 & 2 & 0 & 0 & -1 & 4 & 2 & 0 & 2 & 2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & -1 & 2 & 1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
4 & 2 & 3 & 3 & -1 & -2 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 0 & 1 \\
-3 & -1 & -4 & -3 & 4 & 2 & 3 & 2 & 2 & 0 & 0 & 0 & 3 & 1 & 1 & 2 & 2 & 0 & 1 & -1 & 0 & -1 \\
2 & 2 & 1 & 2 & 1 & -2 & 2 & 1 & 0 & 1 & 0 & -2 & 1 & 1 & -1 & 1 & 0 & -2 & -1 & 1 & 1 & 1 \\
0 & 2 & -2 & 0 & 4 & 1 & 4 & 2 & 2 & 0 & 0 & -2 & 4 & 2 & 0 & 2 & 2 & -2 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 2 & 0 & 1 & 2 & 0 & 2 & 0 & 0 & -1 & 0 & -1 & 2 & 1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 2 & 0 & 3 & 2 & 1 & 1 & 0 & -2 & 2 & 1 & 0 & 1 & 1 & -2 & 0 & 0 & 1 & 1 \\
-2 & -1 & -2 & -2 & 1 & 2 & 1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\
1 & 2 & 0 & 1 & 1 & -1 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 2 & 0 & 1 & 2 & 0 & 2 & 1 & 1 & 0 & 0 & -1 & 2 & 1 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & -1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 \\
4 & 5 & 1 & 4 & 3 & -1 & 2 & 1 & 1 & -1 & 0 & -2 & 3 & 1 & -1 & 1 & 1 & -2 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

TABLE 5.5. The matrix A_2

and $c_2 = 9$ if $r_S \in \widetilde{W}_{5184}$. Since $\gamma\tau^{-1}g_i \in G$, we have

$$(v^\gamma, h_0)_S \le (v^{\gamma\tau^{-1}g_i}, h_0)_S = (v^\gamma, h_0^{g_i\tau})_S = (v^\gamma, h_0)_S + c_i(v^\gamma, r_S)_S.$$

Therefore $(v^\gamma, r_S)_S \ge 0$ holds. □

The properties (1), (2), (3) of D_{S0} stated in Introduction follow from Corollaries 4.4, 4.6, 4.7 and Theorem 6.1.

Proof of Theorem 1.3. By Corollary 4.7, it is enough to show that, for any $\gamma \in \mathrm{Aut}(X)$, there exists $\gamma' \in G$ such that $h_0^{\gamma\gamma'} \in D_{S0}$ holds. Since h_0^γ is nef, this follows from Theorem 6.1. □

7. The Fermat quartic polarizations for g_1 and g_2

A polarization $h \in S$ of degree 4 is said to be a *Fermat quartic polarization* if, by choosing an appropriate basis of $H^0(X, \mathcal{L}_h)$, the morphism $\Phi_h : X \to \mathbb{P}^3$ associated with $|\mathcal{L}_h|$ induces an automorphism of $X \subset \mathbb{P}^3$. It is obvious that h_0^γ is a Fermat quartic polarization for any $\gamma \in \mathrm{Aut}(X)$. Conversely, if h is a Fermat quartic polarization, then the pull-back of h_0 by the automorphism Φ_h of X is h. Therefore the set of Fermat quartic polarizations is the orbit of h_0 by the action of $\mathrm{Aut}(X)$ on S. Consider the Fermat quartic polarizations

$$\begin{aligned} h_1 &:= h_0^{g_1} = h_0 A_1 = [0,1,0,1,2,1,1,0,2,1,-1,1,0,-1,-1,1,1,0,1,0,0,0]_S, \\ h_2 &:= h_0^{g_2} = h_0 A_2 = [1,4,-2,1,6,0,6,3,3,0,0,-3,6,3,0,3,3,-3,0,0,0,0]_S. \end{aligned}$$

Using the equalities

$$\begin{aligned} (7.1)\quad h_1 &= 6h_0 - ([\ell_3] + [\ell_6] + [\ell_8] + [\ell_{14}] + [\ell_{15}] + [\ell_{17}] + [\ell_{19}] + \\ &\quad + [\ell_{22}] + [\ell_{31}] + [\ell_{34}] + [\ell_{63}] + [\ell_{70}] + [\ell_{79}] + [\ell_{92}]), \quad \text{and} \\ (7.2)\quad h_2 &= 15h_0 - (3\,[\ell_3] + 4\,[\ell_6] + [\ell_{13}] + [\ell_{14}] + 3\,[\ell_{18}] + [\ell_{22}] + \\ &\quad + [\ell_{26}] + [\ell_{27}] + 2\,[\ell_{35}] + [\ell_{44}] + 2\,[\ell_{50}] + 3\,[\ell_{92}] + \\ &\quad + [\ell_{93}] + [\ell_{106}] + [\ell_{108}] + 3\,[\ell_{111}]), \end{aligned}$$

we obtain another description of the involutions g_1 and g_2.

Theorem 7.1. *Let (w, x, y) be the affine coordinates of $\mathbb{P}^3$ with $z = 1$, and let*

$$H_{1j}(w,x,y) = H_{1j0}(x,y) + H_{1j1}(x,y)\,w + H_{1j2}(x,y)\,w^2 + H_{1j3}(x,y)\,w^3$$

be polynomials given in Table 7.1. Then the rational map

$$(w,x,y) \mapsto [H_{10} : H_{11} : H_{12} : H_{13}] \in \mathbb{P}^3 \qquad (7.3)$$

gives the involution g_1 of X.

Remark 7.2. We have a similar list of polynomials $H_{20}, H_{21}, H_{22}, H_{23}$ that gives the involution g_2. They are, however, too large to be presented in the paper (see [31]).

Proof of Theorem 7.1. We put

$$Z := \{3, 6, 8, 14, 15, 17, 19, 22, 31, 34, 63, 70, 79, 92\},$$

which is the set of indices of lines on X that appear in the right-hand side of (7.1). The polynomials $H_{10}, H_{11}, H_{12}, H_{13}$ form a basis of the vector space

$$H^0(X, \mathcal{L}_{h_1}) \cong \Gamma(6, \bigcap_{i \in Z} I_i).$$

(See Section 5 for the notation.) We can easily verify that

$$H_{10}^4 + H_{11}^4 + H_{12}^4 + H_{13}^4 \equiv 0 \mod (w^4 + x^4 + y^4 + 1)$$

$$\begin{aligned}
H_{100} = {} & -1 - (1-i)\,x - (1-i)\,x^2 - (1+i)\,y^2 - i\,x^3 - (1-i)\,xy^2 + x^4 - (1-i)\,x^3y \\
& + (1+i)\,x^2y^2 + (1+i)\,xy^3 - (1-i)\,y^4 - (1-i)\,x^5 + (1+i)\,x^4y - (1-i)\,x^3y^2 - i\,x^2y^3 \\
& - i\,xy^4 + i\,y^5 + i\,x^6 - x^5y - (1+i)\,x^4y^2 - (1+i)\,x^3y^3 + (1+i)\,xy^5 + (1-i)\,y^6 \\
H_{101} = {} & (1+i) + x + (1-i)\,y - i\,x^2 - (1+i)\,xy + i\,y^2 + i\,x^3 - x^2y \\
& - i\,xy^2 + y^3 + x^3y + (1-i)\,x^2y^2 - (1-i)\,xy^3 + (1-i)\,y^4 \\
& + x^5 + i\,x^4y + x^3y^2 - i\,x^2y^3 + (1-i)\,xy^4 - (1-i)\,y^5 \\
H_{102} = {} & i + x + (1+i)\,y + x^2 + (1+i)\,y^2 + (1+i)\,x^3 - x^2y \\
& - (1+i)\,xy^2 + i\,y^3 + (1+i)\,x^4 + (1-i)\,x^2y^2 + xy^3 + (1+i)\,y^4 \\
H_{103} = {} & (1-i) - (1-i)\,x + (1-i)\,y - (1+i)\,x^2 - (1-i)\,xy + (1+i)\,x^3 - (1+i)\,y^3
\end{aligned}$$

$$\begin{aligned}
H_{110} = {} & -i + i\,x + y - (1+i)\,x^2 + xy - (1-i)\,y^2 - x^3 - (1-i)\,x^2y + (1+i)\,xy^2 - y^3 \\
& + (1+i)\,x^4 - i\,x^3y - (1-i)\,x^2y^2 + xy^3 + (1-i)\,y^4 - (1+i)\,x^5 + (1-i)\,x^4y + i\,x^2y^3 \\
& - (1+i)\,xy^4 - (1+i)\,y^5 - i\,x^6 + (1+i)\,x^4y^2 + (1+i)\,x^3y^3 + (1-i)\,xy^5 - (1-i)\,y^6 \\
H_{111} = {} & -(1-i) + x + (1+i)\,y - (1+i)\,x^2 - i\,xy - i\,y^2 + (1-i)\,x^3 \\
& - i\,x^2y - y^3 + (1+i)\,x^4 + (1-i)\,x^3y - x^2y^2 + (1+i)\,xy^3 \\
& - i\,y^4 + (1-i)\,x^5 + i\,x^4y + (1-i)\,x^2y^3 + (1-i)\,xy^4 - y^5 \\
H_{112} = {} & -1 + (1+i)\,y + x^2 - (1-i)\,xy - (1+i)\,y^2 - x^2y + (1+i)\,xy^2 \\
& - (1+i)\,y^3 + (1-i)\,x^4 + (1+i)\,x^3y - (1+i)\,x^2y^2 - xy^3 - (1+i)\,y^4 \\
H_{113} = {} & (1+i) - x + y + x^2 - i\,y^2 - (1-i)\,x^3 + i\,x^2y - (1-i)\,xy^2 - i\,y^3
\end{aligned}$$

$$\begin{aligned}
H_{120} = {} & (1+i) + (1+i)\,x + (1+i)\,y + (1-i)\,x^2 + y^2 + (1+i)\,x^3 + (1+i)\,x^2y \\
& - i\,xy^2 - y^3 - (1-i)\,x^3y + (1-i)\,x^2y^2 - (1+i)\,xy^3 + (1-i)\,y^4 \\
& + (1-i)\,x^5 - i\,x^4y + (1-i)\,x^3y^2 - (1+i)\,x^2y^3 + i\,xy^4 - y^5 + x^6 \\
& - (1+i)\,x^5y - (1-i)\,x^4y^2 + x^3y^3 - i\,x^2y^4 - (1-i)\,xy^5 + (1-i)\,y^6 \\
H_{121} = {} & i + x + xy - (1+i)\,y^2 + x^3 - (1+i)\,x^2y - (1-i)\,xy^2 + (1+i)\,y^3 + x^4 - (1-i)\,x^3y \\
& - (1-i)\,x^2y^2 + (1+i)\,xy^3 - (1-i)\,y^4 - (1-i)\,x^5 + (1+i)\,x^3y^2 + (1+i)\,x^2y^3 + (1-i)\,y^5 \\
H_{122} = {} & (1-i) - x - (1+i)\,y + i\,x^2 - (1-i)\,xy - (1+i)\,y^2 - x^3 - (1-i)\,xy^2 \\
& - i\,y^3 - (1+i)\,x^4 - (1-i)\,x^3y - (1+i)\,x^2y^2 - xy^3 + (1+i)\,y^4 \\
H_{123} = {} & 1 - (1+i)\,x + (1-i)\,y + x^2 + i\,xy + i\,y^2 - (1+i)\,x^3 + (1-i)\,x^2y - (1+i)\,xy^2 + (1+i)\,y^3
\end{aligned}$$

$$\begin{aligned}
H_{130} = {} & -(1-i) + i\,x + (1+i)\,y - (1+i)\,x^2 + (1-i)\,xy + (1-i)\,y^2 + x^3 - (1+i)\,x^2y + i\,xy^2 + i\,y^3 \\
& - (1+i)\,x^4 + i\,x^3y + x^2y^2 - (1+i)\,y^4 + (1+i)\,x^5 - (1-i)\,x^4y + (1-i)\,x^3y^2 - x^2y^3 \\
& - (1+i)\,y^5 - (1+i)\,x^6 - (1-i)\,x^5y - (1+i)\,x^4y^2 + i\,x^3y^3 + i\,x^2y^4 + i\,xy^5 + (1+i)\,y^6 \\
H_{131} = {} & -1 - x + (1+i)\,y - (1-i)\,x^2 + (1+i)\,xy - i\,y^2 - (1+i)\,x^3 - i\,x^2y - xy^2 + i\,y^3 \\
& - x^4 - x^3y + xy^3 - (1+i)\,y^4 - (1+i)\,x^5 + x^4y + (1-i)\,x^3y^2 - i\,x^2y^3 + (1+i)\,xy^4 \\
H_{132} = {} & (1+i) + i\,x + y - x^2 + xy + y^2 + i\,x^3 - (1-i)\,x^2y \\
& - (1+i)\,xy^2 - (1-i)\,x^4 - x^2y^2 - i\,xy^3 - (1-i)\,y^4 \\
H_{133} = {} & i - y + x^2 + (1+i)\,xy - (1-i)\,y^2
\end{aligned}$$

TABLE 7.1. Polynomials H_{1j}

holds. Hence the rational map (7.3) induces an automorphism g' of X. We prove $g' = g_1$ by showing that the action g'_* of g' on S is equal to the action $v \mapsto vA_1$ of g_1. We homogenize the polynomials H_{1j} to $\tilde{H}_{1j}(w, x, y, z)$ so that g' is given by

$$[w : x : y : z] \mapsto [\tilde{H}_{10} : \tilde{H}_{11} : \tilde{H}_{12} : \tilde{H}_{13}].$$

Let ℓ_k be a line on X whose index k is not in Z. We calculate a parametric representation

$$[u : v] \mapsto [l_{k0} : l_{k1} : l_{k2} : l_{k3}]$$

of ℓ_k in $\mathbb{P}^3$, where u, v are homogeneous coordinates of $\mathbb{P}^1$ and $l_{k\nu}$ are homogeneous linear polynomials of u, v. We put

$$L^{(k)}_{1j} := \tilde{H}_{1j}(l_{k0}, l_{k1}, l_{k2}, l_{k3})$$

for $j = 0, \dots, 3$, which are homogeneous polynomials of u, v. Let $M^{(k)}$ be the greatest common divisor of $L^{(k)}_{10}, L^{(k)}_{11}, L^{(k)}_{12}, L^{(k)}_{13}$ in $\mathbb{F}_9[u, v]$. Then

$$\rho_k \quad : \quad [u : v] \mapsto [L^{(k)}_{10}/M^{(k)} : L^{(k)}_{11}/M^{(k)} : L^{(k)}_{12}/M^{(k)} : L^{(k)}_{13}/M^{(k)}]$$

is a parametric representation of the image of ℓ_k by g'. (If $k \in Z$, then $L^{(k)}_{1j}$ are constantly equal to 0.) Pulling back the defining homogeneous ideal of $\ell_{k'}$ by ρ_k, we can calculate the intersection number $([\ell_k]^{g'}, [\ell_{k'}])_S$. Since the classes $[\ell_k]$ with $k \notin Z$ span $S \otimes \mathbb{Q}$, we can calculate the action g'_* of g' on S, which turns out to be equal to $v \mapsto vA_1$. □

Remark 7.3. The polynomials $H_{10}, H_{11}, H_{12}, H_{13}$ are found by the following method. Let H'_0, H'_1, H'_2, H'_3 be an arbitrary basis of $\Gamma(6, \cap_{i \in Z} I_i) \cong H^0(X, \mathcal{L}_{h_1})$. Then the normal forms of the quartic monomials of H'_0, H'_1, H'_2, H'_3 are subject to a linear relation of the following form (see [24, n. 3] or [30, Theorem 6.11]):

$$\sum_{i,j=0}^{3} a_{ij} \overline{H'_i H'^3_j} = 0,$$

where the coefficients $a_{ij} \in \mathbb{F}_9$ satisfy $a_{ji} = a^3_{ij}$ and $\det(a_{ij}) \neq 0$; that is, the matrix (a_{ij}) is non-singular Hermitian. We search for $B \in \mathrm{GL}_3(\mathbb{F}_9)$ such that

$$(a_{ij}) = B\ {}^tB^{(3)}$$

holds, where $B^{(3)}$ is obtained from B by applying $x \mapsto x^3$ to the entries, and put

$$(H''_0, H''_1, H''_2, H''_3) = (H'_0, H'_1, H'_2, H'_3)B.$$

Then $H''_0, H''_1, H''_2, H''_3$ satisfy

$$H''^4_0 + H''^4_1 + H''^4_2 + H''^4_3 \equiv 0 \mod (w^4 + x^4 + y^4 + 1).$$

Therefore $(w, x, y) \mapsto [H''_0 : H''_1 : H''_2 : H''_3]$ induces an automorphism g'' of X. Using the method described in the proof of Theorem 7.1, we calculate the matrix A'' such that the action g''_* of g'' on S is given by $v \mapsto vA''$. Next we search for

$\tau \in \mathrm{PGU}_4(\mathbb{F}_9)$ such that $A''T_\tau$ is equal to A_1, where $T_\tau \in \mathrm{O}^+(S)$ is the matrix representation of τ. Then the polynomials

$$(H_{10}, H_{11}, H_{12}, H_{13}) := (H_0'', H_1'', H_2'', H_3'')\tau$$

have the required property.

Remark 7.4. We have calculated the images of the $\mathbb{F}_9$-rational points of X by the morphisms $\psi_i : X \to Y_i$ and $g_i : X \to X$, and confirmed that they are compatible (see [31]).

8. Generators of $\mathrm{O}^+(S)$

Let $F \in \mathrm{O}^+(S)$ denote the isometry of S obtained from the Frobenius action ϕ of $\mathbb{F}_9$ over $\mathbb{F}_3$ on X. Calculating the action of ϕ on the lines ($\ell_1^\phi = \ell_6, \ell_2^\phi = \ell_5, \ell_3^\phi = \ell_8, \ell_4^\phi = \ell_7, \dots$), we see that F is given $v \mapsto vA_F$, where A_F is the matrix presented in Table 8.1. Since $h_0^F = h_0$, we have $D_{S0}^F = D_{S0}$ by Corollary 4.6.

Proposition 8.1. *The automorphism group* $\mathrm{Aut}(D_{S0}) \subset \mathrm{O}^+(S)$ *of* D_{S0} *is the split extension of* $\langle F\rangle \cong \mathbb{Z}/2\mathbb{Z}$ *by* $\mathrm{Aut}(X, h_0)$.

Proof. Since we have calculated the representation (3.2) of $\mathrm{Aut}(X, h_0)$ into $\mathrm{O}^+(S)$, we can verify that $F \notin \mathrm{Aut}(X, h_0)$. Therefore it is enough to show that the order of $\mathrm{Aut}(D_{S0})$ is equal to 2 times $|\mathrm{Aut}(X, h_0)|$. Since $|\mathrm{PGU}_4(\mathbb{F}_9)|$ is equal to 4 times $|\mathrm{PSU}_4(\mathbb{F}_9)|$, this follows from [13, Lemma 2.1] (see also [7, p. 52]). □

Since $([\ell_1], [\ell_1])_S = -2$, the reflection $s_1 : S \otimes \mathbb{R} \to S \otimes \mathbb{R}$ into the hyperplane $([\ell_1])_S^\perp$ is contained in $\mathrm{O}^+(S)$. In the same way as the proof of Theorem 1.3, we obtain the following:

Theorem 8.2. *The autochronous orthogonal group* $\mathrm{O}^+(S)$ *of the Néron-Severi lattice* S *of* X *is generated by* $\mathrm{Aut}(X, h_0) = \mathrm{PGU}_4(\mathbb{F}_9)$, g_1, g_2, F *and* s_1.

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$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 2 & -1 & -2 & -1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
2 & 2 & 2 & 2 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
2 & 1 & 1 & 2 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\
-3 & -2 & -2 & -3 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 3 & 3 & 4 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

TABLE 8.1. Frobenius action on S

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