BORCHERDS’ METHOD FOR ENRIQUES SURFACES:
COMPUTATIONAL DATA

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This note explains the contents of the computational data about the results of the paper [1] (joint work with Simon Brandhorst). The data is available at

http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3andEnriques.html

in the text file L10L26compdata.txt. In this data, we use the Record-format of GAP [2].

We fix a basis $e_1, \ldots, e_{10}$ of $L_{10}$ given in the paper [1].

- **stdNKconfigs** is the list of intersection matrices of the 7 Nikulin-Kondo configurations, sorted by the type I, II, ..., VII.
- **GramL10** is the Gram matrix of $L_{10}$. (Hence the Gram matrix of $S = L_{10}(2)$ is 2 times this matrix.)
- $a_{10}$ is the interior point $a_{10} = e_1^\vee + \cdots + e_{10}^\vee$ of the Vinberg chamber $V$ defined by $\langle x, e_i \rangle_{10} \geq 0$ for $i = 1, \ldots, 10$.
- **Embs** is a list of 17 records, each of which contains the data for one of the 17 primitive embeddings of $S = L_{10}(2)$ into $L_{26}$. The contents of each record in this list are explained below.

Let $\texttt{irec}$ be a record corresponding to a primitive embedding $\iota: S = L_{10}(2) \hookrightarrow L_{26}$. Recall that $R_i$ is the orthogonal complement of the image of $\iota$ in $L_{26}$. Then $\texttt{irec}$ has the following contents.

- **irec.No** is the number of $\iota$ used in Tables of the paper [1].
- **irec.name** is the name of $\iota$, which is one of the strings "12A", "12B", ..., "96C", "infty".
- **irec.GramL26** is a Gram matrix of $L_{26}$ with respect to a certain fixed basis. We use this basis for other data in this record $\texttt{irec}$.
- **irec.embS** is the $10 \times 26$ integer matrix $M$ such that $v \mapsto vM$ gives the embedding $\iota: S \hookrightarrow L_{26}$.
- **irec.embR** is the $16 \times 26$ integer matrix $M'$ such that $v \mapsto vM'$ gives the embedding of $R_i$ into $L_{26}$ with respect to a certain fixed basis of $R_i$.
- **irec.GramR** is the Gram matrix of $R_i$ with respect to the fixed basis.
- **irec.rootsR** is the list of $(-2)$-vectors of $R_i$.
- **irec.rootstypeR** is the ADE-type of $(-2)$-vectors of $R_i$, which is a list of strings of "A1", "A2", ....

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- \( \text{ires.m4R} \) is the number of \((-4)\)-vectors of \( R \).
- \( \text{ires.ogR} \) is the order of \( O(R) \).
- \( \text{ires.projS} \) is the \( 26 \times 10 \) matrix \( P \) such that \( v \mapsto vP \) gives the orthogonal projection \( \text{pr}_S : L_{26} \to S' \).
- \( \text{ires.projR} \) is the \( 26 \times 16 \) matrix \( P' \) such that \( v \mapsto vP' \) gives the orthogonal projection \( \text{pr}_R : L_{26} \to R'_v \).
- \( \text{ires.weyl} \) is the Weyl vector \( w \in L_{26} \) such that \( D : = \{ P(C(w)) \} \) is an induced chamber in \( \mathcal{P}(L_{10}) \), where \( C(w) \subset \mathcal{P}(L_{26}) \) is the Conway chamber corresponding to \( w \).
- \( \text{ires.weylprime} \) is another Weyl vector \( w' \in L_{26} \) such that \( \langle w, w' \rangle_{26} = 1 \), so that \( a_{26} = 2w + w' \) is an interior point of \( C(w) \).
- \( \text{ires.VR} \) is the list of vectors \( v \) of \( R'_v \) such that \( \langle v, v \rangle_{R} = 1 \), that is, the list of vectors in \( V_R \) in Table 2.1 of [1].
- \( \text{ires.walls} \) is the list of \((-2)\)-vectors defining the walls of the induced chamber \( D \). For the embedding of type \text{infty}, this list is a string "infty".
- \( \text{ires.volindex} \) is \( |O(L_{10}) \otimes \mathbb{F}_2| = 4699591897600 \) devided by the number of Vinberg chambers contained in \( D \). For the embedding of type \text{infty}, this list is a string "infty".
- \( \text{ires.orderOL10D} \) is the order of the group of isometries of \( L_{10} \) that maps \( D \) to \( D \), that is, the order of \( O(L_{10}, D) \). For the embedding of type \text{infty}, this item is a string "infty".
- \( \text{ires.generatorsOL10D} \) is a generating set of \( O(L_{10}, D) \). Neither irredundancy nor minimality is proved for this generating set. For the embedding of type \text{infty}, this item is a string "infty".
- \( \text{ires.interiorpt} \) is an interior point \( a_{10} \) of \( D \) that is fixed under the action of \( O(L_{10}, D) \). For the embedding of type \text{infty}, this item is a string "infty".
- \( \text{ires.NK} \) is the Nikulin-Kondo type of the induced chamber \( D \), which is one of the strings "I", "II", ..., "VII", or "not corresponding to Nikulin-Kondo".
- \( \text{ires.NKisom} \) gives an isomorphism from the set of walls of \( D \) to the vertices of the standard Nikulin-Kondo configuration. Suppose that \( D \) is of Nikulin-Kondo type \( \tau \) with \( n \) vertices, where \( n = 12 \) or \( 20 \). Then \( \text{ires.NKisom} = [\nu_1, ..., \nu_n] \) means that the bijection that maps \( i \)th wall in \( \text{ires.walls} \) to the \( \nu_i \)th vertex of the standard Nikulin-Kondo configuration of type \( \tau \) in \text{stdNKconfigs} \) preserves the intersection matrix. If \( \text{ires.NK} \) is "not corresponding to Nikulin-Kondo", then \( \text{ires.NKisom} \) is also "not corresponding to Nikulin-Kondo".
- \( \text{ires.isomto} \): If the induced chamber \( D \) is isomorphic to the induced chamber \( D' \) of a primitive embedding \( \iota' \) of type different from \( \text{ires.name} \), then...
irec.isomto is the pair of the name of \( \iota' \) and a bijection from the set of walls of \( D \) to the set of walls of \( D' \) that preserves the intersection matrix. The bijection is given in the same way as \( \textsf{irec.NKisom} \). If there exists no such primitive embedding \( \iota' \), then \( \textsf{irec.isomto} \) is an empty list \([ \ ]\).

- **irec.wallrecs** is a list of records, each of which contains the data about a wall \( w \) of \( D \). For the embedding of type \textit{infty}, this list is a string “\textit{infty}”.

The contents of each record in this list are explained below.

Let \( \textsf{wrec} \) be a record in \( \textsf{irec.wallrecs} \) corresponding to a wall \( w = D \cap (r)\perp \) of the induced chamber \( D := \iota_r^{-1}(C(w)) \), where \( \iota \) is not of type \textit{infty}. Then \( \textsf{wrec} \) has the following contents. See the proof of Proposition 2.7 of \([1]\) for notation.

- \( \textsf{wrec.r} \) is the \((-2)\)-vector \( r \) of \( L_{10} \) that defines the wall \( w = D \cap (r)\perp \).
- \( \textsf{wrec.rlifts} \) is the list of \((-2)\)-vectors \( \hat{r} \) of \( L_{26} \) such that \( \langle w, \hat{r} \rangle_{26} = 1 \) and \( \iota_r^{-1}(\hat{r}\perp) = (r)\perp \).
- \( \textsf{wrec.embQ} \) is the \( 17 \times 26 \) integer matrix \( M \) such that \( v \mapsto vM \) is the embedding \( Q \hookrightarrow L_{26} \) under a certain fixed basis of \( Q \).
- \( \textsf{wrec.GramQ} \) is the Gram matrix of \( Q \) with respect to the fixed basis of \( Q \).
- \( \textsf{wrec.embQperp} \) is the \( 9 \times 26 \) integer matrix \( M \) such that \( v \mapsto vM \) is the embedding \( Q\perp \hookrightarrow L_{26} \) under a certain fixed basis of \( Q\perp \).
- \( \textsf{wrec.GramQperp} \) is the Gram matrix of \( Q\perp \) with respect to the fixed basis of \( Q\perp \).
- \( \textsf{wrec.rootsQ} \) is the list of \((-2)\)-vectors of \( Q \).
- \( \textsf{wrec.rootstypeQ} \) is the ADE-type of the set of \((-2)\)-vectors of \( Q \).
- \( \textsf{wrec.Sigma} \) is the list of \((-2)\)-vectors in \( \Sigma \).
- \( \textsf{wrec.adjweyl} \) is the Weyl vector \( w' \) corresponding to the Conway chamber \( C(w') \) such that \( \iota_r^{-1}(C(w')) \) is the induced chamber adjacent to \( D \) across the wall \( w = D \cap (r)\perp \).
- \( \textsf{wrec.thegtilde} \) is the isometry \( \tilde{g} \in O(L_{26}, \mathcal{P}) \) such that \( \tilde{g} \) preserves the image of \( \iota : S \hookrightarrow L_{26} \) and that its restriction \( \tilde{g}|S \) to \( S \) maps \( D \) to the induced chamber adjacent to \( D \) across the wall \( w = D \cap (r)\perp \). The existence of this isometry proves Proposition 2.7 of \([1]\).
- \( \textsf{wrec.theg} \) is the isometry \( g = \tilde{g}|S \) of \( L_{10} \). We can check that this isometry is equal to the reflection with respect to \( \textsf{wrec.r} \).

**References**
