BORCHERDS' METHOD FOR ENRIQUES SURFACES: COMPUTATIONAL DATA

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This note explains the contents of the computational data about the results of the paper [1] (joint work with Simon Brandhorst). The data is available at

http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3andEnriques.html

in the text file `L10L26compdata.txt`. In this data, we use the `Record`-format of `GAP` [2].

We fix a basis $e_1, \dots, e_{10}$ of L_{10} given in the paper [1].

- `stdNKconfigs` is the list of intersection matrices of the 7 Nikulin-Kondo configurations, sorted by the type `I`, `II`, ..., `VII`.
- `GramL10` is the Gram matrix of L_{10}. (Hence the Gram matrix of $\mathbf{S} = L_{10}(2)$ is 2 times this matrix.)
- `a10` is the interior point $a_{10} = e_1^\vee + \cdots + e_{10}^\vee$ of the Vinberg chamber V defined by $\langle x, e_i \rangle_{10} \geq 0$ for $i = 1, \dots, 10$.
- `Embs` is a list of 17 records, each of which contains the data for one of the 17 primitive embeddings of $\mathbf{S} = L_{10}(2)$ into L_{26}. The contents of each record in this list are explained below.

Let `irec` be a record corresponding to a primitive embedding $\iota \colon \mathbf{S} = L_{10}(2) \hookrightarrow L_{26}$. Recall that R_ι is the orthogonal complement of the image of ι in L_{26}. Then `irec` has the following contents.

- `irec.No` is the number of ι used in Tables of the paper [1].
- `irec.name` is the name of ι, which is one of the strings `"12A"`, `"12B"`, ..., `"96C"`, `"infty"`.
- `irec.GramL26` is a Gram matrix of L_{26} with respect to a certain fixed basis. We use this basis for other data in this record `irec`.
- `irec.embS` is the 10×26 integer matrix M such that $v \mapsto vM$ gives the embedding $\iota \colon \mathbf{S} \hookrightarrow L_{26}$.
- `irec.embR` is the 16×26 integer matrix M' such that $v \mapsto vM'$ gives the embedding of R_ι into L_{26} with respect to a certain fixed basis of R_ι.
- `irec.GramR` is the Gram matrix of R_ι with respect to the fixed basis.
- `irec.rootsR` is the list of (-2)-vectors of R_ι.
- `irec.rootstypeR` is the ADE-type of (-2)-vectors of R_ι, which is a list of strings of `"A1"`, `"A2"`, ...

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- `irec.m4R` is the number of (-4)-vectors of R_ι.
- `irec.ogR` is the order of $\mathrm{O}(R_\iota)$.
- `irec.projS` is the 26×10 matrix P such that $v\mapsto vP$ gives the orthogonal projection $\mathrm{pr}_{\mathbf{S}}\colon L_{26}\to \mathbf{S}^\vee$.
- `irec.projR` is the 26×16 matrix P' such that $v\mapsto vP'$ gives the orthogonal projection $\mathrm{pr}_R\colon L_{26}\to R_\iota^\vee$.
- `irec.weyl` is the Weyl vector $\mathbf{w}\in L_{26}$ such that $D:=\iota_{\mathcal{P}}^{-1}(C(\mathbf{w}))$ is an induced chamber in $\mathcal{P}(L_{10})$, where $C(\mathbf{w})\subset\mathcal{P}(L_{26})$ is the Conway chamber corresponding to $\mathbf{w}$.
- `irec.weylprime` is another Weyl vector $\mathbf{w}'\in L_{26}$ such that $\langle \mathbf{w},\mathbf{w}'\rangle_{26}=1$, so that $a_{26}=2\mathbf{w}+\mathbf{w}'$ is an interior point of $C(\mathbf{w})$.
- `irec.VR` is the list of vectors v of $R_\iota^\vee$ such that $\langle v,v\rangle_R=-1$, that is, the list of vectors in V_R in Table 2.1 of [1].
- `irec.walls` is the list of (-2)-vectors defining the walls of the induced chamber D. For the embedding of type `infty`, this list is a string `"infty"`.
- `irec.volindex` is $|\mathrm{O}(L_{10})\otimes\mathbb{F}_2|=46998591897600$ devided by the number of Vinberg chambers contained in D. For the embedding of type `infty`, this list is a string `"infty"`.
- `irec.orderOL10D` is tghe oredr of the group of isometries of L_{10} that maps D to D, that is, the order of $\mathrm{O}(L_{10},D)$. For the embedding of type `infty`, this item is a string `"infty"`.
- `irec.generatorsOL10D` is a generating set of $\mathrm{O}(L_{10},D)$. Neither irredundancy nor minimalily is proved for this generating set. For the embedding of type `infty`, this item is a string `"infty"`.
- `irec.interiorpt` is an interior point a_{10} of D that is fixed under the action of $\mathrm{O}(L_{10},D)$. For the embedding of type `infty`, this item is a string `"infty"`.
- `irec.NK` is the Nikulin-Kondo type of the induced chamber D, which is one of the strings `"I"`, `"II"`, ..., `"VII"`, or `"not corresponding to Nikulin-Kondo"`.
- `irec.NKisom` gives an isomorphism from the set of walls of D to the vertices of the standatd Nikulin-Kondo configuration. Suppose that D is of Nikulin-Kondo type τ with n vertices, where $n=12$ or 20. Then `irec.NKisom` $=[\nu_1,\dots,\nu_n]$ means that the bijection that maps ith wall in `irec.walls` to the ν_ith vertex of the standatd Nikulin-Kondo configuration of type τ in `stdNKconfigs` preserves the intersection matrix. If `irec.NK` is`"not corresponding to Nikulin-Kondo"`, then `irec.NKisom` is also `"not corresponding to Nikulin-Kondo"`.
- `irec.isomto`: If the induced chamber D is isomorphic to the induced chamber D' of a primitive embedding ι' of type different from `irec.name`, then

`irec.isomto` is the pair of the name of ι' and a bijection from the set of walls of D to the set of walls of D' that preserves the intersection matrix. The bijection is given in the same way as `irec.NKisom`. If there exists no such primitive embedding ι', then `irec.isomto` is an empty list [].

- `irec.wallrecs` is a list of records, each of which conrains the data about a wall w of D. For the embedding of type `infty`, this list is a string `"infty"`. The contents of each record in this list are explained below.

Let `wrec` be a record in `irec.wallrecs` corresponding to a wall $w = D \cap (r)^\perp$ of the induced chamber $D := \iota_{\mathcal{P}}^{-1}(C(\mathbf{w}))$, where ι is not of type `infty`. Then `wrec` has the following contents. See the proof of Proposition 2.7 of [1] for notation.

- `wrec.r` is the (-2)-vector r of L_{10} that defines the wall $w = D \cap (r)^\perp$.
- `wrec.rlifts` is the list of (-2)-vectors $\tilde{r}$ of L_{26} such that $\langle \mathbf{w}, \tilde{r} \rangle_{26} = 1$ and $\iota_{\mathcal{P}}^{-1}((\tilde{r})^\perp) = (r)^\perp$.
- `wrec.embQ` is the 17×26 integer matrix M such that $v \mapsto vM$ is the embedding $Q \hookrightarrow L_{26}$ under a certain fixed basis of Q.
- `wrec.GramQ` is the Gram matrix of Q with respect to the fixed basis of Q.
- `wrec.embQperp` is the 9×26 integer matrix M such that $v \mapsto vM$ is the embedding $Q^\perp \hookrightarrow L_{26}$ under a certain fixed basis of $Q^\perp$.
- `wrec.GramQperp` is the Gram matrix of $Q^\perp$ with respect to the fixed basis of $Q^\perp$.
- `wrec.rootsQ` is the list of (-2)-vectors of Q.
- `wrec.rootstypeQ` is the ADE-type of the set of (-2)-vectors of Q.
- `wrec.Sigma` is the list of (-2)-vectors in Σ .
- `wrec.adjweyl` is the Weyl vector $\mathbf{w}'$ corresponding to the Conway chamber $C(\mathbf{w}')$ such that $\iota_{\mathcal{P}}^{-1}(C(\mathbf{w}'))$ is the induced chamber adjacent to D across the wall $w = D \cap (r)^\perp$.
- `wrec.thegtilde` is the isometry $\tilde{g} \in \mathrm{O}(L_{26}, \mathcal{P})$ such that $\tilde{g}$ preserves the image of $\iota\colon \mathbf{S} \hookrightarrow L_{26}$ and that its restriction $\tilde{g}|\mathbf{S}$ to $\mathbf{S}$ maps D to the induced chamber adjacent to D across the wall $w = D \cap (r)^\perp$. The existence of this isometry proves Proposition 2.7 of [1].
- `wrec.theg` is the isometry $g = \tilde{g}|\mathbf{S}$ of L_{10}. We can check that this isometry is equal to the reflection with respect to `wrec.r`.

References

[1] Simon Brandhorst and Ichiro Shimada. Borcherds' method for Enriques surfaces, 2019. Preprint, http://www.math.sci.hiroshima-u.ac.jp/~shimada/preprints.html.

[2] The GAP Group. *GAP - Groups, Algorithms, and Programming.* Version 4.8.6; 2016 (http://www.gap-system.org).

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