A NOTE ON CONFIGURATIONS OF \((-2)\)-VECTORS ON ENRIQUES SURFACES

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1. Introduction

This note is a supplement of the joint paper [3] with S. Brandhorst. It was established by Nikulin [7], Kondo [5], and Martin [6] that Enriques surfaces in characteristic $\neq 2$ with finite automorphism group are divided into seven classes $I, II, \ldots, VII$. The configurations of smooth rational curves on these Enriques surfaces are depicted in Kondo [5] by beautiful but complicated graphs.

A lattice of rank $n$ is hyperbolic if the signature is $(1, n-1)$. A positive cone of a hyperbolic lattice $L$ is a connected component of $\{ x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0 \}$. For a positive integer $n$ with $n \equiv 2 \mod 8$, let $L_n$ denote the even unimodular hyperbolic lattice of rank $n$, which is unique up to isomorphism. Borcherds’ method [1, 2] is a method to calculate the automorphism group of an even hyperbolic lattice $S$ by embedding $S$ into $L_{26}$ primitively and using the tessellation of a positive cone of $L_{26}$ by Conway chambers. (See Chapter 27 of [4]. See [3] for the definition of Conway chambers.) This method has been applied to lattices $S_X$ of numerical equivalence classes of divisors of various $K3$ surfaces $X$, and the automorphism group of these $K3$ surfaces are calculated.

The lattice $S_Y$ of numerical equivalence classes of divisors of an Enriques surface $Y$ is isomorphic to $L_{10}$. The universal covering $X \to Y$ of $Y$ by a $K3$ surface $X$ induces a primitive embedding $S_Y(2) \hookrightarrow S_X$, where $S_Y(2)$ is the lattice obtained from $S_Y$ by multiplying the intersection form $\langle , \rangle$ by 2. If $S_X$ is embedded primitively into $L_{26}$ in Borcherds’ method, then $S_Y(2)$ is also embedded primitively into $L_{26}$. In [3], hoping to apply Borcherds’ method to Enriques surfaces systematically, we have classified all primitive embeddings of $L_{10}(2)$ into $L_{26}$. It turns out that there exist exactly 17 primitive embeddings

12A, 12B, 20A, \ldots, 20F, 40A, \ldots, 40E, 96A, 96B, 96C, infty

up to the action of the orthogonal groups of $L_{10}$ and $L_{26}$. Let $P_{10}$ be a positive cone of $L_{10}$. For each of these primitive embeddings except for the type infty, we obtain a finite polyhedral cone in $P_{10}$ bounded by hyperplanes perpendicular to $(-2)$-vectors in $L_{10}$ such that $P_{10}$ is tessellated by the image of reflections of this finite polyhedral cone with respect to the walls. The set of walls of this finite polyhedral cone defines a configuration of $(-2)$-vectors of $L_{10}$. The 7 configurations I, II, \ldots, VII of Nikulin-Kondo appear among these 16 configurations.

In this note, we give a combinatorial description for each of these configurations. The result includes new descriptions of the Nikulin-Kondo configurations, which we hope are handier than the picturesque graphs of [5] in some situations.

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An explicit computational data is available at [10]. We used GAP [11] for the calculation.

**Conventions.** (1) A configuration is a pair \((\Gamma, \mu)\) of a finite set \(\Gamma\) and a mapping \(\mu: \Gamma \times \Gamma \to \mathbb{Z}\) such that \(\mu(x, y) = \mu(y, x)\) for all \(x, y \in \Gamma\). In this note, we always assume that

\[ \mu(x, x) = -2 \quad \text{for all} \quad x \in \Gamma. \]  

The automorphism group of a configuration \((\Gamma, \mu)\) is the group of permutations of \(\Gamma\) that preserve \(\mu\). The size of a configuration \((\Gamma, \mu)\) is \(|\Gamma|\).

(2) The cyclic group of order \(n\) is denoted by \(C_n\). The symmetric group of degree \(n\) is denoted by \(S_n\), and the alternating group of degree \(n\) is denoted by \(A_n\). Let \(I_n\) denote the identity matrix of size \(n\). Let \(1_n\) and \(0_n\) be the square matrix of size \(n\) whose components are all 1 and all 0, respectively.

2. **Combinatorial descriptions**

2.1. **12A.** The configuration of type 12A is the configuration of Nikulin-Kondo type I (Fig. 1.4 of [5]). The automorphism group is isomorphic to \(C_2 \times C_2\).

2.2. **12B.** The configuration of type 12B is the configuration of Nikulin-Kondo type II (Fig. 2.4 of [5]). The automorphism group is isomorphic to \(C_2 \times S_4\).

2.3. **20A.** The configuration of type 20A is isomorphic to the configuration of Nikulin-Kondo type V (Fig. 5.5 of [5]).

Let \(A\) be the set \{1, 2, 3, 4\}, and \(B\) the set of subsets \{\(i, j\)\} of \(A\) with size 2. Let \(A_1\) and \(A_2\) be two copies of \(A\) with the natural bijection to \(A\) denoted by \(a \mapsto \bar{a}\). Let \(B_1\) and \(B_2\) be two copies of \(B\) with the natural bijection to \(B\) denoted by \(b \mapsto \bar{b}\). We then put

\[ \Gamma := A_1 \sqcup A_2 \sqcup B_1 \sqcup B_2, \]

and define a symmetric function \(\mu: \Gamma \times \Gamma \to \mathbb{Z}\) satisfying (1.1) as follows.

- Suppose that \(a, a' \in A_1\) with \(a \neq a'\). Then \(\mu(a, a') = 0\).
- Suppose that \(a \in A_1\) and \(a' \in A_2\). Then
  \[ \mu(a, a') = \begin{cases} 2 & \text{if } \bar{a} = \bar{a'}, \\ 0 & \text{otherwise}. \end{cases} \]
- Suppose that \(a, a' \in A_2\) with \(a \neq a'\). Then \(\mu(a, a') = 2\).
- Suppose that \(a \in A_1\) and \(b \in B_1\). Then \(\mu(a, b) = 0\).
- Suppose that \(a \in A_1\) and \(b \in B_2\). Then
  \[ \mu(a, b) = \begin{cases} 1 & \text{if } \bar{a} \in \bar{b}, \\ 0 & \text{otherwise}. \end{cases} \]
- Suppose that \(a \in A_2\) and \(b \in B_1\). Then
  \[ \mu(a, b) = \begin{cases} 2 & \text{if } \bar{a} \in \bar{b}, \\ 0 & \text{otherwise}. \end{cases} \]
- Suppose that \(a \in A_2\) and \(b \in B_2\). Then \(\mu(a, b) = 0\).
- Suppose that \(b, b' \in B_1\) with \(b \neq b'\). Then
  \[ \mu(a, b) = \begin{cases} 2 & \text{if } \bar{b} \cap \bar{b'} = \emptyset, \\ 1 & \text{otherwise}. \end{cases} \]
Suppose that $b \in B_1$ and $b' \in B_2$. Then
\[
\mu(a, b) = \begin{cases} 
2 & \text{if } \bar{b} \cap \bar{b}' = \emptyset, \\
0 & \text{otherwise.}
\end{cases}
\]

Suppose that $b, b' \in B_2$ with $b \neq b'$. Then $\mu(b, b') = 0$.

Then $(\Gamma, \mu)$ defines the configuration of type $20A$.

**Remark 2.1.** The automorphism group of $(\Gamma, \mu)$ is isomorphic to $\mathfrak{S}_4$, acting naturally on $A$.

2.4. $20B$. The configuration of type $20B$ is isomorphic to the configuration of Nikulin-Kondo type III (Fig. 3.5 of [5]).

We put $P := \{1, 2, 3, 4\}$. Let $Q_1$ and $Q_2$ be quadrangles. For $i = 1, 2$, let $VQ_i$ be the set of vertices of $Q_i$, and let $EQ_i$ be the set of edges of $Q_i$. Let $EQ_i = \{a_i, a_i'\} \cup \{b_i, b_i'\}$ be the decomposition such that $a_i$ and $a_i'$ (resp. $b_i$ and $b_i'$) have no common vertex. We then put
\[
\Gamma := P \sqcup VQ_1 \sqcup VQ_2 \sqcup EQ_1 \sqcup EQ_2,
\]
and define a symmetric function $\mu: \Gamma \times \Gamma \to \mathbb{Z}$ satisfying (1.1) as follows.

- Suppose that $p_1, p_2 \in P$ with $p_1 \neq p_2$. Then $\mu(p_1, p_2) = 0$.
- Suppose that $p \in P$ and $v \in VQ_1 \sqcup VQ_2$. Then $\mu(p, v) = 0$.
- Suppose that $p \in P$ and $e_1 \in EQ_1$. Then
  \[
  \mu(p, e_1) = \begin{cases} 
1 & \text{if } (p \in \{1, 2\} \text{ and } e_1 \in \{a_1, a_1'\}) \text{ or } (p \in \{3, 4\} \text{ and } e_1 \in \{b_1, b_1'\}), \\
0 & \text{otherwise.}
\end{cases}
\]
- Suppose that $p \in P$ and $e_2 \in EQ_2$. Then
  \[
  \mu(p, e_2) = \begin{cases} 
1 & \text{if } (p \in \{1, 3\} \text{ and } e_2 \in \{a_2, a_2'\}) \text{ or } (p \in \{2, 4\} \text{ and } e_2 \in \{b_2, b_2'\}), \\
0 & \text{otherwise.}
\end{cases}
\]
- Suppose that $v_1, v_2 \in VQ_1 \sqcup VQ_2$ with $v_1 \neq v_2$. Then
  \[
  \mu(v_1, v_2) = \begin{cases} 
0 & \text{if } v_1 \text{ and } v_2 \text{ are the end-points of an edge,} \\
2 & \text{otherwise.}
\end{cases}
\]
- Suppose that $v \in VQ_1 \sqcup VQ_2$ and $e \in EQ_1 \sqcup EQ_2$. Then
  \[
  \mu(v, e) = \begin{cases} 
2 & \text{if } v \text{ is an end-point of } e, \\
0 & \text{otherwise.}
\end{cases}
\]
- Suppose that $e_1, e_2 \in EQ_1 \sqcup EQ_2$ with $e_1 \neq e_2$. Then $\mu(e_1, e_2) = 0$.

Then $(\Gamma, \mu)$ defines the configuration of type $20B$.

**Remark 2.2.** The automorphism group of $(\Gamma, \mu)$ is the group of the automorphism of the disjoint union $Q_1 \sqcup Q_2$ of two quadrangles, that is, $D_8^2 \rtimes C_2$. 
2.5. **20C and 20D.** The configurations of type **20C** and of type **20D** are isomorphic, and they are isomorphic to the configuration of Nikulin-Kondo type VII (Fig. 7.7 of [5]).

Let \( A = \{1, \ldots, 5\} \), and let \( B \) be the set of non-ordered pairs \( \{(ij), (kl)\} \) of disjoint subsets \( (ij) = \{i, j\} \) and \( (kl) = \{k, l\} \) of \( A \) with size 2. For \( b = \{(ij), (kl)\} \in B \), let \( \bar{b} \in A \) denote the unique element of \( A \) that is not contained in \( (ij) \cup (kl) \).

We then put 
\[
\Gamma := A \sqcup B,
\]
and define a symmetric function \( \mu : \Gamma \times \Gamma \to \mathbb{Z} \) satisfying (1.1) as follows.

- Suppose that \( a, a' \in A \) with \( a \neq a' \). Then we have \( \mu(a, a') = 2 \).
- Suppose that \( a \in A \) and \( b \in B \). Then we have
  \[
  \mu(a, b) := \begin{cases} 
  2 & \text{if } a = \bar{b}, \\
  0 & \text{otherwise}. 
  \end{cases}
  \]

- Suppose that \( b, b' \in B \) with \( b \neq b' \). Then we have
  \[
  \mu(b, b') := \begin{cases} 
  1 & \text{if } b \cap b' \neq \emptyset, \\
  0 & \text{otherwise}. 
  \end{cases}
  \]

Then \((\Gamma, \mu)\) defines the configurations of type **20C** and type **20D**.

**Remark 2.3.** The automorphism group of \((\Gamma, \mu)\) is isomorphic to \( S_5 \).

2.6. **20E.** The configuration of type **20E** is isomorphic to the configuration of Nikulin-Kondo type VI (Fig. 6.4 of [5]). The description below of this configuration was obtained in [9].

Let \( A \) be the set of subsets of \( \{1, \ldots, 5\} \) with size 3. Let \( A_1 \) and \( A_2 \) be two copies of \( A \) with the natural bijection to \( A \) denoted by \( a \mapsto \bar{a} \). We then put
\[
\Gamma := A_1 \sqcup A_2,
\]
and define a symmetric function \( \mu : \Gamma \times \Gamma \to \mathbb{Z} \) satisfying (1.1) as follows.

- Suppose that \( a, a' \in A_1 \) with \( a \neq a' \). Then
  \[
  \mu(a, a') = \begin{cases} 
  1 & \text{if } |a \cap a'| = 1, \\
  0 & \text{otherwise}. 
  \end{cases}
  \]

- Suppose that \( a, a' \in A_2 \) with \( a \neq a' \). Then
  \[
  \mu(a, a') = \begin{cases} 
  1 & \text{if } |a \cap a'| = 2, \\
  0 & \text{otherwise}. 
  \end{cases}
  \]

- Suppose that \( a \in A_1 \) and \( a' \in A_2 \). Then
  \[
  \mu(a, a') = \begin{cases} 
  2 & \text{if } \bar{a} = \bar{a}', \\
  0 & \text{otherwise}. 
  \end{cases}
  \]

Then \((\Gamma, \mu)\) defines the configuration of type **20E**.

**Remark 2.4.** The sub-configuration \((A_1, \mu|_{A_1})\) is isomorphic to the Petersen graph, and the sub-configuration \((A_2, \mu|_{A_2})\) is isomorphic to the complement of the Petersen graph. The automorphism group of \((\Gamma, \mu)\) is equal to the automorphism group of the Petersen graph, which is isomorphic to \( S_5 \).
2.7. 20F. The configuration of type 20F is isomorphic to the configuration of Nikulin-Kondo type IV (Fig. 4.4 of [5]). The description below of this configuration was obtained in [8].

Let $\Gamma$ be the set of vertices of the Petersen graph $P$, and let $\Gamma$ be the set with 20 vertices with a map $\rho: \Gamma \to \Gamma$ such that $|\rho^{-1}(\bar{v})| = 2$ for every $\bar{v} \in \Gamma$. We fix a numbering $v_1, v_2$ of the elements in each fiber $\rho^{-1}(\bar{v}) = \{v_1, v_2\}$ of $\rho$. We then define a symmetric function $\mu: \Gamma \times \Gamma \to \mathbb{Z}$ satisfying (1.1) as follows.

- We have $\mu(v, v') = 0$ if $\rho(v) = \rho(v')$.
- We have $\mu(v, v') = 0$ if $\rho(v)$ and $\rho(v')$ are not connected in $P$.
- We have $\mu(v, v') = 1$ if $\rho(v)$ and $\rho(v')$ are connected by a thin line in Figure 2.1.
- Suppose that $\bar{v}$ and $\bar{v}'$ are connected by a thick line in Figure 2.1. Let $\rho^{-1}(\bar{v}) = \{v_1, v_2\}$ and $\rho^{-1}(\bar{v}') = \{v'_1, v'_2\}$ be the fibers with the fixed numberings. Then

$$\mu(v_i, v'_j) = \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Then the isomorphism class of the configuration $(\Gamma, \mu)$ does not depend on the choice of numberings of two elements in fibers of $\rho$, and $(\Gamma, \mu)$ defines the configuration of type 20F.

Remark 2.5. The group $\text{Aut}(\Gamma, \mu)$ is of order 640. The action of $\text{Aut}(\Gamma, \mu)$ on $\Gamma$ preserves the fibers of $\rho: \Gamma \to \Gamma$, and we have a natural homomorphism from $\text{Aut}(\Gamma, \mu)$ to the automorphism group $\text{Aut}(P)$ of the Petersen graph, which is isomorphic to $S_5$. Thus we obtain an exact sequence

$$0 \longrightarrow C_2^5 \longrightarrow \text{Aut}(\Gamma, \mu) \longrightarrow G_{20} \longrightarrow 1,$$

where $G_{20}$ is the subgroup of $\text{Aut}(P) \cong S_5$ consisting of elements that preserve the thick edges in Figure 2.1. As a subgroup of $S_5$, the group $G_{20}$ is conjugate to the subgroup generated by $(12345)$ and $(2354)$.

2.8. 40A. Let $C_+$ and $C_-$ be two copies of the cubes $I^3 \subset \mathbb{R}^3$, where $I \subset \mathbb{R}$ is the unit interval. Let $\varepsilon$ be $+$ or $\varepsilon$. A vertex of $C_+$ is written as $(a_x, a_y, a_z, \varepsilon)$, where $a_x, a_y, a_z \in \{0, 1\}$, and a face of $C_+$ is written as $(w = a, \varepsilon)$, where $w \in \{x, y, z\}$ and $a \in \{0, 1\}$. Let $V$ be the set of vertices of $C_\pm$, and let $F$ be the set of faces of $C_\pm$. 
Let $P$ be the set of pairs of a face $f_+ = (w = a_+) \in \mathcal{C}_+$ and a face $f_− = (w = a_-) \in \mathcal{C}_-$ that are parallel. Each element of $P$ is written as \((w = a_+, w = a_-)\), where \(w \in \{x, y, z\}\) and \(a_\pm \in \{0, 1\}\). We have \(|V| = 16, |F| = 12, |P| = 12\). We put
\[
\Gamma := V \cup F \cup P,
\]
and define a symmetric function \(\mu : \Gamma \times \Gamma \rightarrow \mathbb{Z}\) satisfying (1.1) as follows.

- Suppose that \(v_1, v_2 \in V\) with \(v_1 \neq v_2\). Then
  \[
  \mu(v_1, v_2) = \begin{cases} 
  0 & \text{if } v_1v_2 \text{ is an edge of } \mathcal{C}_+ \text{ or } \mathcal{C}_-, \\
  4 & \text{if } v_1v_2 \text{ is a diagonal of } \mathcal{C}_+ \text{ or } \mathcal{C}_-, \\
  2 & \text{otherwise.}
  \end{cases}
  \]

- Suppose that \(v \in V\) and \(f \in F\). Then
  \[
  \mu(v, f) = \begin{cases} 
  2 & \text{if } v \in f, \\
  0 & \text{otherwise.}
  \end{cases}
  \]

- Suppose that \(v \in V\) and \(p = (f_+, f_-) \in P\). Then
  \[
  \mu(v, p) = \begin{cases} 
  2 & \text{if } v \in f_+ \cup f_-, \\
  0 & \text{otherwise.}
  \end{cases}
  \]

- Suppose that \(f_1, f_2 \in F\) with \(f_1 \neq f_2\). Let \(f_i\) be \((w_i = a_i, \varepsilon_i)\), where \(w_i \in \{x, y, z\}, a_i \in \{0, 1\}\), and \(\varepsilon_i \in \{+, −\}\). Then
  \[
  \mu(f_1, f_2) = \begin{cases} 
  1 & \text{if } \varepsilon_1 \neq \varepsilon_2 \text{ and } w_1 \neq w_2, \\
  0 & \text{otherwise.}
  \end{cases}
  \]

- Suppose that \(f = (w = a, \varepsilon) \in F\) and \(p = (f'_+, f'_-) \in P\). Let \(\bar{f}\) be the unique face of \(\mathcal{C}_\varepsilon\) that is disjoint from \(f\). Then
  \[
  \mu(f, p) = \begin{cases} 
  2 & \text{if } \bar{f} = f'_+ \text{ or } \bar{f} = f'_-, \\
  0 & \text{otherwise.}
  \end{cases}
  \]

- Suppose that \(p_1, p_2 \in P\) with \(p_1 \neq p_2\). Let \(\text{faces}(p_i)\) denote the set of 2 faces contained in \(p_i\), and let \(\text{verts}(p_i)\) denote the set of 8 vertices contained in the two faces of \(p_i\).
  \[
  \mu(p_1, p_2) = \begin{cases} 
  2 & \text{if } \text{verts}(p_1) \cap \text{verts}(p_2) = \emptyset, \\
  0 & \text{if } \text{faces}(p_1) \cap \text{faces}(p_2) \neq \emptyset, \\
  1 & \text{otherwise.}
  \end{cases}
  \]

Then \((\Gamma, \mu)\) defines the configuration of type 40A.

**Remark 2.6.** The automorphism group \(\text{Aut}(\Gamma, \mu)\) is of order 768, and \(V, F, P\) are the orbits of the action on \(\Gamma\). Let \(V_+\) and \(V_-\) be the set of vertices of \(\mathcal{C}_+\) and \(\mathcal{C}_-\), regarded as graphs with edges being the edges of the cubes. The automorphism group of the graph \(V_+\) is of order 48. The stabilizer subgroup \(\text{Stab}(V_+)\) of \(V_+\) in \(\text{Aut}(\Gamma, \mu)\) is of index 2, the natural homomorphism \(\text{Stab}(V_+) \rightarrow \text{Aut}(V_+)\) is surjective, and its kernel is isomorphic to \(\mathcal{C}_2^3\) acting on \(V_-\) as \(((a_x, a_y, a_z), −) \mapsto ((±a_x, ±a_y, ±a_z), −)\).
2.9. 40B and 40C. The configurations of type 40B and of 40C are isomorphic.

We put $F := \{1, 2, 3, 4\}$. Let $P$ be the set $F \times F$ with the projections $pr_1 : P \to F$ and $pr_2 : P \to F$. Let $B$ be the set of bijections $f : F \to F$. We put

$$\Gamma := P \sqcup B,$$

and define a symmetric function $\mu : \Gamma \times \Gamma \to \mathbb{Z}$ satisfying (1.1) as follows.

- Suppose that $p, p' \in P$ with $p \neq p'$. Then
  $$\mu(p, p') = \begin{cases} 1 & \text{if } pr_1(p) = pr_1(p') \text{ or } pr_2(p) = pr_2(p'), \\ 0 & \text{otherwise}. \end{cases}$$

- Suppose that $p \in P$ and $f \in B$. Then
  $$\mu(p, f) = \begin{cases} 2 & \text{if } f(pr_1(p)) = pr_2(p), \\ 0 & \text{otherwise}. \end{cases}$$

- Suppose that $f, f' \in B$ with $f \neq f'$. Then $\gamma := ff'^{-1}$ is a permutation of $F$. Let $\tau(\gamma)$ denote the lengths of cycles in the cycle decomposition of $\gamma \in \mathfrak{S}_4$. Then
  $$\mu(f, f') = \begin{cases} 2 & \text{if } \tau(\gamma) = 4, \\ 2 & \text{if } \tau(\gamma) = 2 + 2, \\ 1 & \text{if } \tau(\gamma) = 3 + 1, \\ 0 & \text{if } \tau(\gamma) = 2 + 1 + 1. \end{cases}$$

Then $(\Gamma, \mu)$ defines the configurations of type 40B and 40C.

Remark 2.7. The group $\text{Aut}(\Gamma, \mu)$ is isomorphic to $(\mathfrak{S}_4 \times \mathfrak{S}_4) \rtimes C_2$, which acts on $P$ in the natural way.

2.10. 40D and 40E. The configurations of type 40D and of 40E are isomorphic.

A subset $(ij) := \{i, j\}$ of size 2 of $\{1, \ldots, 6\}$ is called a duad, and a subset $(ijk) := \{i, j, k\}$ of size 3 of $\{1, \ldots, 6\}$ is called a trio. A syntheme is a non-ordered set $(ij)(kl)(mn) := \{(ij), (kl), (mn)\}$ of 3 duads such that $\{i, j, k, l, m, n\} = \{1, \ldots, 6\}$. A double trio is a non-ordered pair $(ijk)(lmn) := \{(ijk), (lmn)\}$ of complementary trios. Let $D, S$ and $T$ be the set of duads, synthemes, and double trios, respectively. We have $|D| = 15$, $|S| = 15$, and $|T| = 10$. We then put

$$\Gamma := D \sqcup S \sqcup T,$$

and define a symmetric function $\mu : \Gamma \times \Gamma \to \mathbb{Z}$ satisfying (1.1) as follows.

- Suppose that $\delta_1, \delta_2 \in D$ with $\delta_1 \neq \delta_2$. Then
  $$\mu(\delta_1, \delta_2) = \begin{cases} 1 & \text{if } |\delta_1 \cap \delta_2| = 1, \\ 0 & \text{if } |\delta_1 \cap \delta_2| = 0. \end{cases}$$

- Suppose that $\delta \in D$ ans $\sigma \in S$. Then
  $$\mu(\delta, \sigma) = \begin{cases} 2 & \text{if } \delta \in \sigma, \\ 0 & \text{if } \delta \notin \sigma. \end{cases}$$

- Suppose that $\delta \in D$ ans $\tau = \{t_1, t_2\} \in T$, where $t_1$ and $t_2$ are trios. Then
  $$\mu(\delta, \tau) = \begin{cases} 2 & \text{if } \delta \subset t_1 \text{ or } \delta \subset t_2, \\ 0 & \text{otherwise}. \end{cases}$$
• Suppose that \( \sigma_1, \sigma_2 \in S \) with \( \sigma_1 \neq \sigma_2 \). Then
\[
\mu(\sigma_1, \sigma_2) = \begin{cases} 
1 & \text{if } \sigma_1 \cap \sigma_2 = \emptyset, \\
0 & \text{otherwise.}
\end{cases}
\]

• Suppose that \( \sigma \in S \) and \( \tau \in T \). Then
\[
\mu(\sigma, \tau) = \begin{cases} 
2 & \text{if } |\delta \cap t| = 1 \text{ for any duad } \delta \in \sigma \text{ and any trio } t \in \tau, \\
0 & \text{otherwise.}
\end{cases}
\]

• Suppose that \( \tau_1, \tau_2 \in T \) with \( \tau_1 \neq \tau_2 \). Then \( \mu(\tau_1, \tau_2) = 2 \).

Then \((\Gamma, \mu)\) defines the configurations of type 40D and 40E.

Remark 2.8. By construction, the symmetric group \( S_6 \) acts on \((\Gamma, \mu)\), and \( D, S, T \) are the orbits. The full automorphism group of the configuration \((\Gamma, \mu)\) is isomorphic to the automorphism group \( \text{Aut}(S_6) \) of the alternating group \( S_6 \). The group \( \text{Aut}(S_6) \) contains \( A_6 \) as a normal subgroup of index 4 such that \( \text{Aut}(S_6)/A_6 \) is isomorphic to \( C_2^2 \), and contains \( S_6, \text{PGL}_2(9) \) and \( M_{10} \) as subgroups of index 2. (See, for example, Section 1.5, Chapter 10 of [4].) We can construct \( \text{Aut}(S_6) \) from \( S_6 \) by adding an automorphism \( \theta \) that induces the non-trivial outer automorphism of \( S_6 \). Correspondingly, the action of \( \text{Aut}(S_6) \) on \((\Gamma, \mu)\) fuses the duads \( D \) and the synthemes \( S \), and decomposes \( \Gamma \) into two orbits \( D \sqcup S \) and \( T \).

2.11. 96A. Recall that \( 0_n \) is the \( n \times n \) zero matrix, and \( 1_n \) is the \( n \times n \) matrix with all components 1. We consider the matrix
\[
\Sigma_{16} := \begin{bmatrix}
-2I_4 & 1_4 & 2I_4 & 0_4 \\
1_4 & -2I_4 & 0_4 & 2I_4 \\
2I_4 & 0_4 & -2I_4 & 1_4 \\
0_4 & 2I_4 & 1_4 & -2I_4
\end{bmatrix}.
\]

We put
\[
d := \begin{bmatrix} -2 & 2 \\
2 & -2 \end{bmatrix}, \quad t_+ := \begin{bmatrix} 2 & 0 \\
0 & 2 \end{bmatrix}, \quad t_- := \begin{bmatrix} 0 & 2 \\
2 & 0 \end{bmatrix},
\]

and
\[
D_8 := \begin{bmatrix}
D_8 & T_8 & 1_8 & 0_8 \\
T_8 & D_8 & 0_8 & 1_8 \\
1_8 & 0_8 & D_8 & T_8 \\
0_8 & 1_8 & T_8 & D_8
\end{bmatrix}.
\]

We then consider the matrix
\[
\Sigma_{32} := \begin{bmatrix}
D_8 & T_8 & 1_8 & 0_8 \\
T_8 & D_8 & 0_8 & 1_8 \\
1_8 & 0_8 & D_8 & T_8 \\
0_8 & 1_8 & T_8 & D_8
\end{bmatrix}.
\]

For \( k = 16 \) and \( k = 32 \), let \((\Gamma_k, \mu_k)\) be the configuration of size \( k \) with the symmetric bilinear form \( \mu_k : \Gamma_k \times \Gamma_k \to \mathbb{Z} \) given by the matrix \( \Sigma_k \) defined above. Then there exist exactly 64 sub-configurations \((\Gamma', \mu_{32})^\prime \) of \((\Gamma_{32}, \mu_{32})\) with \( \Gamma' \subset \Gamma_{32} \) that are isomorphic to \((\Gamma_{16}, \mu_{16})\). We denote by \( \Gamma_{64} \) the set of sub-configurations.
of \((\Gamma_{32}, \mu_{32})\) isomorphic to \((\Gamma_{16}, \mu_{16})\), and define \(\mu_{64}: \Gamma_{64} \times \Gamma_{64} \to \mathbb{Z}\) by

\[
\mu_{64}(\Gamma', \Gamma'') := \begin{cases} 
6 & \text{if } |\Gamma' \cap \Gamma''| = 0, \\
4 & \text{if } |\Gamma' \cap \Gamma''| = 4, \\
2 & \text{if } |\Gamma' \cap \Gamma''| = 8, \\
0 & \text{if } |\Gamma' \cap \Gamma''| = 12, \\
-2 & \text{if } |\Gamma' \cap \Gamma''| = 16.
\end{cases}
\]

We then put \(\Gamma := \Gamma_{32} \cup \Gamma_{64}\), and define a symmetric function \(\mu: \Gamma \times \Gamma \to \mathbb{Z}\) satisfying (1.1) as follows.

- Suppose that \(v, v' \in \Gamma_{32}\). Then \(\mu(v, v') := \mu_{32}(v, v')\).
- Suppose that \(\Gamma', \Gamma'' \in \Gamma_{64}\). Then \(\mu(\Gamma', \Gamma'') := \mu_{64}(\Gamma', \Gamma'')\).
- Suppose that \(v \in \Gamma_{32}\) and \(\Gamma' \in \Gamma_{64}\). Then \(\mu(v, \Gamma') := \begin{cases} 
2 & \text{if } v \in \Gamma', \\
0 & \text{otherwise}.
\end{cases}\)

Then \((\Gamma, \mu)\) defines the configuration of type 96A.

**Remark 2.9.** The order of the automorphism group of \((\Gamma, \mu)\) is 147456. The natural homomorphism \(\text{Aut}(\Gamma_{32}, \mu_{32}) \to \text{Aut}(\Gamma, \mu)\) is an isomorphism. The set \(\Gamma_{32}\) is regarded as the indexes \(\{1, \ldots, 32\}\) of row vectors of the matrix \(\Sigma_{32}\). We have a decomposition

\(\Gamma_{32} = o_1 \sqcup \cdots \sqcup o_4,\) \(o_i := \{8(i-1) + 1, \ldots, 8(i-1) + 8\}\).

The action of \(\text{Aut}(\Gamma_{32}, \mu_{32})\) on \(\Gamma_{32}\) preserves this decomposition, and hence we have a homomorphism

\(\pi: \text{Aut}(\Gamma_{32}, \mu_{32}) \to \mathfrak{S}_4\)

to the permutation group of \(o_1, \ldots, o_4\). The image is isomorphic to \(C_2^2\). Each \(o_i\) is equipped with a structure of the configuration by \(\mu_{32}|o_i: o_i \times o_i \to \mathbb{Z}\), or equivalently, by the matrix \(D_{8}\). The automorphism group \(\text{Aut}(o_i)\) of this configuration \((o_i, \mu_{32}|o_i)\) is isomorphic to \(C_2^2 \times \mathfrak{S}_4\). Let \(G_{192}\) denote the subgroup \(\text{Aut}(o_i) \cap \mathfrak{A}_8\) of \(\text{Aut}(o_i)\), where the intersection is taken in the full permutation group \(\mathfrak{S}_8\) of \(o_i\). Then the natural homomorphism

\(\text{Ker}(\pi) \to \text{Aut}(o_1) \times \text{Aut}(o_3)\)
is injective, and the image is equal to \(G_{192} \times G_{192}\). Thus we have an exact sequence

\(1 \to G_{192} \times G_{192} \to \text{Aut}(\Gamma, \mu) \to C_2^2 \to 0.\)

**2.12. 96B and 96C.** The configurations of type 96B and of 96C are isomorphic.

We put

\(m := \begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix},\) \(t_+ := \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},\) \(t_- := \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.\)

We then define an \(8 \times 8\) matrix \(D\) by

\[
D := \begin{bmatrix} m & t_+ & t_+ & t_+ \\ t_+ & m & t_+ & t_+ \\ t_+ & t_+ & m & t_+ \\ t_+ & t_+ & t_+ & m \end{bmatrix}.
\]
and a $24 \times 24$ matrix $T$ by

\begin{equation}
T := \begin{bmatrix}
D & 1_8 & 1_8 \\
1_8 & D & 1_8 \\
1_8 & 1_8 & D
\end{bmatrix}.
\end{equation}

Let $S$ be the set of 18 square matrices $S_1, \ldots, S_{18}$ of size 4 with components in \{+, −\} obtained from $S_1$ in Table 2.1 by permuting rows and columns. For a $3 \times 3$ matrix

\[
\nu := \begin{bmatrix}
i_{11} & i_{12} & i_{13} \\
i_{21} & i_{22} & i_{23} \\
i_{31} & i_{32} & i_{33}
\end{bmatrix}
\]

with components $i_{αβ}$ in $\{1, \ldots, 18\}$, let $S[\nu]$ denote the $24 \times 24$ matrix obtained from $\nu$ by first replacing each $i_{αβ}$ with the member $S_{i_{αβ}}$ of $S$ indexed by $i_{αβ}$ and then replacing $+$ with $t_+$ and $-$ with $t_-$. We put

\[
\nu_1 := \begin{bmatrix} 9 & 8 & 7 \\ 6 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}, \quad \nu_2 := \begin{bmatrix} 5 & 9 & 7 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \quad \nu_3 := \begin{bmatrix} 4 & 8 & 9 \\ 9 & 5 & 4 \\ 2 & 1 & 3 \end{bmatrix},
\]

\[
\nu_4 := \begin{bmatrix} 8 & 9 & 1 \\ 6 & 1 & 5 \\ 1 & 5 & 9 \end{bmatrix}, \quad \nu_5 := \begin{bmatrix} 8 & 1 & 9 \\ 1 & 9 & 5 \\ 6 & 5 & 1 \end{bmatrix}, \quad \nu_6 := \begin{bmatrix} 9 & 8 & 1 \\ 1 & 5 & 9 \\ 6 & 1 & 5 \end{bmatrix}.
\]

Then the $96 \times 96$ symmetric matrix

\begin{equation}
\begin{bmatrix}
T & S[\nu_1] & S[\nu_2] & S[\nu_3] \\
T & S[\nu_4] & S[\nu_5] \\
T & S[\nu_6] \\
T
\end{bmatrix}
\end{equation}

defines the configurations of type 96B and 96C.
Remark 2.10. The group $\Sigma_4$ acts on $\mathcal{S}$ as $S \mapsto \sigma S$ for $S \in \mathcal{S}$ and $\sigma \in \Sigma_4$, where $\sigma S$ is obtained from $S$ by permuting rows of $S$ by $\sigma$. Let $G_{\text{row}}$ be the subgroup of the full permutation group $\Sigma(\mathcal{S})$ of $\mathcal{S}$ generated by the action of $\Sigma_4$ on rows and the flipping $+ \leftrightarrow -$. Then $|G_{\text{row}}| = 48$, and $\mathcal{S}$ is decomposed by $G_{\text{row}}$ into 3 orbits, each of which is of size 6. Similarly, we define $G_{\text{col}}$ to be the subgroup of $\Sigma(\mathcal{S})$ generated by the action of $\Sigma_4$ on columns and the flipping. Then $|G_{\text{col}}| = 48$ and $\mathcal{S}$ is decomposed by $G_{\text{col}}$ into 3 orbits of size 6. The intersection of any orbit of $G_{\text{row}}$ and any orbit of $G_{\text{col}}$ consists of two matrices that are interchanged by the flipping.

Let $\mathcal{M}$ be the set of $3 \times 3$ matrices with components in the set $\{1, \ldots, 18\}$ of indexes of $\mathcal{S}$. The groups $G_{\text{row}}$ and $G_{\text{col}}$ act on $\{1, \ldots, 18\}$ as described in the previous paragraph. Let $\mathcal{G}$ be the subgroup of the full permutation group of $\mathcal{M}$ generated by the following permutations:

- the permutations of 3 rows,
- choosing a row and making an element of $G_{\text{row}}$ act on the 3 components of the row,
- the permutations of 3 columns, and
- choosing a column and making an element of $G_{\text{col}}$ act on the 3 components of the column.

Then we confirm that there exists one and only one orbit $O$ of the action of $\mathcal{G}$ on $\mathcal{M}$ with the following property: for every $\nu \in O$, each row of $\nu$ consists of 3 distinct elements, and each column of $\nu$ consists of 3 distinct elements. We have $|O| = 23887872$.

The 6 matrices $\nu_1, \ldots, \nu_6$ above belong to this orbit $O$. We tried to characterize the 6-tuple $\nu_1, \ldots, \nu_6$ of elements of $O$ combinatorially, but we could not find a nice description.

Remark 2.11. The automorphism group of $(\Gamma, \mu)$ is of order 221184. The set $\Gamma$ is decomposed into 48 pairs $\{v, v'\}$ with $\mu(v, v') = 4$. Let $P_{48}$ be the set of these pairs. The kernel of the natural homomorphism

$$\pi: \text{Aut}(\Gamma, \mu) \to \Sigma(P_{48})$$

is isomorphic to $C_2$. The set $P_{48}$ is decomposed into the disjoint union of 4 subsets $t_1, \ldots, t_4$ of size 12, each of which corresponds to the diagonal block $T$ of the matrix (2.3). The natural homomorphism

$$\rho: \text{Im} \pi \to \Sigma_4$$

is surjective. Hence $\text{Ker} \rho$ is of order 4608. The kernel of the natural homomorphism

$$\sigma: \text{Ker} \rho \to \Sigma(t_1)$$

is isomorphic to $C_2^6$, and hence $\text{Im} \sigma$ is of order 1152. The set $t_1$ is then decomposed into the disjoint union of 3 subsets $d_1, \ldots, d_3$ of size 4, each of which corresponds to the diagonal block $D$ of the matrix (2.2). The natural homomorphism

$$\tau: \text{Im} \sigma \to \Sigma_3$$

is surjective. Hence $\text{Ker} \tau$ is of order 192, which is isomorphic to $C_2^6 \cdot C_3$. 
References


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