RATIONAL DOUBLE POINTS ON ENRIQUES SURFACES

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ABSTRACT. We classify, up to some lattice-theoretic equivalence, all possible configurations of rational double points that can appear on a surface birational to a complex Enriques surface.

1. Introduction

We work over the complex number field $\mathbb{C}$.

The automorphism group of an Enriques surface changes in a complicated way under specializations of the surface ([3, Section 3.1], [24, Remark 7.17]), and smooth rational curves on the surface control the change of the automorphism group. Hence the study of configurations of smooth rational curves are important for the explicit description of variations of automorphism groups on the family of Enriques surfaces (see Nikulin [16]). On the other hand, in the investigation of $\mathbb{Q}$-homology projective planes, Hwang-Keum-Ohashi [11] and Schütt [18] classified all possible maximal root systems of smooth rational curves on Enriques surfaces.

In this paper, we classify all root systems of smooth rational curves on Enriques surfaces up to certain equivalence relation. The list we obtain (see Theorem 1.7 and Table 1.1) includes, of course, the 31 root systems of rank 9 classified in [11] and [18]. Our method is purely lattice-theoretic and algorithmic. The main tool is the generalized Borcherds algorithm ([5], [6], [22]) to calculate the orthogonal group of a hyperbolic lattice. An advantage of our method is that we can obtain the whole result by a single set of algorithms, and that we are exempted from the case-by-case investigation of possible root systems.

A lattice L of rank n is said to be *hyperbolic* if the real quadratic space $L \otimes \mathbb{R}$ is of signature $(1, n-1)$. Let L be a hyperbolic lattice. A *positive cone* of L is one of the two connected components of $\{x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0\}$. Let $\mathrm{O}^+(L)$ denote the group of isometries of L that preserve a positive cone.

A *root* of a lattice L is a vector of square-norm -2. We say that L is a *root lattice* if L is generated by roots. An ADE-*configuration of roots* of L is a finite set Φ of roots of L such that each connected component of the Dynkin diagram of Φ is of type A_l, D_m, or E_n. The ADE-*type* $\tau(\Phi)$ of an ADE-configuration Φ of roots is the ADE-type of the Dynkin diagram of Φ. It is well-known that a negative-definite root lattice R has an ADE-configuration Φ_R of roots that form a basis of R. We define the ADE-type $\tau(R)$ of R to be $\tau(\Phi_R)$. Conversely, for an ADE-type t, we have a negative-definite root lattice $R(t)$, unique up to isomorphism, such that $\tau(R(t)) = t$.

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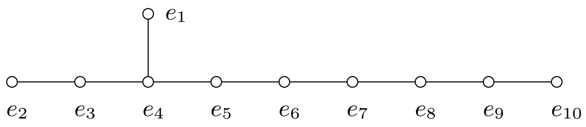

FIGURE 1.1. Dynkin diagram of the basis E_{10} of L_{10}

Let L_{10} be an even unimodular hyperbolic lattice of rank 10, which is unique up to isomorphism, and which has a basis $E_{10} := \{e_1, \dots, e_{10}\}$ consisting of roots whose Dynkin diagram is given in Figure 1.1.

For a smooth projective surface Z, we denote by S_Z the lattice of numerical equivalence classes of divisors of Z, and by $\mathcal{P}_Z$ the positive cone of the hyperbolic lattice S_Z containing an ample class. Note that, if Y is an Enriques surface, then S_Y is isomorphic to L_{10}.

An RDP-*Enriques surface* is a pair (Y, ρ) of an Enriques surface Y and a birational morphism $\rho\colon Y \to \overline{Y}$ to a surface $\overline{Y}$ that has only rational double points as its singularities. Let (Y, ρ) be an RDP-Enriques surface, and let $C_1, \dots, C_n$ be the smooth rational curves contracted by ρ. We denote by $\Phi_\rho \subset S_Y$ the set of classes of $C_1, \dots, C_n$. Then Φ_ρ is an ADE-configuration of roots of S_Y, and $\tau(\Phi_\rho)$ is the ADE-type of the rational double points on $\overline{Y}$.

Definition 1.1. Two RDP-Enriques surfaces (Y, ρ) and (Y', ρ') are said to be *equivalent* if there exists an isometry $S_Y \xrightarrow{\sim} S_{Y'}$ of lattices that maps $\mathcal{P}_Y$ to $\mathcal{P}_{Y'}$ and Φ_ρ to $\Phi_{\rho'}$ bijectively.

In this paper, we classify all equivalence classes of RDP-Enriques surfaces (Y, ρ). The ADE-type $\tau(\Phi_\rho)$ is a principal invariant of the equivalence class, but this invariant is not enough to distinguish all the equivalence classes.

Our first main theorem is the following purely lattice-theoretic result. Let Φ_f be an ADE-configuration of roots in L_{10}. We denote by R_f the root sublattice of L_{10} generated by Φ_f, and by $\overline{R}_f$ the primitive closure of R_f in L_{10}.

Definition 1.2. Two ADE-configurations of roots $\Phi_f \subset L_{10}$ and $\Phi_{f'} \subset L_{10}$ are said to be *equivalent* if there exists an isometry $g \in \mathrm{O}^+(L_{10})$ that maps Φ_f to $\Phi_{f'}$ bijectively. (See also Definition 3.1 for another formulation.)

Theorem 1.3. (1) *For any* ADE-*configuration* Φ_f *of roots in* L_{10}, *the lattice* $\overline{R}_f$ *is a root lattice. Two* ADE-*configurations of roots* $\Phi_f \subset L_{10}$ *and* $\Phi_{f'} \subset L_{10}$ *are equivalent if and only if* $\tau(\Phi_f) = \tau(\Phi_{f'})$ *and* $\tau(\overline{R}_f) = \tau(\overline{R}_{f'})$.

(2) *Let* $(t, \bar{t})$ *be a pair of* ADE-*types. Then there exists an* ADE-*configuration of roots* $\Phi_f \subset L_{10}$ *such that* $(\tau(\Phi_f), \tau(\overline{R}_f)) = (t, \bar{t})$ *if and only if the following hold;*

(i) $R(t)$ *is of rank* < 10,
(ii) $R(\bar{t})$ *is an even overlattice of* $R(t)$, *and*
(iii) *the Dynkin diagram of* $\bar{t}$ *is a sub-diagram of the Dynkin diagram of* E_{10}.

There exist exactly 184 equivalence classes of ADE-configurations of roots in L_{10}. They are given in Table 1.1. In the case where $R_f = \overline{R}_f$, the item $\tau(\overline{R}_f)$ is simply denoted by Φ.

Corollary 1.4. *Let* (Y, ρ) *be an* RDP-*Enriques surface, and* R_ρ *the sublattice of* S_Y *generated by the set* Φ_ρ *of classes of smooth rational curves contracted by* ρ. *Then* *the primitive closure $\overline{R}_\rho$ of R_ρ in S_Y is a root lattice. Two* RDP-*Enriques surfaces (Y, ρ) and (Y', ρ') are equivalent if and only if $\tau(\Phi_\rho) = \tau(\Phi_{\rho'})$ and $\tau(\overline{R}_\rho) = \tau(\overline{R}_{\rho'})$.*

The finite abelian group $\overline{R}_\rho/R_\rho$ can be used to distinguish the topological type of the smooth part $\overline{Y}^\circ := \overline{Y} \setminus \mathrm{Sing}(\overline{Y})$ of $\overline{Y}$. In [12], Keum and Zhang studied configurations of ADE-type cA_l, and the topological fundamental groups $\pi_1(\overline{Y}^\circ)$. Rams and Schütt [17] corrected a result of [12] and showed that there exist at least three equivalence classes of RDP-Enriques surfaces with $\tau(\Phi_\rho) = 4A_2$ that are distinguished by $\pi_1(\overline{Y}^\circ)$.

Our next problem is to determine all equivalence classes of ADE-configurations $\Phi_f \subset L_{10}$ that can be realized as the ADE-configuration $\Phi_\rho \subset S_Y \cong L_{10}$ associated with an RDP-Enriques surface (Y, ρ). Let (Y, ρ) be an RDP-Enriques surface, and let $C_1, \dots, C_n$ be the smooth rational curves on Y contracted by ρ, so that $\Phi_\rho = \{[C_1], \dots, [C_n]\}$. Let $\pi\colon X \to Y$ denote the universal covering of Y, and $\varepsilon\colon X \to X$ the deck-transformation of the double covering π. Then $\pi^*\colon S_Y \to S_X$ is injective, and the image $\pi^* S_Y$ is equal to the invariant sublattice in S_X of the action of ε in S_X. The pull-back of C_i by π splits into the disjoint union of two smooth rational curves C_i' and C_i'' on X. We put

$$\Phi_\rho^\sim := \{[C_1'], [C_1''], \dots, [C_n'], [C_n'']\} \quad \subset \quad S_X,$$

that is, $\Phi_\rho^\sim$ is the set of classes of smooth rational curves on X contracted by $\rho \circ \pi\colon X \to \overline{Y}$. We denote by M_ρ the sublattice of S_X generated by $\pi^* S_Y$ and $\Phi_\rho^\sim$. Then the rank of M_ρ is equal to $10 + n$. We then denote by $\overline{M}_\rho$ the primitive closure of M_ρ in S_X. Note that the action of ε on S_X preserves the sublattices M_ρ and $\overline{M}_\rho$. We put

$$Q_{(Y,\rho)} := \overline{M}_\rho / M_\rho.$$

Definition 1.5. Let (Y', ρ') be another RDP-Enriques surface with the universal covering $\pi'\colon X' \to Y'$. We say that (Y, ρ) and (Y', ρ') are *strongly equivalent* if there exists an isometry

$$\mu\colon \overline{M}_\rho \xrightarrow{\sim} \overline{M}_{\rho'}$$

with the following properties; the isometry μ maps $\pi^* S_Y$ to $\pi'^* S_{Y'}$ isomorphically, and the isometry $\mu_Y\colon S_Y \xrightarrow{\sim} S_{Y'}$ induced by μ maps $\mathcal{P}_Y$ to $\mathcal{P}_{Y'}$ and Φ_ρ to $\Phi_{\rho'}$ bijectively. An isometry $\mu\colon \overline{M}_\rho \xrightarrow{\sim} \overline{M}_{\rho'}$ satisfying these conditions is called a *strong-equivalence isometry.*

It is obvious that, if (Y, ρ) and (Y', ρ') are strongly equivalent, then they are equivalent. It is also obvious that a strong-equivalence isometry $\overline{M}_\rho \xrightarrow{\sim} \overline{M}_{\rho'}$ is compatible with the actions of the Enriques involutions on $\overline{M}_\rho$ and on $\overline{M}_{\rho'}$. The following lemma is proved in Section 4.2.

Lemma 1.6. *Let $\mu\colon \overline{M}_\rho \xrightarrow{\sim} \overline{M}_{\rho'}$ be a strong-equivalence isometry. Then μ maps $\Phi_\rho^\sim$ to $\Phi_{\rho'}^\sim$ bijectively. In particular, if (Y, ρ) and (Y', ρ') are strongly equivalent, then we have $Q_{(Y,\rho)} \cong Q_{(Y',\rho')}$.*

Our second main result is as follows.

Theorem 1.7. *There exist exactly* 265 *strong equivalence classes of* RDP-*Enriques surfaces (Y, ρ) with $\Phi_\rho \neq \emptyset$. The invariants $\tau(\Phi_\rho)$, $\tau(\overline{R}_\rho)$, and $Q_{(Y,\rho)}$ are given in Table 1.1.*

No.	$\tau(\Phi_f)$	$\tau(\overline{R}_f)$	$Q_{(Y,\rho)}$
1	A_1	Φ	0
2	$2A_1$	Φ	0
3	A_2	Φ	0
4	$3A_1$	Φ	0
5	A_1+A_2	Φ	0
6	A_3	Φ	0
7	$4A_1$	Φ	0
8	$4A_1$	D_4	$0, 2$
9	$2A_1+A_2$	Φ	0
10	A_1+A_3	Φ	0
11	$2A_2$	Φ	0
12	A_4	Φ	0
13	D_4	Φ	0
14	$5A_1$	Φ	0
15	$5A_1$	A_1+D_4	$0, 2$
16	$3A_1+A_2$	Φ	0
17	$2A_1+A_3$	Φ	0
18	$2A_1+A_3$	D_5	$0, 2$
19	A_1+2A_2	Φ	0
20	A_1+A_4	Φ	0
21	A_1+D_4	Φ	0
22	A_2+A_3	Φ	0
23	A_5	Φ	0
24	D_5	Φ	0
25	$6A_1$	$2A_1+D_4$	$0, 2$
26	$6A_1$	D_6	$2, 22$
27	$4A_1+A_2$	Φ	0
28	$4A_1+A_2$	A_2+D_4	$0, 2$
29	$3A_1+A_3$	Φ	0
30	$3A_1+A_3$	A_1+D_5	$0, 2$
31	$2A_1+2A_2$	Φ	0
32	$2A_1+A_4$	Φ	0
33	$2A_1+D_4$	Φ	0
34	$2A_1+D_4$	D_6	$0, 2$
35	$A_1+A_2+A_3$	Φ	0
36	A_1+A_5	Φ	0
37	A_1+A_5	E_6	$0, 2$
38	A_1+D_5	Φ	0
39	$3A_2$	Φ	$0, 3$
40	$3A_2$	E_6	0
41	A_2+A_4	Φ	0
42	A_2+D_4	Φ	0
43	$2A_3$	Φ	0
44	$2A_3$	D_6	$0, 2, 2$
45	A_6	Φ	0
46	D_6	Φ	0
47	E_6	Φ	0
48	$7A_1$	A_1+D_6	2
49	$7A_1$	E_7	22
50	$5A_1+A_2$	$A_1+A_2+D_4$	0
51	$4A_1+A_3$	$2A_1+D_5$	$0, 2$
52	$4A_1+A_3$	A_3+D_4	$0, 2$
53	$4A_1+A_3$	D_7	$2, 2, 2, 22$
54	$3A_1+2A_2$	Φ	0
55	$3A_1+A_4$	Φ	0
56	$3A_1+D_4$	A_1+D_6	$0, 2$
57	$3A_1+D_4$	E_7	$2, 22$
58	$2A_1+A_2+A_3$	Φ	0
59	$2A_1+A_2+A_3$	A_2+D_5	$0, 2$
60	$2A_1+A_5$	Φ	0
61	$2A_1+A_5$	A_1+E_6	$0, 2$
62	$2A_1+D_5$	Φ	0
63	$2A_1+D_5$	D_7	$0, 2$
64	A_1+3A_2	Φ	$0, 3$
65	A_1+3A_2	A_1+E_6	0
66	$A_1+A_2+A_4$	Φ	0
67	$A_1+A_2+D_4$	Φ	0
68	A_1+2A_3	Φ	0
69	A_1+2A_3	A_1+D_6	$0, 2, 2, 4$
70	A_1+2A_3	E_7	$0, 2$
71	A_1+A_6	Φ	0
72	A_1+D_6	Φ	0
73	A_1+D_6	E_7	$0, 2$
74	A_1+E_6	Φ	0
75	$2A_2+A_3$	Φ	0
76	A_2+A_5	Φ	$0, 3$
77	A_2+A_5	E_7	0
78	A_2+D_5	Φ	0
79	A_3+A_4	Φ	0
80	A_3+D_4	Φ	0
81	A_3+D_4	D_7	$0, 2, 2$
82	A_7	Φ	0
83	A_7	E_7	$0, 2, 2$
84	D_7	Φ	0
85	E_7	Φ	0
86	$8A_1$	A_1+E_7	$-$
87	$8A_1$	D_8	22
88	$8A_1$	E_8	222
89	$6A_1+A_2$	A_2+D_6	$-$
90	$5A_1+A_3$	A_1+D_7	2
91	$4A_1+A_4$	A_4+D_4	0
92	$4A_1+D_4$	A_1+E_7	2
93	$4A_1+D_4$	D_8	$2, 2$
94	$4A_1+D_4$	E_8	$22, 22$
95	$3A_1+A_2+A_3$	$A_1+A_2+D_5$	0
96	$3A_1+A_5$	$2A_1+E_6$	$0, 2$
97	$3A_1+D_5$	A_1+D_7	0
98	$2A_1+3A_2$	$2A_1+E_6$	0
99	$2A_1+A_2+A_4$	Φ	0
100	$2A_1+A_2+D_4$	A_2+D_6	0
101	$2A_1+2A_3$	A_1+E_7	$0, 2, 4$
102	$2A_1+2A_3$	A_3+D_5	$0, 2$
103	$2A_1+2A_3$	D_8	$2, 2, 2, 22, 22, 4, 42$
104	$2A_1+2A_3$	E_8	$2, 2, 2, 22$
105	$2A_1+A_6$	Φ	0
106	$2A_1+D_6$	A_1+E_7	$0, 2$
107	$2A_1+D_6$	D_8	$0, 2$
108	$2A_1+D_6$	E_8	$2, 2, 2, 22$
109	$2A_1+E_6$	Φ	0
110	$A_1+2A_2+A_3$	Φ	0
111	$A_1+A_2+A_5$	Φ	$0, 3$
112	$A_1+A_2+A_5$	A_1+E_7	0
113	$A_1+A_2+A_5$	A_2+E_6	$0, 2, 3, 6$
114	$A_1+A_2+A_5$	E_8	$0, 2$
115	$A_1+A_2+D_5$	Φ	0
116	$A_1+A_3+A_4$	Φ	0
117	$A_1+A_3+D_4$	A_1+D_7	$0, 2$
118	A_1+A_7	Φ	0
119	A_1+A_7	A_1+E_7	$0, 2, 2, 4, 4$
120	A_1+A_7	E_8	$0, 2, 2$
121	A_1+D_7	Φ	0
122	A_1+E_7	Φ	0
123	A_1+E_7	E_8	$0, 2$
124	$4A_2$	A_2+E_6	$0, 3$
125	$4A_2$	E_8	0
126	$2A_2+A_4$	Φ	0
127	A_2+2A_3	A_2+D_6	$0, 2$
128	A_2+A_6	Φ	0
129	A_2+D_6	Φ	0
130	A_2+E_6	Φ	$0, 3$
131	A_2+E_6	E_8	0
132	A_3+A_5	Φ	0
133	A_3+D_5	Φ	0
134	A_3+D_5	D_8	$0, 2, 2, 4$
135	A_3+D_5	E_8	$0, 2$
136	$2A_4$	Φ	$0, 5$
137	$2A_4$	E_8	0
138	A_4+D_4	Φ	0
139	A_8	Φ	$0, 3$
140	A_8	E_8	0

TABLE 1.1. ADE-configurations of roots in L_{10} (continues)

No.	$\tau(\Phi_f)$	$\tau(\overline{R}_f)$	$Q_{(Y,\rho)}$
141	$2D_4$	D_8	$0, 2$
142	$2D_4$	E_8	$2, 2, 22$
143	D_8	Φ	0
144	D_8	E_8	$0, 2, 2$
145	E_8	Φ	0
146	$9A_1$	$A_1 + E_8$	$-$
147	$7A_1 + A_2$	$A_2 + E_7$	$-$
148	$6A_1 + A_3$	D_9	$-$
149	$5A_1 + D_4$	$A_1 + E_8$	$-$
150	$4A_1 + D_5$	D_9	$-$
151	$3A_1 + A_2 + D_4$	$A_2 + E_7$	$-$
152	$3A_1 + 2A_3$	$A_1 + E_8$	2
153	$3A_1 + D_6$	$A_1 + E_8$	2
154	$2A_1 + A_2 + A_5$	$A_1 + E_8$	0
155	$2A_1 + A_3 + A_4$	$A_4 + D_5$	0
156	$2A_1 + A_3 + D_4$	D_9	$2, 2$
157	$2A_1 + A_7$	$A_1 + E_8$	$0, 2, 4$
158	$2A_1 + D_7$	D_9	0
159	$2A_1 + E_7$	$A_1 + E_8$	0
160	$A_1 + 4A_2$	$A_1 + E_8$	$-$
161	$A_1 + A_2 + 2A_3$	$A_2 + E_7$	0
162	$A_1 + A_2 + A_6$	Φ	0
163	$A_1 + A_2 + D_6$	$A_2 + E_7$	0
164	$A_1 + A_2 + E_6$	$A_1 + E_8$	0
165	$A_1 + A_3 + A_5$	$A_3 + E_6$	$0, 2$
166	$A_1 + A_3 + D_5$	$A_1 + E_8$	0
167	$A_1 + 2A_4$	$A_1 + E_8$	0
168	$A_1 + A_8$	Φ	$0, 3$
169	$A_1 + A_8$	$A_1 + E_8$	0
170	$A_1 + 2D_4$	$A_1 + E_8$	2
171	$A_1 + D_8$	$A_1 + E_8$	$0, 2$
172	$A_1 + E_8$	Φ	0
173	$3A_2 + A_3$	$A_3 + E_6$	0
174	$2A_2 + A_5$	$A_2 + E_7$	$0, 3$
175	$A_2 + A_7$	$A_2 + E_7$	$0, 2$
176	$A_2 + E_7$	Φ	0
177	$3A_3$	D_9	$2, 22$
178	$A_3 + D_6$	D_9	$0, 2$
179	$A_3 + E_6$	Φ	0
180	$A_4 + A_5$	Φ	0
181	$A_4 + D_5$	Φ	0
182	A_9	Φ	0
183	$D_4 + D_5$	D_9	0
184	D_9	Φ	0

TABLE 1.1. ADE-configurations of roots in L_{10} (continued)

In Table 1.1, the group $Q_{(Y,\rho)}$ is written in the following abbreviations:

$$0 = \{0\},\ \ n = \mathbb{Z}/n\mathbb{Z}\ \ (n \le 6),\ \ 22 = (\mathbb{Z}/2\mathbb{Z})^2,\ \ 222 = (\mathbb{Z}/2\mathbb{Z})^3,\ \ 42 = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

When the fourth column $Q_{(Y,\rho)}$ is empty $(-)$, there exist no RDP-Enriques surfaces (Y, ρ) such that $\tau(\Phi_\rho)$ and $\tau(\overline{R}_\rho)$ are given in the second and the third columns.

Corollary 1.8. *There exist exactly* 175 *equivalence classes of* RDP*-Enriques surfaces* (Y, ρ) *with* $\Phi_\rho \neq \emptyset$.

Example 1.9. There exist no RDP-Enriques surfaces (Y, ρ) with singularities of type $\tau(\Phi_\rho) = 6A_1 + A_2$.

Example 1.10. RDP-Enriques surfaces (Y, ρ) with singularities of type $\tau(\Phi_\rho) = 2A_1 + 2A_3$ are divided into 4 equivalence classes with

$$\tau(\overline{R}_\rho) = A_1 + E_7,\ \ A_3 + D_5,\ \ D_8,\ \ E_8,$$

and into $3 + 2 + 7 + 4 = 16$ strong equivalence classes.

An important class of involutions of $K3$ surfaces other than Enriques involutions comes from the sextic double plane models. The classification of ADE-types of rational double points on normal sextic double planes was given by Yang [27]. A finer classification of rational double points on normal sextic double planes was given in [20] in the relation with the topology of Zariski pairs of plane curves (see [2], [21]). This classification was further refined to the complete description of connected components of the equisingular families of irreducible sextic plane curves with only simple singularities by Akyol and Degtyarev [1]. The present article may be regarded as the Enriques counterpart of these studies of plane sextic curves.

This paper is organized as follows. In Section 2, we review preliminary results about lattices. Some algorithms for negative-definite root lattices are presented, and the notion of chambers in a hyperbolic lattice is introduced. In Section 3, we study the lattice L_{10} in detail, and prove Theorem 1.3 by the generalized Borcherds algorithm ([5], [6], [22]). The classical result due to Vinberg [26] plays an important role. In Section 4, we give a method to enumerate all the strong equivalence classes of RDP-Enriques surfaces, explain how to carry out this method, and prove Theorem 1.7.

For the computation, we used GAP [9]. A computational data is available from the author's webpage [25]. In particular, the isomorphism class of the lattice $\overline{M}_\rho$ for each strongly equivalence class of RDP-Enriques surfaces is given explicitly in [25]. Since we have $\overline{M}_\rho = S_X$ when (Y, ρ) corresponds to a general point of an irreducible component of the moduli of RDP-Enriques surfaces with a fixed ADE-type, this data will be useful in the study of the moduli of RDP-Enriques surfaces.

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2. Preliminaries

2.1. ADE-configuration. A *Dynkin configuration* is a finite set Φ with a map

$$\langle\ ,\ \rangle\colon \Phi \times \Phi \to \{-2, 0, 1\}$$

such that $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in \Phi$, $\langle x, x\rangle = -2$ for all $x \in \Phi$, and $\langle x, y\rangle \in \{0, 1\}$ for all $x, y \in \Phi$ with $x \neq y$. With a Dynkin configuration $\Phi = \{r_1, \dots, r_n\}$, we associate its *Dynkin diagram*, which is a graph whose set of vertices is Φ and whose set of edges is the set of pairs $\{r_i, r_j\}$ such that $\langle r_i, r_j\rangle = 1$. We say that a Dynkin configuration is an ADE-*configuration* if every connected component of its Dynkin diagram is of type A_l $(l \geq 1)$, D_m $(m \geq 4)$, or E_n $(n = 6, 7, 8)$. We define the ADE-*type* $\tau(\Phi)$ of an ADE-configuration Φ to be the sum of the types of the connected components of its Dynkin diagram. Let Φ and Φ' be Dynkin configurations. An isomorphism from Φ to Φ' is a bijection $\Phi \xrightarrow{\sim} \Phi'$ that preserves $\langle\ ,\ \rangle$. The isomorphism class of an ADE-configuration is uniquely determined by its ADE-type. We denote by $\mathrm{Aut}(\Phi)$ the group of automorphisms of Φ, which we let act on Φ from the *right*.

2.2. Lattice. Let L be a free $\mathbb{Z}$-module of finite rank, and R a submodule of L. The *primitive closure* $\overline{R}$ of R in L is the intersection of $R \otimes \mathbb{Q}$ and L in $L \otimes \mathbb{Q}$. We say that R is *primitive in* L if $R = \overline{R}$.

A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form

$$\langle\ ,\ \rangle\colon L \times L \to \mathbb{Z}.$$

Let L be a lattice. We say that L is *even* if $\langle x, x\rangle \in 2\mathbb{Z}$ for all $x \in L$. The group of isometries of L is denoted by $\mathrm{O}(L)$. We let $\mathrm{O}(L)$ act on L from the *right*. An *embedding* of a Dynkin configuration Φ into L is an injection $\Phi \hookrightarrow L$ that preserves $\langle\ ,\ \rangle$. We define the *dual lattice* $L^\vee$ of L by

$$L^\vee := \{\ v \in L \otimes \mathbb{Q} \ \mid\ \langle x, v\rangle \in \mathbb{Z} \ \text{ for all } \ x \in L\ \}.$$

The finite abelian group $A_L := L^\vee/L$ is called the *discriminant group* of L. The group $\mathrm{O}(L)$ acts on A_L from the right. We say that L is *unimodular* if A_L is trivial. The *signature* of L is the signature of the real quadratic space $L \otimes \mathbb{R}$. Suppose that L is of rank $n > 0$. We say that L is *hyperbolic* if the signature is $(1, n-1)$, and is *negative-definite* if the signature is $(0, n)$.

2.3. **Roots and reflections.** Let L be an even lattice. A *root* of L is a vector $r \in L$ such that $\langle r, r\rangle = -2$. The set of roots of L is denoted by $\mathrm{Roots}(L)$. A root r of L defines an isometry

$$s_r \colon x \mapsto x + \langle x, r\rangle r$$

of L, which is called the *reflection* associated with r. We denote by $W(L)$ the subgroup of $\mathrm{O}(L)$ generated by all the reflections associated with the roots, and call it the *Weyl group* of L. We say that L is a *root lattice* if L is generated by roots. Let Φ be a subset of $\mathrm{Roots}(L)$. We denote by $W(\Phi, L)$ the subgroup of $W(L)$ generated by all the reflections s_r associated with $r \in \Phi$.

2.4. **Negative-definite root lattice.** Let Φ be an ADE-configuration. Extending $\langle\ ,\ \rangle \colon \Phi \times \Phi \to \mathbb{Z}$ by linearity to the bilinear form on the free $\mathbb{Z}$-module generated by Φ, we obtain a negative-definite root lattice of rank $|\Phi|$, which we will denote by $\langle \Phi\rangle$. Conversely, let R be a negative-definite root lattice. Then R has a basis Φ_R consisting of roots that form an ADE-configuration, which we call an ADE-*basis* of R. We define the ADE-type $\tau(R)$ of R to be the ADE-type $\tau(\Phi_R)$ of an ADE-basis Φ_R of R. In the following, we describe the set of all ADE-bases of a negative-definite root lattice. See [8, Chapter 1] or [10, Chapter 1] for the proof.

Let R be a negative-definite root lattice. For a root r of R, we denote by $r^\perp$ the hyperplane of $R \otimes \mathbb{R}$ defined by $\langle x, r\rangle = 0$. We then put

$$(R\otimes\mathbb{R})^\circ := (R\otimes\mathbb{R}) \ \setminus \ \bigcup r^\perp,$$

where r runs through the finite set $\mathrm{Roots}(R)$. Let Γ be a connected component of $(R\otimes\mathbb{R})^\circ$, and $\overline{\Gamma}$ the closure of Γ in $R\otimes\mathbb{R}$. Then the set Φ_Γ of all $r \in \mathrm{Roots}(R)$ such that $\langle x, r\rangle > 0$ for any $x \in \Gamma$ and that $r^\perp \cap \overline{\Gamma}$ contains a non-empty open subset of $r^\perp$ form an ADE-basis of R, and the mapping $\Gamma \mapsto \Phi_\Gamma$ gives a bijection from the set of connected components of $(R\otimes\mathbb{R})^\circ$ to the set of ADE-bases of R. For $x \in (R\otimes\mathbb{R})^\circ$, we denote by $\Gamma(x)$ the connected component of $(R\otimes\mathbb{R})^\circ$ containing x. Let $\Phi = \{r_1, \dots, r_n\}$ be an ADE-basis of R. We put

$$c := r_1^\vee + \cdots + r_n^\vee \ \ \in \ \ (R\otimes\mathbb{R})^\circ,$$

where $r_1^\vee, \dots, r_n^\vee$ are the basis of $R^\vee$ dual to the basis $r_1, \dots, r_n$ of R. Then the connected component of $(R\otimes\mathbb{R})^\circ$ corresponding to the ADE-basis Φ is $\Gamma(c)$. It is obvious that we have a natural embedding $\mathrm{Aut}(\Phi) \hookrightarrow \mathrm{O}(R)$ whose image is

$$\mathrm{Stab}(\Gamma(c), R) := \{\ g \in \mathrm{O}(R) \ \mid\ \Gamma(c)^g = \Gamma(c)\ \} = \{\ g \in \mathrm{O}(R)\ \mid\ c^g = c\ \}.$$

Since $W(R)$ acts on the set of connected components of $(R\otimes\mathbb{R})^\circ$ simple-transitively, we have a *splitting* exact sequence

$$1 \ \to\ W(R) \ \to\ \mathrm{O}(R) \ \stackrel{\kappa}{\to}\ \mathrm{Aut}(\Phi) \ \to\ 1. \tag{2.1}$$

Algorithm 2.1. Let u and v be points of $(R\otimes\mathbb{Q}) \cap (R\otimes\mathbb{R})^\circ$. This algorithm finds the unique element $g \in W(R)$ that maps $\Gamma(u)$ to $\Gamma(v)$. Let ξ be a sufficiently general element of $R\otimes\mathbb{Q}$, and let ε be a sufficiently small positive rational number. We consider the open line segment in $R\otimes\mathbb{R}$ drawn by the point

$$p(t) := u + t(v + \varepsilon\xi),$$

where t moves in the set of positive real numbers. Let $\{r_1, \dots, r_N\}$ be the set of roots r_i of R such that $\langle u, r_i\rangle < 0$ and $\langle v, r_i\rangle > 0$. For each r_i, let t_i be the unique rational number such that $\langle p(t_i), r_i\rangle = 0$. Since the perturbation vector ξ is general, we can assume that $t_1, \dots, t_N$ are distinct. We sort $r_1, \dots, r_N$ in such a way that $t_1 < \dots < t_N$. Then $g := s_{r_1} \dots s_{r_N} \in W(R)$ satisfies $\Gamma(u)^g = \Gamma(v)$. ∎

As applications of Algorithm 2.1, we obtain the following algorithms. As noted above, the stabilizer subgroup $\mathrm{Stab}(\Gamma(c), R)$ is canonically identified with $\mathrm{Aut}(\Phi)$.

Algorithm 2.2. Let an isometry $g \in \mathrm{O}(R)$ be given. The image $\kappa(g) \in \mathrm{Aut}(\Phi)$ of g by the homomorphism κ in (2.1) is calculated by applying Algorithm 2.1 to $u = c^g$ and $v = c$. We find $h \in W(R)$ such that $\Gamma(c)^{gh} = \Gamma(c)$. Hence we have $\kappa(g) = gh$. ∎

Algorithm 2.3. Applying Algorithm 2.1 to $u = c$ and $v = -c$, we find the element $l \in W(R)$ such that $\Gamma(c)^l = -\Gamma(c)$. This element l is the *longest element* of the Coxeter group $W(R)$. (See [10, Section 1.8].) ∎

Another method to obtain an ADE-basis of R is as follows. We put

$$\mathrm{Hom}(R, \mathbb{R})^\circ := \{\ \ell \in \mathrm{Hom}(R, \mathbb{R})\ \mid\ \ell(r) \neq 0 \quad \text{for any } r \in \mathrm{Roots}(R)\ \},$$

and for $\ell \in \mathrm{Hom}(R, \mathbb{R})^\circ$, we put $\mathrm{Roots}(R)_{\ell > 0} := \{r \in \mathrm{Roots}(R) \mid \ell(r) > 0\}$.

Definition 2.4. Let S be a subset of $\mathrm{Roots}(R)_{\ell>0}$. We say that $r \in S$ is *indecomposable in S* if r is not written as a linear combination $\sum a_i r_i$ of elements $r_i \in S$ with $a_i \in \mathbb{Z}_{\geq 0}$ such that $\sum a_i > 1$.

Let $\Phi_{\ell>0}$ be the set of roots $r \in \mathrm{Roots}(R)_{\ell>0}$ indecomposable in $\mathrm{Roots}(R)_{\ell>0}$. Then $\Phi_{\ell>0}$ is an ADE-basis of R, and the mapping $\ell \mapsto \Phi_{\ell>0}$ gives a bijection from the set of connected components of $\mathrm{Hom}(R, \mathbb{R})^\circ$ to the set of ADE-bases of R.

2.5. **Even hyperbolic lattice.** Let L be an even hyperbolic lattice. A *positive cone* is one of the two connected components of the space $\{x \in L \otimes \mathbb{R} \mid \langle x, x\rangle > 0\}$. Let $\mathcal{P}$ be a positive cone. We denote by $\mathrm{O}^+(L)$ the stabilizer subgroup of $\mathcal{P}$ in $\mathrm{O}(L)$. For a non-zero vector $v \in L \otimes \mathbb{R}$, we put

$$H_v^+ := \{\ x \in \mathcal{P}\ \mid\ \langle v, x\rangle \geq 0\ \}, \quad (v)^\perp := \{\ x \in \mathcal{P}\ \mid\ \langle v, x\rangle = 0\ \}.$$

Then $(v)^\perp \neq \emptyset$ if and only if $\langle v, v\rangle < 0$. Note that the Weyl group $W(L)$ acts on $\mathcal{P}$. A *standard fundamental domain of the action of $W(L)$ on $\mathcal{P}$* is the closure in $\mathcal{P}$ of a connected component of

$$\mathcal{P}\ \setminus\ \bigcup (r)^\perp,$$

where r runs through $\mathrm{Roots}(L)$. Then $W(L)$ acts on the set of standard fundamental domains simple-transitively. Let Δ be a standard fundamental domain. We put $\mathrm{Aut}(\Delta) := \{g \in \mathrm{O}^+(L) \mid \Delta^g = \Delta\}$. Then we have a *splitting* exact sequence

$$1 \;\to\; W(L) \;\to\; \mathrm{O}^+(L) \;\to\; \mathrm{Aut}(\Delta) \;\to\; 1, \tag{2.2}$$

and we have a tessellation

$$\mathcal{P} = \bigcup_{g \in W(L)} \Delta^g. \tag{2.3}$$

2.6. **Chambers.** Let L be an even hyperbolic lattice, and $\mathcal{P}$ a positive cone of L. A closed subset $\mathcal{C}$ of $\mathcal{P}$ is called a *chamber* if the interior $\mathcal{C}^\circ$ of $\mathcal{C}$ in $\mathcal{P}$ is non-empty and there exists a set of vectors $\mathcal{H} \subset L \otimes \mathbb{R}$ such that the family of hyperplanes $\{(v)^\perp \mid v \in \mathcal{H}\}$ of $\mathcal{P}$ is locally finite in $\mathcal{P}$ and that

$$\mathcal{C} = \bigcap_{v \in \mathcal{H}} H_v^+. \tag{2.4}$$

Let $\mathcal{C}$ be a chamber defined by a set of vectors $\mathcal{H} \subset L \otimes \mathbb{R}$ as in (2.4). A closed subset F of $\mathcal{C}$ is called a *face* of $\mathcal{C}$ if $F \neq \emptyset$, $F \cap \mathcal{C}^\circ = \emptyset$, and there exists a subset $\mathcal{H}_F$ of $\mathcal{H}$ such that

$$F = \mathcal{C} \cap \bigcap_{v \in \mathcal{H}_F} (v)^\perp.$$

If F is a face, then we have a unique linear subspace V of $L \otimes \mathbb{R}$ such that $V \cap F$ contains a non-empty open subset of V. We say that V or $\mathcal{P} \cap V$ the *supporting linear subspace* of F. The *codimension* of a face is defined to be the codimension of its supporting linear subspace in $L \otimes \mathbb{R}$ or in $\mathcal{P}$. A face of codimension 1 is called a *wall.* Let F be a wall of $\mathcal{C}$. A vector $v \in L \otimes \mathbb{R}$ is said to *define the wall* F if $\mathcal{C}$ is contained in H_v^+ and $F = \mathcal{C} \cap (v)^\perp$ holds.

2.7. **Discriminant form and overlattices.** Let L be an even lattice. Recall that $A_L := L^\vee / L$. Then the natural $\mathbb{Q}$-valued symmetric bilinear form on $L^\vee$ defines a finite quadratic form

$$q_L \colon A_L \to \mathbb{Q}/2\mathbb{Z},$$

which we call the *discriminant form* of L. See Nikulin [14] for the basic properties of the discriminant form. We have a natural homomorphism $\mathrm{O}(L) \to \mathrm{O}(q_L)$, where $\mathrm{O}(q_L)$ is the automorphism group of the finite quadratic form q_L. An *even overlattice* of L is a submodule M of $L^\vee$ containing L such that the restriction of the natural $\mathbb{Q}$-valued symmetric bilinear form on $L^\vee$ makes M an even lattice. By definition, we have the following:

Proposition 2.5. *The map $M \mapsto M/L$ gives a bijection from the set of even overlattices M of L to the set of totally isotropic subgroups of q_L.*

Note that $\mathrm{O}(L)$ acts on the set of even overlattices of L from the right.

Proposition 2.6. *Suppose that the signature of L is (s_+, s_-). Let H be an even unimodular lattice of signature (h_+, h_-). Then L can be embedded into H primitively if and only if there exists an even lattice of signature $(h_+ - s_+, h_- - s_-)$ whose discriminant form is isomorphic to $-q_L$.*

Remark 2.7. The signature $(h_+ - s_+, h_- - s_-)$ and the isomorphism class of the discriminant form $-q_L$ determine a genus of even lattices. There exist various versions of the criterion to determine whether a genus given by signature and a finite quadratic form is empty or not. See, for example, Nikulin [14], Conway-Sloane [7, Chapter 15], Miranda-Morrison [13]. This criterion has been applied to many problems of $K3$ surfaces. See, for example, [1, 19, 20, 23, 27].

3. The lattice L_{10}

For an embedding $f \colon \Phi \hookrightarrow L_{10}$ of an ADE-configuration Φ into L_{10}, we denote by Φ_f the image of Φ by f. Then Definition 3.1 below coincides with Definition 1.2.

Definition 3.1. Let $f\colon \Phi \hookrightarrow L_{10}$ and $f'\colon \Phi' \hookrightarrow L_{10}$ be embeddings of ADE-configurations Φ and Φ'. We say that f and f' are *equivalent* if there exist an isomorphism $\gamma\colon \Phi \xrightarrow{\sim} \Phi'$ and an isometry $g \in \mathrm{O}^+(L_{10})$ that make the following diagram commutative:

$$\begin{array}{ccc} \Phi & \overset{f}{\hookrightarrow} & L_{10} \\ \gamma\downarrow & & \downarrow g \\ \Phi' & \overset{f'}{\hookrightarrow} & L_{10}. \end{array}$$

The purpose of this section is to prove Theorem 1.3.

3.1. **Negative-definite primitive root sublattices of L_{10}.**

Definition 3.2. Let $\mathcal{N}$ denote the set of all negative-definite primitive root sublattices of L_{10}, on which $\mathrm{O}^+(L_{10})$ acts from the right.

We calculate the set $\mathcal{N}/\mathrm{O}^+(L_{10})$ of orbits of this action. Recall that the lattice L_{10} has a basis $E_{10} = \{e_1, \dots, e_{10}\}$ consisting of roots that form the Dynkin diagram in Figure 1.1. We fix, once and for all, the positive cone $\mathcal{P}_{10}$ that contains the vector

$$c_0 := e_1^\vee + \dots + e_{10}^\vee$$

of square-norm 1240, where $e_1^\vee, \dots, e_{10}^\vee$ are the basis of $L_{10}^\vee = L_{10}$ dual to the basis $e_1, \dots, e_{10}$. For simplicity, we put, for a subset S of $L_{10} \otimes \mathbb{R}$,

$$[S]^\perp := \{\, v \in L_{10} \mid \langle v, x\rangle = 0 \ \text{ for all } \ x \in S \,\}, \tag{3.1}$$

$$(S)^\perp := \{\, v \in \mathcal{P}_{10} \mid \langle v, x\rangle = 0 \ \text{ for all } \ x \in S \,\}. \tag{3.2}$$

We consider the chamber

$$\Delta_0 := \{\, x \in \mathcal{P}_{10} \mid \langle x, e_i\rangle \ge 0 \ \text{ for all } \ i = 1.\dots, 10 \,\},$$

which contains c_0 in its interior. Vinberg [26] proved the following:

Theorem 3.3 (Vinberg)**.** *Each e_i defines a wall of Δ_0. The chamber Δ_0 is a standard fundamental domain of the action of $W(L_{10})$ on $\mathcal{P}_{10}$.*

Definition 3.4. We call a standard fundamental domain of the action of $W(L_{10})$ on $\mathcal{P}_{10}$ a *Vinberg chamber*. We denote by $\mathcal{V}$ the set of all Vinberg chambers.

The following easy fact is used frequently in this section: Let $r_1, \dots, r_n$ be roots of L_{10} such that the linear subspace $\mathcal{P}' := (\{r_1, \dots, r_n\})^\perp$ of $\mathcal{P}_{10}$ is non-empty. Let Δ be a Vinberg chamber. If $\mathcal{P}' \cap \Delta$ contains a non-empty open subset of $\mathcal{P}'$, then $\mathcal{P}' \cap \Delta$ is a face of Δ, and its supporting linear subspace is $\mathcal{P}'$.

Since the Dynkin diagram in Figure 1.1 has no symmetries, we see from (2.2) that $\mathrm{O}^+(L_{10})$ is equal to $W(L_{10})$. In particular, we have the following:

Proposition 3.5. *The map $g \mapsto \Delta_0^g$ is a bijection from $\mathrm{O}^+(L_{10})$ to $\mathcal{V}$.*

We denote by $\gamma\colon \mathcal{V} \to \mathrm{O}^+(L_{10})$ the inverse map of $g \mapsto \Delta_0^g$, that is, $\gamma(\Delta)$ is the unique element of $\mathrm{O}^+(L_{10})$ such that

$$\Delta_0^{\gamma(\Delta)} = \Delta.$$

The following lemma is easy to prove:

Lemma 3.6. (1) *A subset* Σ *of* E_{10} *is an* ADE*-configuration of roots if and only if* $\Sigma \neq \emptyset$, $\Sigma \neq E_{10}$, *and* $\Sigma \neq \{e_1, \dots, e_9\}$. (2) *Let* $\overline{\mathcal{P}}_{10}$ *and* $\overline{\Delta}_0$ *denote the closure of* $\mathcal{P}_{10}$ *and* Δ_0 *in* $L_{10} \otimes \mathbb{R}$, *respectively. Then* $\overline{\Delta}_0 \cap (\overline{\mathcal{P}}_{10} \setminus \mathcal{P}_{10})$ *is equal to the half-line* $\mathbb{R}_{\geq 0} e_{10}^{\vee} = ([\{e_1, \dots, e_9\}]^{\perp} \otimes \mathbb{R}) \cap \overline{\mathcal{P}}_{10}$.

We consider the set

$$\mathcal{S} := 2^{E_{10}} \setminus \{\emptyset, E_{10}, \{e_1, \dots, e_9\}\}.$$

Since $\overline{\Delta}_0$ is the cone over a 9-dimensional simplex, we see that

$$\Sigma \quad \mapsto \quad F_{\Sigma} := (\Sigma)^{\perp} \cap \Delta_0$$

is a bijection from $\mathcal{S}$ to the set of faces of Δ_0. For an element Σ of $\mathcal{S}$, we denote by $\langle \Sigma \rangle$ the sublattice of L_{10} generated by Σ. Since Σ is a subset of the basis E_{10} of L_{10}, the sublattice $\langle \Sigma \rangle$ is primitive in L_{10}. Therefore we obtain the following:

Lemma 3.7. *Let* Σ *be an element of* $\mathcal{S}$. *Then* $\langle \Sigma \rangle \in \mathcal{N}$ *and* $\langle \Sigma \rangle = [F_{\Sigma}]^{\perp}$. □

Lemma 3.8. *Let* R *be a negative-definite root sublattice of* L_{10}, *and* $\overline{R}$ *the primitive closure of* R *in* L_{10}. *Then there exists an isometry* $g \in \mathrm{O}^+(L_{10})$ *such that* $\overline{R}^g = \langle \Sigma \rangle$ *for some* $\Sigma \in \mathcal{S}$. *In particular, the lattice* $\overline{R}$ *is a root lattice.*

Proof. Suppose that R is generated by roots $r_1, \dots, r_n$. Then

$$\mathcal{P}(R) := (\{r_1, \dots, r_n\})^{\perp} = ([R]^{\perp} \otimes \mathbb{R}) \cap \mathcal{P}_{10}$$

is non-empty, because $\mathcal{P}(R)$ is a positive cone of the hyperbolic lattice $[R]^{\perp}$. Since $\mathcal{P}(R)$ is contained in a hyperplane $(r_1)^{\perp}$, we see that $\mathcal{P}(R)$ is disjoint from the interior of any Vinberg chamber. Since $\mathcal{P}_{10}$ is tessellated by Vinberg chambers, there exists a Vinberg chamber Δ such that $\mathcal{P}(R) \cap \Delta$ contains a non-empty open subset of $\mathcal{P}(R)$. Therefore $\mathcal{P}(R) \cap \Delta$ is a face of Δ. Then $\gamma(\Delta)^{-1} \in \mathrm{O}^+(L_{10})$ maps $\mathcal{P}(R) \cap \Delta$ to a face F_{Σ} of Δ_0 associated with some $\Sigma \in \mathcal{S}$. Hence $\gamma(\Delta)^{-1}$ maps $\overline{R} = [\mathcal{P}(R) \cap \Delta]^{\perp}$ to $\langle \Sigma \rangle = [F_{\Sigma}]^{\perp}$. □

Corollary 3.9. *The map* $\Sigma \mapsto \langle \Sigma \rangle$ *induces a surjection from* $\mathcal{S}$ *to* $\mathcal{N}/\mathrm{O}^+(L_{10})$.

For $\Sigma \in \mathcal{S}$, we put

$$\mathcal{P}(\Sigma) := (\Sigma)^{\perp} = ([\Sigma]^{\perp} \otimes \mathbb{R}) \cap \mathcal{P}_{10}.$$

Definition 3.10. For $\Sigma, \Sigma' \in \mathcal{S}$, we write $\Sigma \sim \Sigma'$ if there exists an isometry $g \in \mathrm{O}^+(L_{10})$ such that $\langle \Sigma \rangle^g = \langle \Sigma' \rangle$, or equivalently, $\mathcal{P}(\Sigma)^g = \mathcal{P}(\Sigma')$.

In the following, we fix an element $\Sigma \in \mathcal{S}$, and calculate the set $\{\Sigma' \in \mathcal{S} \mid \Sigma' \sim \Sigma\}$ and a finite generating set of the stabilizer subgroup

$$\mathrm{Stab}(\langle \Sigma \rangle, L_{10}) := \{g \in \mathrm{O}^+(L_{10}) \mid \langle \Sigma \rangle^g = \langle \Sigma \rangle\} = \{g \in \mathrm{O}^+(L_{10}) \mid \mathcal{P}(\Sigma)^g = \mathcal{P}(\Sigma)\}$$

of $\langle \Sigma \rangle$ in $\mathrm{O}^+(L_{10})$. We put

$$\mathcal{V}_{\Sigma} := \{\ \Delta \in \mathcal{V} \ \mid \ \mathcal{P}(\Sigma) \cap \Delta \text{ contains a non-empty open subset of } \mathcal{P}(\Sigma)\ \}.$$

Definition 3.11. If $\Delta \in \mathcal{V}_{\Sigma}$, then $\mathcal{P}(\Sigma) \cap \Delta$ is a chamber in the positive cone $\mathcal{P}(\Sigma)$ of the hyperbolic lattice $[\Sigma]^{\perp}$. A closed subset D of $\mathcal{P}(\Sigma)$ is said to be an *induced chamber* if D is written as $\mathcal{P}(\Sigma) \cap \Delta$ by some $\Delta \in \mathcal{V}_{\Sigma}$.

By definition, the positive cone $\mathcal{P}(\Sigma)$ of the hyperbolic lattice $[\Sigma]^\perp$ is covered by induced chambers in such a way that, if D and D' are distinct induced chambers, then their interiors in $\mathcal{P}(\Sigma)$ are disjoint. Let Δ be an arbitrary element of $\mathcal{V}_\Sigma$. Looking at the tessellation of $\mathcal{P}_{10}$ by Vinberg chambers locally around an interior point of the induced chamber $\mathcal{P}(\Sigma)\cap\Delta$, we obtain the following:

Lemma 3.12. *Let $W(\Sigma, L_{10})$ denote the subgroup of $\mathrm{O}^+(L_{10})$ generated by the reflections s_r associated with the roots $r \in \Sigma$. Let Δ be an element of $\mathcal{V}_\Sigma$. Then $W(\Sigma, L_{10})$ acts on the set $\{\Delta' \in \mathcal{V}_\Sigma \,|\, \mathcal{P}(\Sigma)\cap\Delta = \mathcal{P}(\Sigma)\cap\Delta'\}$ simple-transitively.*

Note that, if $\Delta \in \mathcal{V}_\Sigma$, then $\gamma(\Delta)^{-1}$ maps the induced chamber $\mathcal{P}(\Sigma)\cap\Delta$ of $\mathcal{P}(\Sigma)$ to a face of Δ_0. Hence we can define a mapping $\sigma\colon \mathcal{V}_\Sigma \to \mathcal{S}$ by the property

$$(\mathcal{P}(\sigma(\Delta))\cap\Delta_0)^{\gamma(\Delta)} = \mathcal{P}(\Sigma)\cap\Delta, \tag{3.3}$$

or equivalently, the equality ${F_{\sigma(\Delta)}}^{\gamma(\Delta)} = \mathcal{P}(\Sigma)\cap\Delta$. Taking the orthogonal complement of the both sides, we obtain

$$\langle\sigma(\Delta)\rangle^{\gamma(\Delta)} = \langle\Sigma\rangle \tag{3.4}$$

Lemma 3.13. *We have $\{\Sigma' \in \mathcal{S} \,|\, \Sigma' \sim \Sigma\} = \{\sigma(\Delta) \,|\, \Delta \in \mathcal{V}_\Sigma\}$.*

Proof. By (3.4), we have $\sigma(\Delta) \sim \Sigma$ for any $\Delta \in \mathcal{V}_\Sigma$. Suppose that $\Sigma' \sim \Sigma$, and let $h \in \mathrm{O}^+(L_{10})$ be an isometry such that $\mathcal{P}(\Sigma')^h = \mathcal{P}(\Sigma)$. We put $\Delta := \Delta_0^h$ so that $\gamma(\Delta) = h$. Since $(\mathcal{P}(\Sigma')\cap\Delta_0)^h = \mathcal{P}(\Sigma)\cap\Delta$ contains a non-empty open subset of $\mathcal{P}(\Sigma)$, we have $\Delta \in \mathcal{V}_\Sigma$ and $\Sigma' = \sigma(\Delta)$ by the defining property (3.3) of σ. □

Using Lemma 3.12, we can prove the following:

Lemma 3.14. *For $\Delta, \Delta' \in \mathcal{V}_\Sigma$, if $\mathcal{P}(\Sigma)\cap\Delta = \mathcal{P}(\Sigma)\cap\Delta'$, then $\sigma(\Delta) = \sigma(\Delta')$.* □

Let Δ be an element of $\mathcal{V}_\Sigma$, and let $D := \mathcal{P}(\Sigma)\cap\Delta$ be the corresponding induced chamber of $\mathcal{P}(\Sigma)$. Looking at the isomorphism

$$F_{\sigma(\Delta)} = (\mathcal{P}(\sigma(\Delta))\cap\Delta_0) \xrightarrow{\sim} \mathcal{P}(\Sigma)\cap\Delta = D$$

induced by $\gamma(\Delta) \in \mathrm{O}^+(L_{10})$, we see that the set of walls of D is equal to

$$\{\ {F_\Xi}^{\gamma(\Delta)}\ \mid\ \Xi \in \mathcal{S} \text{ satisfies } \Sigma \subset \Xi \text{ and } |\Sigma| + 1 = |\Xi|\ \}.$$

Definition 3.15. Let $w := {F_\Xi}^{\gamma(\Delta)}$ be a wall of the induced chamber D of $\mathcal{P}(\Sigma)$. An induced chamber *adjacent to D across the wall w* is the unique induced chamber D' of $\mathcal{P}(\Sigma)$ such that $D \ne D'$ and that w is a wall of D'. We say that an induced chamber D' is *adjacent to D* if D' is adjacent to D across some wall of D.

Algorithm 3.16. Let D and w be as in Definition 3.15. By the following method, we can obtain a Vinberg chamber $\Delta' \in \mathcal{V}_\Sigma$ and an isometry $\gamma(\Delta') \in \mathrm{O}^+(L_{10})$ such that $\mathcal{P}(\Sigma)\cap\Delta'$ is the induced chamber adjacent to D across w. Let Ξ be the element of $\mathcal{S}$ such that $w = {F_\Xi}^{\gamma(\Delta)}$, and let $W(\Xi, L_{10})$ denote the subgroup of $\mathrm{O}^+(L_{10})$ generated by the reflections s_r associated with the roots $r \in \Xi$. Let ξ be the longest element of the Coxeter group $W(\Xi, L_{10}) \cong W(\langle\Xi\rangle)$, which can be calculated by Algorithm 2.3. Then, if $\mathcal{U}$ is a sufficiently small neighborhood in $\mathcal{P}_{10}$ of a general point of the face F_Ξ of Δ_0, we have

$$\Delta_0^\xi \cap \mathcal{U} = \{\ x \in \mathcal{U}\ \mid\ \langle x, r\rangle \le 0\ \text{ for all }\ r \in \Xi\ \},$$

that is, the Vinberg chamber Δ_0^ξ is opposite to Δ_0 with respect to the face F_Ξ. Then $\Delta' := \Delta_0^{\xi\gamma(\Delta)}$ is the desired Vinberg chamber, and we have $\gamma(\Delta') = \xi\gamma(\Delta)$. ∎

We present the main algorithm of this section, which is based on the generalized Borcherds method ([5], [6], [22]).

Algorithm 3.17. Let an element Σ of $\mathcal{S}$ be given. This algorithm calculates the set $\{\Sigma' \in \mathcal{S} \mid \Sigma' \sim \Sigma\}$ and a generating set of the stabilizer subgroup $\mathrm{Stab}(\langle\Sigma\rangle, L_{10})$ of $\langle\Sigma\rangle$ in $\mathrm{O}^+(L_{10})$. We set

$$\mathcal{D} := [\Delta_0], \quad \gamma(\mathcal{D}) := [\mathrm{id}], \quad \sigma(\mathcal{D}) := [\Sigma], \quad \mathcal{G} := \{\}, \quad i := 0,$$

where id is the identity of $\mathrm{O}^+(L_{10})$. During the calculation, we have the following:

- (i) $\mathcal{D}$ is a list $[\Delta_0, \dots, \Delta_j]$ of elements of $\mathcal{V}_\Sigma$ such that $\sigma(\Delta_\mu) \neq \sigma(\Delta_\nu)$ if $\mu \neq \nu$, where $\sigma\colon \mathcal{V}_\Sigma \to \mathcal{S}$ is defined by (3.3),
- (ii) $\gamma(\mathcal{D})$ is the list $[\gamma(\Delta_0), \dots, \gamma(\Delta_j)]$ of elements of $\mathrm{O}^+(L_{10})$,
- (iii) $\sigma(\mathcal{D})$ is the list $[\sigma(\Delta_0), \dots, \sigma(\Delta_j)]$ of distinct elements of $\mathcal{S}$, and
- (iv) $\mathcal{G}$ is a set of elements of $\mathrm{Stab}(\langle\Sigma\rangle, L_{10})$.

While $i + 1 \leq j + 1 = |\mathcal{D}|$, we execute the following calculation.

(1) Let Δ_i be the $(i+1)$st element of $\mathcal{D}$. By Algorithm 3.16, we calculate Vinberg chambers $\Delta^{(1)}, \dots, \Delta^{(k)}$ in $\mathcal{V}_\Sigma$ such that $\{\mathcal{P}(\Sigma) \cap \Delta^{(\kappa)} \mid \kappa = 1, \dots, k\}$ is the set of induced chambers in $\mathcal{P}(\Sigma)$ adjacent to the induced chamber $\mathcal{P}(\Sigma) \cap \Delta_i$. Note that, in Algorithm 3.16, we also calculate $\gamma(\Delta^{(\kappa)}) \in \mathrm{O}^+(L_{10})$ for $\kappa = 1, \dots, k$.

(2) For $\kappa = 1, \dots, k$, we calculate $\Sigma^{(\kappa)} := \sigma(\Delta^{(\kappa)})$ by $\gamma(\Delta^{(\kappa)})$.

(2-1) If $\Sigma^{(\kappa)} \notin \sigma(\mathcal{D})$, then we add $\Delta^{(\kappa)}$ to $\mathcal{D}$, $\gamma(\Delta^{(\kappa)})$ to $\gamma(\mathcal{D})$, and $\Sigma^{(\kappa)}$ to $\sigma(\mathcal{D})$ at the end of each list.

(2-2) If $\Sigma^{(\kappa)}$ appears at the $(m+1)$st position of $\sigma(\mathcal{D})$, then we have $\Sigma^{(\kappa)} = \sigma(\Delta_m)$, where $\Delta_m \in \mathcal{D}$. We add $g := \gamma(\Delta^{(\kappa)})^{-1} \cdot \gamma(\Delta_m)$ to $\mathcal{G}$. Note that, since $\mathcal{P}(\Sigma^{(\kappa)})^{\gamma(\Delta^{(\kappa)})} = \mathcal{P}(\Sigma) = \mathcal{P}(\sigma(\Delta_m))^{\gamma(\Delta_m)}$, we have $g \in \mathrm{Stab}(\langle\Sigma\rangle, L_{10})$.

(3) We increment i to $i + 1$.

Since $|\mathcal{D}| = |\sigma(\mathcal{D})|$ cannot exceed $|\mathcal{S}| = 1021$, this algorithm terminates. When it terminates, we output $\mathcal{D}$ as $\mathcal{D}_\Sigma$, $\sigma(\mathcal{D})$ as $\sigma(\mathcal{D}_\Sigma)$, and $\mathcal{G}$ as $\mathcal{G}_\Sigma$. ∎

Proposition 3.18. *We have* $\sigma(\mathcal{D}_\Sigma) = \{\Sigma' \in \mathcal{S} \mid \Sigma' \sim \Sigma\}$*. Moreover, the finite set* $\mathcal{G}_\Sigma$ *together with* $W(\Sigma, L_{10})$ *generates* $\mathrm{Stab}(\langle\Sigma\rangle, L_{10})$*.*

Proof. Let G be the subgroup of $\mathrm{O}^+(L_{10})$ generated by the union of $\mathcal{G}_\Sigma$ and $W(\Sigma, L_{10})$. Note that $G \subset \mathrm{Stab}(\langle\Sigma\rangle, L_{10})$, and hence $\mathcal{P}(\Sigma)^g = \mathcal{P}(\Sigma)$ holds for all $g \in G$. First we show that, for any $\Delta \in \mathcal{V}_\Sigma$, there exists an element $g \in G$ such that $\Delta^g \in \mathcal{D}_\Sigma$. An *adjacent sequence* is a sequence $D_{(0)}, \dots, D_{(N)}$ of induced chambers of $\mathcal{P}(\Sigma)$ such that $D_{(\nu-1)}$ and $D_{(\nu)}$ are adjacent for each $\nu = 1, \dots, N$. The number N is called the *length* of the adjacent sequence $D_{(0)}, \dots, D_{(N)}$. For $\Delta \in \mathcal{V}_\Sigma$, let $d(\Delta)$ be the minimum of the lengths of adjacent sequences from $\mathcal{P}(\Sigma) \cap \Delta_0$ to $\mathcal{P}(\Sigma) \cap \Delta$. Since $\mathcal{P}(\Sigma)$ is connected and covered by induced chambers, we have $d(\Delta) < \infty$ for any $\Delta \in \mathcal{V}_\Sigma$. Suppose that

$$\mathcal{B} := \{\ \Delta \in \mathcal{V}_\Sigma \ \mid\ \Delta^g \notin \mathcal{D}_\Sigma \text{ for any } g \in G\ \}$$

is non-empty. Let $\Delta_{\min} \in \mathcal{B}$ be an element such that $N := d(\Delta_{\min}) \leq d(\Delta)$ holds for any $\Delta \in \mathcal{B}$. We put $\Delta_{(0)} := \Delta_0$ and $\Delta_{(N)} := \Delta_{\min}$. Then we have an adjacent sequence

$$\mathcal{P}(\Sigma) \cap \Delta_{(0)},\ \mathcal{P}(\Sigma) \cap \Delta_{(1)},\ \dots,\ \mathcal{P}(\Sigma) \cap \Delta_{(N-1)},\ \mathcal{P}(\Sigma) \cap \Delta_{(N)}$$

of length N from $\mathcal{P}(\Sigma) \cap \Delta_0$ to $\mathcal{P}(\Sigma) \cap \Delta_{\min}$. The minimality of $N = d(\Delta_{\min})$ implies that there exists an element g of G such that $\Delta^g_{(N-1)} \in \mathcal{D}_\Sigma$. We put $\Delta_i := \Delta^g_{(N-1)}$.

Since $\mathcal{P}(\Sigma)^g = \mathcal{P}(\Sigma)$, we see that $\mathcal{P}(\Sigma) \cap \Delta^g_{(N)}$ is an induced chamber adjacent to $\mathcal{P}(\Sigma) \cap \Delta_i$. Therefore, when Algorithm 3.17 processed $\Delta_i \in \mathcal{D}$, there must be a Vinberg chamber $\Delta^{(\kappa)}$ among $\Delta^{(1)}, \dots, \Delta^{(k)}$ such that $\mathcal{P}(\Sigma) \cap \Delta^{(\kappa)} = \mathcal{P}(\Sigma) \cap \Delta^g_{(N)}$. By Lemma 3.12, we have an element $g' \in W(\Sigma, L_{10}) \subset G$ such that $\Delta^{gg'}_{(N)} = \Delta^{(\kappa)}$. Since $\Delta_{\min} \in \mathcal{B}$ implies that $\Delta^{(\kappa)} = \Delta^{gg'}_{\min}$ is not in $\mathcal{D}_\Sigma$, the case (2-2) must have occurred. Therefore $\sigma(\Delta^{(\kappa)}) = \sigma(\Delta^{gg'}_{\min})$ should have been added to $\sigma(\mathcal{D}_\Sigma)$, and there exists an element $\Delta_m \in \mathcal{D}_\Sigma$ such that

$$g'' := \gamma(\Delta^{(\kappa)})^{-1} \cdot \gamma(\Delta_m) \quad \in \quad \mathcal{G}_\Sigma.$$

Then $gg'g'' \in G$ and $\Delta^{gg'g''}_{\min} = \Delta_m \in \mathcal{D}_\Sigma$, which is a contradiction. Thus $\mathcal{B} = \emptyset$ is proved.

Next we prove $\sigma(\mathcal{D}_\Sigma) = \{\Sigma' \in \mathcal{S} \mid \Sigma' \sim \Sigma\}$. Lemma 3.13 implies that all $\Sigma' \in \sigma(\mathcal{D}_\Sigma)$ satisfies $\Sigma' \sim \Sigma$. Suppose that $\Sigma' \in \mathcal{S}$ satisfies $\Sigma' \sim \Sigma$, and let $h \in \mathrm{O}^+(L_{10})$ be an isometry such that $\langle \Sigma \rangle = \langle \Sigma' \rangle^h$. We put $\Delta := \Delta_0^h$. Since $(\mathcal{P}(\Sigma') \cap \Delta_0)^h = \mathcal{P}(\Sigma) \cap \Delta$, we have $\Delta \in \mathcal{V}_\Sigma$. Since $\mathcal{B} = \emptyset$, we have an element $g \in G$ such that $\Delta^g \in \mathcal{D}_\Sigma$. Since $\mathcal{P}(\Sigma)^g = \mathcal{P}(\Sigma)$, we have $(\mathcal{P}(\Sigma') \cap \Delta_0)^{hg} = \mathcal{P}(\Sigma) \cap \Delta^g$, which implies $\Sigma' = \sigma(\Delta^g)$. Therefore Σ' belongs to $\sigma(\mathcal{D}_\Sigma)$.

We prove $G \supset \mathrm{Stab}(\langle \Sigma \rangle, L_{10})$. Let h be an element of $\mathrm{Stab}(\langle \Sigma \rangle, L_{10})$. Then we have $\Delta_0^h \in \mathcal{V}_\Sigma$. Since $\mathcal{B} = \emptyset$, we have an element $g \in G$ such that $\Delta_0^{hg} \in \mathcal{D}_\Sigma$. Since $\mathcal{P}(\Sigma)^{hg} = \mathcal{P}(\Sigma)$, we have $(\mathcal{P}(\Sigma) \cap \Delta_0)^{hg} = \mathcal{P}(\Sigma) \cap \Delta_0^{hg}$, and therefore $\sigma(\Delta_0^{hg}) = \Sigma$. Since elements of $\sigma(\mathcal{D}_\Sigma)$ are all distinct, we have $\Delta_0^{hg} = \Delta_0$ and hence $h = g^{-1} \in G$ holds. □

We execute Algorithm 3.17 for all $\Sigma \in \mathcal{S}$, and confirm that $\Sigma \sim \Sigma'$ holds if and only if their ADE-types $\tau(\Sigma)$ and $\tau(\Sigma')$ coincide. Hence we obtain the following:

Theorem 3.19. *The set* $\mathcal{N}/\mathrm{O}^+(L_{10})$ *is identified with the set* $\tau(\mathcal{S})$ *of* ADE-*types of elements of* $\mathcal{S}$. *In particular, we have* $|\mathcal{N}/\mathrm{O}^+(L_{10})| = 86$.

3.2. **Stabilizer subgroups.** Note that $W(\Sigma, L_{10})$ is a normal subgroup of the stabilizer subgroup $\mathrm{Stab}(\langle \Sigma \rangle, L_{10})$. We put

$$\mathrm{Stab}(\Sigma, L_{10}) := \{\ g \in \mathrm{O}^+(L_{10}) \ \mid \ \Sigma^g = \Sigma\ \},$$

which is a subgroup of $\mathrm{Stab}(\langle \Sigma \rangle, L_{10})$ and is mapped isomorphically to the quotient group $\mathrm{Stab}(\langle \Sigma \rangle, L_{10})/W(\Sigma, L_{10})$. Hence the rows of the following commutative diagram are *splitting* exact sequences:

$$\begin{array}{ccccccccc} 1 & \to & W(\Sigma, L_{10}) & \to & \mathrm{Stab}(\langle \Sigma \rangle, L_{10}) & \overset{\tilde{\kappa}}{\to} & \mathrm{Stab}(\Sigma, L_{10}) & \to & 1 \\ & & \downarrow \wr & & \downarrow \mathrm{res} & & \downarrow \mathrm{res} & & \\ 1 & \to & W(\langle \Sigma \rangle) & \to & \mathrm{O}(\langle \Sigma \rangle) & \overset{\kappa}{\to} & \mathrm{Aut}(\Sigma) & \to & 1, \end{array} \tag{3.5}$$

where the vertical arrows are the restriction homomorphisms.

Corollary 3.20. *The group* $\mathrm{Stab}(\langle \Sigma \rangle, L_{10})$ *is generated by the union of* $W(\Sigma, L_{10})$ *and* $\tilde{\kappa}(\mathcal{G}_\Sigma)$, *and the group* $\mathrm{Stab}(\Sigma, L_{10})$ *is generated by* $\tilde{\kappa}(\mathcal{G}_\Sigma)$.

We can calculate $\tilde{\kappa}(\mathcal{G}_\Sigma)$ by the same method as Algorithm 2.2. We can also calculate a generating set $\kappa(\mathrm{res}(\mathcal{G}_\Sigma)) = \mathrm{res}(\tilde{\kappa}(\mathcal{G}_\Sigma))$ of the subgroup

$$H_\Sigma := \mathrm{Im}(\ \mathrm{Stab}(\langle \Sigma \rangle, L_{10}) \overset{\mathrm{res}}{\to} \mathrm{O}(\langle \Sigma \rangle) \overset{\kappa}{\to} \mathrm{Aut}(\Sigma)\ ) \tag{3.6}$$

of $\mathrm{Aut}(\Sigma)$.

3.3. **Embeddings of an ADE-configuration.** Let Φ be an ADE-configuration with $|\Phi| < 10$. Let $\mathrm{Emb}(\Phi)$ denote the set of all embeddings of Φ into L_{10}. Then $\mathrm{Aut}(\Phi)$ acts on $\mathrm{Emb}(\Phi)$ from the left, and $\mathrm{O}^+(L_{10})$ acts on $\mathrm{Emb}(\Phi)$ from the right. In this section, we calculate the set $\mathrm{Aut}(\Phi)\backslash\mathrm{Emb}(\Phi)/\mathrm{O}^+(L_{10})$, and prove Theorem 1.3.

Let $f\colon \Phi \hookrightarrow L_{10}$ be an embedding. We denote by Φ_f the image of f, by R_f the sublattice of L_{10} generated by Φ_f, and by $\overline{R}_f$ the primitive closure of R_f in L_{10}. Then $\overline{R}_f$ corresponds to an even overlattice $R_{(f)}$ of $\langle\Phi\rangle$ via the isometry $\langle\Phi\rangle \xrightarrow{\sim} R_f$ given by f, and we have $\overline{R}_f \in \mathcal{N}$ by Lemma 3.8. Recall that $\tau(\mathcal{S})$ is the set of ADE-types of all $\Sigma \in \mathcal{S}$. We consider the following condition on an even overlattice $\overline{R}$ of $\langle\Phi\rangle$.

($\sharp$) $\overline{R}$ is a root lattice whose ADE-type belongs to $\tau(\mathcal{S})$.

Then Theorem 3.19 implies the following:

(a) For any $f \in \mathrm{Emb}(\Phi)$, the even overlattice $R_{(f)}$ of $\langle\Phi\rangle$ satisfies ($\sharp$), and there exist an element $g \in \mathrm{O}^+(L_{10})$ and an element $\Sigma \in \mathcal{S}$ such that $\overline{R}_{fg} = \langle\Sigma\rangle$.

(b) Suppose that an even overlattice $\overline{R}$ of $\langle\Phi\rangle$ satisfies ($\sharp$). Then there exist an element $\Sigma \in \mathcal{S}$ and an isometry $\bar{f}\colon \overline{R} \xrightarrow{\sim} \langle\Sigma\rangle$. The restriction of $\bar{f}$ to $\Phi \subset \overline{R}$ gives an embedding $f \in \mathrm{Emb}(\Phi)$.

Therefore we can calculate $\mathrm{Aut}(\Phi)\backslash\mathrm{Emb}(\Phi)/\mathrm{O}^+(L_{10})$ by the following method.

(1) Let $\mathcal{L}(\Phi)$ denote the set of even overlattices of $\langle\Phi\rangle$, on which $\mathrm{Aut}(\Phi)$ acts from the right. We calculate the set $\mathcal{L}(\Phi)/\mathrm{Aut}(\Phi)$ of orbits of this action, and for each orbit $o \in \mathcal{L}(\Phi)/\mathrm{Aut}(\Phi)$, we choose a representative $\overline{R} \in o$ and calculate the stabilizer subgroup $\mathrm{Stab}(\overline{R}, \Phi)$ of $\overline{R}$ in the finite group $\mathrm{Aut}(\Phi)$.

(2) For $\overline{R} \in \mathcal{L}(\Phi)$, let $[\overline{R}] \in \mathcal{L}(\Phi)/\mathrm{Aut}(\Phi)$ denote the orbit containing $\overline{R}$. Similarly, for $\Sigma \in \mathcal{S}$, let $[\Sigma] \in \mathcal{N}/\mathrm{O}^+(L_{10})$ denote the orbit containing $\langle\Sigma\rangle \in \mathcal{N}$. We define a subset $\mathcal{I}_\Phi$ of $\mathcal{L}(\Phi)/\mathrm{Aut}(\Phi) \times \mathcal{N}/\mathrm{O}^+(L_{10})$ by

$$\mathcal{I}_\Phi := \{\ ([\overline{R}], [\Sigma])\ \mid\ \overline{R} \text{ satisfies } (\sharp) \text{ and } \tau(\overline{R}) = \tau(\Sigma)\ \}.$$

For each pair $([\overline{R}], [\Sigma]) \in \mathcal{I}_\Phi$, we define a set $\overline{\mathrm{emb}}([\overline{R}], [\Sigma])$ as follows. Let $\mathrm{Isom}(\overline{R}, \langle\Sigma\rangle)$ denote the set of all isomorphisms from $\overline{R}$ to $\langle\Sigma\rangle$, on which $\mathrm{Stab}(\overline{R}, \Phi)$ acts from the left and $\mathrm{Stab}(\langle\Sigma\rangle, L_{10})$ acts from the right. We put

$$\overline{\mathrm{emb}}([\overline{R}], [\Sigma]) := \mathrm{Stab}(\overline{R}, \Phi)\backslash\mathrm{Isom}(\overline{R}, \langle\Sigma\rangle)/\mathrm{Stab}(\langle\Sigma\rangle, L_{10}).$$

Then the set $\mathrm{Aut}(\Phi)\backslash\mathrm{Emb}(\Phi)/\mathrm{O}^+(L_{10})$ is the disjoint union of $\overline{\mathrm{emb}}([\overline{R}], [\Sigma])$, where $([\overline{R}], [\Sigma])$ runs through the set $\mathcal{I}_\Phi$.

3.3.1. *Execution of task* (1). The task (1) above can be easily carried out by Proposition 2.5. We calculate the set of all totally isotropic subgroups of the discriminant form $q_{\langle\Phi\rangle}$ up to the action of $\mathrm{Aut}(\Phi)$. Executing this calculation for all 157 ADE-configurations Φ with $|\Phi| < 10$, we obtain the following:

Theorem 3.21. *Let Φ be an* ADE-*configuration with $|\Phi| < 10$. Suppose that $\overline{R}_1$ and $\overline{R}_2$ are even overlattices of $\langle\Phi\rangle$. If $\overline{R}_1$ and $\overline{R}_2$ are root lattices with the same* ADE-*type, then there exists an automorphism $g \in \mathrm{Aut}(\Phi)$ such that $\overline{R}_1^g = \overline{R}_2$.*

Example 3.22. Suppose that $\Phi = \{r_1, \dots, r_n\}$ is of ADE-type nA_1. Then the discriminant group $A_{\langle\Phi\rangle}$ of $\langle\Phi\rangle$ is an n-dimensional $\mathbb{F}_2$-vector space with basis

$\dim \mathcal{C}$	$\mathcal{C}$	$\tau(\overline{R})$	Condition ($\sharp$)
0	0	$8A_1$	no
1	$\langle[5678]\rangle$	$4A_1+D_4$	no
1	$\langle[12\dots8]\rangle$	not a root lattice	no
2	$\langle[3678],\ \ [4578]\rangle$	$2A_1+D_6$	no
2	$\langle[1678],\ \ [2345]\rangle$	$2D_4$	no
3	$\langle[2678],\ \ [3578],\ \ [4568]\rangle$	A_1+E_7	yes
3	$\langle[1678],\ \ [2578],\ \ [3478]\rangle$	D_8	yes
4	$\langle[1678],\ \ [2578],\ \ [3568],\ \ [4567]\rangle$	E_8	yes

TABLE 3.1. Even overlattices of the root lattice of type $8A_1$

$r_i^\vee := -r_i/2 \bmod \langle\Phi\rangle$ $(i=1,\dots,n)$, and $q_{\langle\Phi\rangle}$ is given by

$$q_{\langle\Phi\rangle}(a_1 r_1^\vee + \dots + a_n r_n^\vee) = -(a_1^2+\dots+a_n^2)/2 \bmod 2\mathbb{Z}, \quad \text{where } a_1,\dots,a_n \in \mathbb{F}_2.$$

Hence a subspace $\mathcal{C}$ of $A_{\langle\Phi\rangle} = \mathbb{F}_2^n$ is totally isotropic with respect to $q_{\langle\Phi\rangle}$ if and only if $\mathcal{C}$ is a doubly-even linear code in $\mathbb{F}_2^n$. (See Ebeling [8] for the terminologies on codes.) We classify these codes up to the action of $\mathrm{Aut}(\Phi) \cong \mathfrak{S}_n$ by a brute-force method. In the case $n=8$, there exist exactly 8 doubly-even linear codes in $\mathbb{F}_2^8$ up to $\mathfrak{S}_8$. The corresponding even overlattices of $\langle\Phi\rangle$ are given in Table 3.1, where $[i_1\dots i_k]$ denotes the codeword $(r_{i_1}^\vee+\dots+r_{i_k}^\vee) \bmod \langle\Phi\rangle$ in $\mathbb{F}_2^8$.

3.3.2. *Execution of task* (2). Suppose that $([\overline{R}],[\Sigma])$ is an element of $\mathcal{I}_\Phi$. We describe a method to calculate the set $\overline{\mathrm{emb}}([\overline{R}],[\Sigma])$. We can find an ADE-basis of the even overlattice $\overline{R}$ by the method described in Section 2.4, and hence we can write an element $\varphi_0 \in \mathrm{Isom}(\overline{R},\langle\Sigma\rangle)$ explicitly. Using φ_0 as a reference point, we identify $\mathrm{Isom}(\overline{R},\langle\Sigma\rangle)$ with $\mathrm{O}(\langle\Sigma\rangle)$. Moreover, since we have a natural injective homomorphism $\mathrm{Stab}(\overline{R},\Phi) \hookrightarrow \mathrm{O}(\overline{R})$, we can regard $\mathrm{Stab}(\overline{R},\Phi)$ as a subgroup of $\mathrm{O}(\langle\Sigma\rangle)$.

Remark 3.23. By Algorithm 2.2, we can choose $\varphi_0 \in \mathrm{Isom}(\overline{R},\langle\Sigma\rangle)$ in such a way that the image by φ_0 of the connected component

$$\{\, x \in \overline{R}\otimes\mathbb{R} \mid \langle x,r\rangle > 0 \text{ for all } r\in\Phi \,\}$$

of $(\overline{R}\otimes\mathbb{R})^\circ = (\langle\Phi\rangle\otimes\mathbb{R})^\circ$ contains the connected component

$$\{\, y \in \langle\Sigma\rangle\otimes\mathbb{R} \mid \langle y,r\rangle > 0 \text{ for all } r\in\Sigma \,\}$$

of $(\langle\Sigma\rangle\otimes\mathbb{R})^\circ$. See Section 2.4.

Let H_Φ denote the image of $\mathrm{Stab}(\overline{R},\Phi) \subset \mathrm{O}(\langle\Sigma\rangle)$ by the quotient homomorphism $\mathrm{O}(\langle\Sigma\rangle) \to \mathrm{Aut}(\Sigma)$ by $W(\langle\Sigma\rangle)$, which can be calculated by Algorithm 2.2. Since $\mathrm{Stab}(\langle\Sigma\rangle, L_{10})$ contains $W(\Sigma, L_{10})$, we see that

$$\overline{\mathrm{emb}}([\overline{R}],[\Sigma]) := \mathrm{Stab}(\overline{R},\Phi)\backslash\mathrm{O}(\langle\Sigma\rangle)\,/\,\mathrm{Stab}(\langle\Sigma\rangle,L_{10}) = H_\Phi\backslash\mathrm{Aut}(\Sigma)\,/\,H_\Sigma,$$

where H_Σ is defined by (3.6). By this computation, we obtain the following:

Theorem 3.24. *The set* $\overline{\mathrm{emb}}([\overline{R}],[\Sigma])$ *consists of a single element for any* ADE-*configuration* Φ *with* $|\Phi| < 10$ *and any pair* $([\overline{R}],[\Sigma]) \in \mathcal{I}_\Phi$.

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. The assertion (1) is a corollary of Lemma 3.8. The assertions (2) and (3) follow from Theorems 3.19, 3.21 and 3.24. □

3.4. **The stabilizer subgroup of Φ_f in $\mathrm{O}^+(L_{10})$.** Suppose that $f \in \mathrm{Emb}(\Phi)$ and $\Sigma \in \mathcal{S}$ satisfy $\overline{R}_f = \langle\Sigma\rangle$, and that f induces an element $\varphi_0 \in \mathrm{Isom}(\overline{R}, \langle\Sigma\rangle)$ satisfying the condition in Remark 3.23. We put

$$\mathrm{Stab}(\Phi_f, L_{10}) := \{\ g \in \mathrm{O}^+(L_{10})\ \mid\ \Phi_f^g = \Phi_f\ \}.$$

It is obvious that $\mathrm{Stab}(\Phi_f, L_{10}) \subset \mathrm{Stab}(\langle\Sigma\rangle, L_{10})$. The purpose of this section is to calculate a finite generating set $\mathcal{G}'_\Phi$ of $\mathrm{Stab}(\Phi_f, L_{10})$. This set $\mathcal{G}'_\Phi$ will be used in the next section for the classifications of strongly equivalence classes of RDP-Enriques surfaces.

The following general algorithm is used several times in this section.

Algorithm 3.25. Let G be a group generated by $\gamma_1, \dots, \gamma_N \in G$. Suppose that G acts on a finite set S, and let s_0 be an element of S. This algorithm calculates a set $\mathcal{R}_0$ of elements of G such that $g \mapsto s_0^g$ gives a bijection from $\mathcal{R}_0$ to the orbit $s_0^G := \{s_0^g \mid g \in G\}$ of s_0 under the action of G. This algorithm also calculates a finite generating set $\mathcal{G}_0$ of the stabilizer subgroup

$$\mathrm{Stab}(s_0, G) := \{\ g \in G\ \mid\ s_0^g = s_0\ \}.$$

We set $\Gamma := \{\gamma_1, \dots, \gamma_N, \gamma_1^{-1}, \dots, \gamma_N^{-1}\}$. We then put $h_0 := \mathrm{id} \in G$, and

$$\mathcal{R} := [h_0], \quad \mathcal{O} := [s_0], \quad \mathcal{G} := \{\ \}, \quad i := 0.$$

During the calculation, we have the following:

(i) $\mathcal{R}$ is a list $[h_0, h_1, \dots, h_j]$ of elements of G, and $\mathcal{O}$ is the list $[s_0, s_1, \dots, s_j]$ of *distinct* elements of s_0^G such that $s_\mu = s_0^{h_\mu}$ holds for $\mu = 0, \dots, j$, and
(ii) $\mathcal{G}$ is a set of elements of $\mathrm{Stab}(s_0, G)$.

While $i + 1 \le j + 1 = |\mathcal{R}| = |\mathcal{O}|$, we execute the following calculation.

(1) Let h_i be the $(i+1)$st element of $\mathcal{R}$, and let $s_i = s_0^{h_i}$ be the $(i+1)$st element of $\mathcal{O}$. For each $\gamma \in \Gamma$, we execute the following:
  (1-1) If $s_i^\gamma = s_0^{h_i\gamma} \notin \mathcal{O}$, then we add $h_i\gamma$ to $\mathcal{R}$ and s_i^γ to $\mathcal{O}$ at the end of each list, whereas
  (1-2) if $s_i^\gamma = s_0^{h_i\gamma}$ is equal to the $(m+1)$st element $s_m = s_0^{h_m}$ of $\mathcal{O}$, then we add $h_i\gamma h_m^{-1} \in \mathrm{Stab}(s_0, G)$ to $\mathcal{G}$.
(2) We increment i to $i + 1$.

Since $|\mathcal{O}| = |\mathcal{R}|$ cannot exceed $|S|$, this algorithm terminates. When it terminates, it outputs $\mathcal{R}$ as $\mathcal{R}_0$, $\mathcal{O}$ as $\mathcal{O}_0$, and $\mathcal{G}$ as $\mathcal{G}_0$. ▮

Proposition 3.26. *The set* $\mathcal{O}_0$ *is equal to* s_0^G, *and* $\mathcal{G}_0$ *generates* $\mathrm{Stab}(s_0, G)$.

Proof. The proof is similar to and simpler than the proof of Proposition 3.18. The details are left to the reader. □

Remark 3.27. We have calculated a finite generating set $\tilde{\kappa}(\mathcal{G}_\Sigma) \cup W(\Sigma, L_{10})$ of $\mathrm{Stab}(\langle\Sigma\rangle, L_{10})$, and hence we can obtain a finite generating set of $\mathrm{Stab}(\Phi_f, L_{10})$ by applying Algorithm 3.25 to the action of $\mathrm{Stab}(\langle\Sigma\rangle, L_{10})$ on the set of ADE-configurations of roots in $\langle\Sigma\rangle$. This method, however, takes too much time and memory for many $f \in \mathrm{Emb}(\Phi)$. Therefore we use the following method.

We put

$$V := \langle\Sigma\rangle \otimes \mathbb{R} = \langle\Phi_f\rangle \otimes \mathbb{R}.$$

Then $\mathrm{Stab}(\langle\Sigma\rangle, L_{10})$ acts on V via the restriction homomorphism res (see (3.5)). We denote the action of $g \in \mathrm{Stab}(\langle\Sigma\rangle, L_{10})$ on V simply by $v \mapsto v^g$ without writing res. For an ADE-configuration Ψ of roots of $\langle\Sigma\rangle$, we put

$$\Gamma(\Psi) := \{\ y \in V \ \mid\ \langle y, r\rangle \ge 0 \ \text{ for all }\ r \in \Psi\ \}.$$

By the assumption on f (see Remark 3.23), we have

$$\Gamma(\Sigma) \subset \Gamma(\Phi_f).$$

We calculate the finite set

$$\mathcal{W}(\Phi_f, \Sigma) := \{\ w \in W(\Sigma, L_{10}) \ \mid\ \Gamma(\Sigma)^w \subset \Gamma(\Phi_f)\ \}.$$

Recall that $\tilde{\kappa}(\mathcal{G}_\Sigma)$ is a generating set of the subgroup $\mathrm{Stab}(\Sigma, L_{10})$ of $\mathrm{Stab}(\langle\Sigma\rangle, L_{10})$. Therefore, by Algorithm 3.25, we can calculate a finite subset

$$\mathcal{R}_{\Phi_f,\Sigma} = \{h_1, \dots, h_N\}$$

of $\mathrm{Stab}(\Sigma, L_{10})$ such that $g \mapsto \Phi_f^g$ gives a bijection from $\mathcal{R}_{\Phi_f,\Sigma}$ to the orbit

$$\mathcal{O}(\Phi_f) := \{\ \Phi_f^g \ \mid\ g \in \mathrm{Stab}(\Sigma, L_{10})\ \}.$$

We set $\mathcal{G}' := \{\quad\}$. When the calculation below terminates, this set $\mathcal{G}'$ is the desired generating set $\mathcal{G}'_\Phi$ of $\mathrm{Stab}(\Phi_f, L_{10})$. Recall that the upper row of (3.5) is a splitting exact sequence. Hence each element g of $\mathrm{Stab}(\langle\Sigma\rangle, L_{10})$ is uniquely written as $\tilde{\kappa}(g)w$, where $w \in W(\Sigma, L_{10})$. We consider the coset decomposition

$$\mathrm{Stab}(\langle\Sigma\rangle, L_{10}) = \bigsqcup_{w \in W(\Sigma, L_{10})} \mathrm{Stab}(\Sigma, L_{10}) \cdot w,$$

and for each $w \in W(\Sigma, L_{10})$, we consider the set

$$\Xi(w) := (\mathrm{Stab}(\Sigma, L_{10}) \cdot w) \cap \mathrm{Stab}(\Phi_f, L_{10}).$$

If $w \notin \mathcal{W}(\Phi_f, \Sigma)$, then we obviously have $\Xi(w) = \emptyset$. Therefore we assume that $w \in \mathcal{W}(\Phi_f, \Sigma)$. We then calculate the image $\mathcal{O}(\Phi_f)^w$ of the orbit $\mathcal{O}(\Phi_f)$ by w. If $\mathcal{O}(\Phi_f)^w$ does not contain Φ_f, then we have $\Xi(w) = \emptyset$. Therefore we assume that there exists an element h_i of $\mathcal{R}_{\Phi_f,\Sigma}$ such that $\Phi_f^{h_i w} = \Phi_f$. We add $h_i w$ in $\mathcal{G}'$. Every element of $\Xi(w)$ is written uniquely as $h_i g w$, where $g \in \mathrm{Stab}(\Sigma, L_{10})$. Since $h_i g w = h_i w (w^{-1} g w)$, an element $g \in \mathrm{Stab}(\Sigma, L_{10})$ satisfies $h_i g w \in \Xi(w)$ if and only if $(\Phi_f^{w^{-1}})^g = \Phi_f^{w^{-1}}$. Using the finite generating set $\tilde{\kappa}(\mathcal{G}_\Sigma)$ of $\mathrm{Stab}(\Sigma, L_{10})$ and Algorithm 3.25, we calculate a finite generating set $\mathcal{G}''(w)$ of

$$\mathrm{Stab}(\Phi_f^{w^{-1}}, \Sigma, L_{10}) := \{\ g \in \mathrm{Stab}(\Sigma, L_{10}) \ \mid\ (\Phi_f^{w^{-1}})^g = \Phi_f^{w^{-1}}\ \}.$$

We then append $w^{-1}\mathcal{G}''(w)w$ to $\mathcal{G}'$. Thus a finite generating set $\mathcal{G}'_\Phi$ of $\mathrm{Stab}(\Phi_f, L_{10})$ is computed.

The group $\mathrm{Stab}(\Phi_f, L_{10})$ is, in general, infinite. However we have examples as follows.

Example 3.28. We consider the case where $\tau(\Phi) = 8A_1$ and $\tau(\Sigma) = E_8$ (no. 88 of Table 1.1). In this case, the method above still takes too much computation time, but we can calculate $\mathrm{Stab}(\Phi_f, L_{10})$ as follows. Since $\overline{R}_f = \langle\Sigma\rangle$ is unimodular, L_{10} is the orthogonal direct-sum of $\langle\Sigma\rangle$ and a hyperbolic plane U. Hence $\mathrm{Stab}(\langle\Sigma\rangle, L_{10})$ is contained in the subgroup $\mathrm{O}(\langle\Sigma\rangle) \times \mathrm{O}^+(U)$ of $\mathrm{O}^+(L_{10})$. Note that $\mathrm{O}^+(U)$ is of order 2. Let g_U be the non-trivial element of $\mathrm{O}^+(U)$. Then the kernel of the natural homomorphism

$$\mathrm{Stab}(\Phi_f, L_{10}) \to \mathrm{Aut}(\Phi)$$

is generated by $(\mathrm{id}, g_U) \in \mathrm{O}(\langle\Sigma\rangle) \times \mathrm{O}^+(U)$, and the image of this homomorphism is $\{\sigma \in \mathrm{Aut}(\Phi) \cong \mathfrak{S}_8 \mid \mathcal{C}^\sigma = \mathcal{C}\}$, where $\mathcal{C}$ is the doubly-even linear code in $A_{\langle\Phi\rangle} \cong \mathbb{F}_2^8$ corresponding to the even overlattice $\overline{R} \cong \langle\Sigma\rangle$, that is, $\mathcal{C}$ is an extended Hamming code [8, Chapter 1]. Therefore we have $|\mathrm{Stab}(\Phi_f, L_{10})| = 2688$, and we can easily obtain a generating set of $\mathrm{Stab}(\Phi_f, L_{10})$.

The case where $\tau(\Phi) = 9A_1$ and $\tau(\Sigma) = A_1 + E_8$ (no. 146 of Table 1.1) is also treated in the similar method. We have $|\mathrm{Stab}(\Phi_f, L_{10})| = 1344$ in this case.

4. Geometric realizability

Let Φ be an ADE-configuration with $|\Phi| < 10$. An embedding $f\colon \Phi \hookrightarrow L_{10}$ is said to be *geometrically realized by an* RDP*-Enriques surface* (Y, ρ) if there exists an isometry $S_Y \xrightarrow{\sim} L_{10}$ that maps $\mathcal{P}_Y$ to $\mathcal{P}_{10}$ and Φ_ρ to Φ_f bijectively. If an embedding $f'\colon \Phi' \hookrightarrow L_{10}$ is equivalent (in the sense of Definition 3.1) to a geometrically realizable embedding $f\colon \Phi \hookrightarrow L_{10}$, then f' is also geometrically realizable. The purpose of this section is to introduce a lattice $\overline{M}_f$ corresponding to the lattice $\overline{M}_\rho$ associated with an RDP-Enriques surface (Y, ρ), and give a criterion for the geometric realizability. By means of these tools, we classify the strong equivalence classes of RDP-Enriques surfaces, and prove Theorem 1.7.

For a lattice L, let $L(2)$ denote the lattice obtained from L by multiplying the intersection form by 2. The orthogonal direct-sum of lattices L and L' is denoted by $L \oplus L'$.

4.1. **Geometry of an Enriques involution.** An involution $\varepsilon\colon X \to X$ of a $K3$ surface X is said to be an *Enriques involution* if ε has no fixed points, or equivalently, the quotient $X/\langle\varepsilon\rangle$ is an Enriques surface. The following is due to Nikulin [15].

Proposition 4.1 (Nikulin). *Let X be a $K3$ surface, and let $\varepsilon\colon X \to X$ be an involution, which acts on $H^2(X, \mathbb{Z})$ from the right by the pull-back. Suppose that ε acts on $H^0(X, \Omega_X^2)$ as the multiplication by -1. Let S_X^+ denote the primitive sublattice $\{v \in S_X \mid v^\varepsilon = v\}$ of S_X. Then ε is an Enriques involution if and only if* $\mathrm{rank}(S_X^+) = 10$ *and $A_{S_X^+} \cong \mathbb{F}_2^{10}$, where $A_{S_X^+}$ is the discriminant group of S_X^+*

Let $\varepsilon\colon X \to X$ be an Enriques involution, and let $\pi\colon X \to Y$ denote the universal covering of the Enriques surface $Y := X/\langle\varepsilon\rangle$. Then S_Y is isomorphic to L_{10}, and $\pi^*\colon S_Y \to S_X$ induces an isometry $S_Y(2) \cong \pi^* S_Y = S_X^+$. We also have $\pi^{*-1}(\mathcal{P}_X) = \mathcal{P}_Y$. By an isometry $S_Y \cong L_{10}$ that maps $\mathcal{P}_Y$ to $\mathcal{P}_{10}$, the notion of Vinberg chambers in $\mathcal{P}_Y$ makes sense. We also use the notation (3.1) and (3.2) for S_Y. We put

$$\begin{aligned} \mathrm{Nef}(X) &:= \{\, x \in \mathcal{P}_X \mid \langle x, C'\rangle \ge 0 \text{ for all curves } C' \subset X \,\}, \\ \mathrm{Nef}(Y) &:= \{\, x \in \mathcal{P}_Y \mid \langle x, C\rangle \ge 0 \text{ for all curves } C \subset Y \,\}. \end{aligned}$$

It is well-known that $\mathrm{Nef}(X)$ is a standard fundamental domain of the action of $W(S_X)$ on $\mathcal{P}_X$. It is obvious that $\pi^{*-1}(\mathrm{Nef}(X)) = \mathrm{Nef}(Y)$. Since $\mathrm{Nef}(Y)$ is bounded by the hyperplanes $([C])^\perp$ of $\mathcal{P}_Y$ perpendicular to the classes of smooth rational curves C on Y, and a smooth rational curve on Y has self-intersection number -2, the cone $\mathrm{Nef}(Y)$ is tessellated by Vinberg chambers.

In the following, we fix an ample class $a \in S_Y$ such that $\langle r, a\rangle \neq 0$ for any root r of S_Y. For example, we choose an interior point of a Vinberg chamber contained in $\mathrm{Nef}(Y)$. In particular, if R is a negative-definite root sublattice of S_Y, then $\langle -, a\rangle$ is an element of $\mathrm{Hom}(R, \mathbb{R})^\circ$ in the notation of Section 2.4. Since $\pi^* a$ is ample on X, we have $\langle r', \pi^* a\rangle \neq 0$ for any root r' of S_X. We put

$$N_Y := \{\ v \in S_X \ \mid\ \langle v, y\rangle = 0 \ \text{ for all }\ y \in \pi^* S_Y\ \}.$$

If N_Y contained a root, then there would be an effective divisor of X contracted by π, which is absurd. Hence we have the following:

Lemma 4.2. *The negative-definite even lattice N_Y contains no roots.* □

Note that ε acts on N_Y as the multiplication by -1. We put

$$\mathcal{T}_Y := \{\ t \in N_Y \ \mid\ \langle t, t\rangle = -4\ \}.$$

If $r \in \mathrm{Roots}(S_Y)$ and $t \in \mathcal{T}_Y$, then $(\pi^* r + t)/2 \in S_X \otimes \mathbb{Q}$ is of square-norm -2. A root r of S_Y is said to be *liftable* if $r' := (\pi^* r + t)/2$ is contained in S_X for some $t \in \mathcal{T}_Y$, and when this is the case, the root r' of S_X is called a *lift* of r.

Lemma 4.3. *For a liftable root $r \in S_Y$, there exist exactly two lifts r' and r'' of r, and they satisfy $\pi^* r = r' + r''$, $r'' = r'^{\varepsilon}$, and $\langle r', r''\rangle = 0$.*

Proof. If $r' = (\pi^* r + t)/2$ with $t \in \mathcal{T}_Y$ is a lift of r, then so is $r'' := (\pi^* r - t)/2$. Suppose that $t' \in \mathcal{T}_Y$ gives a lift $(\pi^* r + t')/2$ of r. Then we have $(t - t')/2 \in N_Y$. By Lemma 4.2, we see that $\langle t, t'\rangle = -4$ or $\langle t, t'\rangle \geq 4$. By the Cauchy-Schwarz inequality for N_Y, we have $|\langle t, t'\rangle| \leq 4$. Therefore we have $t' = \pm t$. □

An effective divisor D of Y is said to be *splitting* if there exists an effective divisor D' of X such that $\pi^*(D) = D' + \varepsilon(D')$ and $\langle D', \varepsilon(D')\rangle = 0$. Note that an effective divisor of Y is splitting if each connected component of its support is simply connected. In particular, a smooth rational curve on Y is splitting.

Lemma 4.4. *Let r be a root of S_Y such that $\langle r, a\rangle > 0$. Then r is the class of a splitting effective divisor of Y if and only if r is liftable.*

Proof. Suppose that $r = [D]$, where D is a splitting effective divisor, and suppose that $\pi^*(D) = D' + \varepsilon(D')$ with $\langle D', \varepsilon(D')\rangle = 0$. Then $t := [D'] - [\varepsilon(D')]$ belongs to $\mathcal{T}_Y$ and $[D'] = (\pi^* r + t)/2 \in S_X$. Therefore r is liftable. Conversely, suppose that r has lifts r' and $r'' = r'^{\varepsilon}$. Since r' is a root of S_X and $\langle r', \pi^* a\rangle > 0$, there exists an effective divisor D' of X such that $r' = [D']$. Then we have $\pi^* r = r' + r''$, $r'' = [\varepsilon(D')]$ and $\langle D', \varepsilon(D')\rangle = 0$ by Lemma 4.3. Let D be the effective divisor of Y such that $\pi^*(D) = D' + \varepsilon(D')$. Then D is splitting and we have $r = [D]$. □

Let H be a nef divisor of Y such that $\langle H, H\rangle > 0$, and let $h \in \mathrm{Nef}(Y) \cap S_Y$ be the class of H. If k is a sufficiently large and divisible integer, then the complete linear system $|kH|$ is base-point free and the Stein factorization of the morphism $Y \to \mathbb{P}^N$ induced by $|kH|$ gives rise to an RDP-Enriques surface

$$\rho_h \colon Y \to \overline{Y}.$$

We calculate the set $\Phi_h := \Phi_{\rho_h}$ of the classes of smooth rational curves contracted by ρ_h. Let $[h]^\perp$ denote the orthogonal complement in S_Y of the sublattice $[h] := \mathbb{Z}h$ generated by h. Then $[h]^\perp$ is negative-definite. We put

$$\Pi_h^+ := \{\, r \in \mathrm{Roots}([h]^\perp) \mid \langle r, a\rangle > 0 \,\},\;\; L_h^+ := \{\, r \in \Pi_h^+ \mid r \text{ is liftable} \,\}, \tag{4.1}$$

and consider the root sublattice $\langle L_h^+\rangle$ of $[h]^\perp$ generated by L_h^+ (possibly of rank 0).

Proposition 4.5. *The* ADE*-configuration* Φ_h *of the* RDP*-Enriques surface* (Y, ρ_h) *is equal to the* ADE*-basis of* $\langle L_h^+\rangle$ *associated with the linear form* $\langle -, a\rangle$ *by the correspondence described in Section 2.4.*

Proof. By Lemma 4.4, every element r of L_h^+ is the class of a splitting effective divisor D_r contracted by ρ_h. Since the class of any smooth rational curve contracted by ρ_h is in L_h^+, the divisor D_r is irreducible if and only if $r \in L_h^+$ is indecomposable in L_h^+ in the sense of Definition 2.4. Therefore Φ_h is equal to the set Φ'_h of indecomposable elements of L_h^+. In particular, the set Φ'_h is an ADE-configuration of roots of S_Y, and the vectors of Φ'_h are linearly independent. Since every element of L_h^+ is a linear combination of the indecomposable elements, we have $\langle \Phi'_h\rangle = \langle L_h^+\rangle$. Since the ADE-basis of the root lattice $\langle L_h^+\rangle$ associated with the linear form $\langle -, a\rangle \in \mathrm{Hom}(\langle L_h^+\rangle, \mathbb{R})^\circ$ is unique, we obtain the proof. □

4.2. **Lattices associated with an** RDP**-Enriques surface.** Let (Y, ρ) be an RDP-Enriques surface, and $a \in S_Y$ an ample class such that $\langle a, r\rangle \neq 0$ holds for any $r \in \mathrm{Roots}(S_Y)$. Let $C_1, \dots, C_n$ be the smooth rational curves on Y contracted by $\rho\colon Y \to \overline{Y}$, so that $\Phi_\rho = \{[C_1], \dots, [C_n]\}$. Since the Dynkin diagram of Φ_ρ is a disjoint union of trees, we have the following:

Lemma 4.6. *Let D be an effective divisor of Y contracted by ρ. Then each connected component of the support of D is simply connected. In particular, D is splitting.* □

Let $\pi\colon X \to Y$ be the universal covering, and let C'_i and C''_i be the two connected components of $\pi^{-1}(C_i)$. Lemma 4.6 implies that, interchanging C'_i and C''_i if necessary, we can assume that

$$\langle C_i, C_j\rangle = \langle C'_i, C'_j\rangle = \langle C''_i, C''_j\rangle, \quad \text{and} \quad \langle C'_i, C''_j\rangle = 0, \tag{4.2}$$

hold for all i, j. We put

$$\phi([C_i]) := [C'_i] - [C''_i] \;\; \in \;\; N_Y.$$

Note that $[C'_i]$ and $[C''_i]$ are the lifts of $[C_i]$ associated with $\phi([C_i]) \in \mathcal{T}_Y$.

Remark 4.7. Let c be the number of connected components of the Dynkin diagram of Φ_ρ. Then there exist exactly 2^c possibilities for the choice of the map ϕ.

By (4.2), we have

$$\langle \phi([C_i]), \phi([C_j])\rangle = 2\langle [C_i], [C_j]\rangle, \tag{4.3}$$

and hence ϕ defines an embedding

$$\phi\colon \langle \Phi_\rho\rangle(2) \hookrightarrow N_Y. \tag{4.4}$$

Recall that R_ρ is the sublattice of S_Y generated by Φ_ρ, and that $\overline{R}_\rho$ is the primitive closure of R_ρ in S_Y. Recall also that $\Phi_\rho^\sim$ is the subset $\{[C'_1], [C''_1], \dots, [C'_n], [C''_n]\}$ of $\mathrm{Roots}(S_X)$, that M_ρ is the sublattice of S_X generated by $\pi^* S_Y$ and $\Phi_\rho^\sim$, and that

$\overline{M}_\rho$ is the primitive closure of M_ρ in S_X. Then M_ρ is an even overlattice of the orthogonal direct-sum

$$B_\rho := \pi^* S_Y \oplus \operatorname{Im}\phi.$$

Thus we obtain a sequence of inclusions

$$\text{(4.5)} \qquad \pi^*\Phi_\rho \subset \pi^* R_\rho \subset \pi^*\overline{R}_\rho \subset \pi^* S_Y \subset B_\rho \subset M_\rho \subset \overline{M}_\rho.$$

Lemma 4.8. *We have* $\Phi_\rho^\sim = \{\, r' \in \operatorname{Roots}(\overline{M}_\rho) \mid r' + r'^\varepsilon \in \pi^*\Phi_\rho \,\}$.

Proof. The inclusion $\subset$ is obvious. Suppose that $r' \in \operatorname{Roots}(\overline{M}_\rho)$ satisfies $r' + r'^\varepsilon = \pi^* r$ for some $r \in \Phi_\rho$. Since $\langle r, a\rangle > 0$, we have $\langle r', \pi^* a\rangle > 0$, and hence we have an effective divisor D' of X such that $r' = [D']$. Let D be the effective divisor of Y such that $\pi^* D = D' + \varepsilon(D')$. Then $r = [D] \in \Phi_\rho$ and hence $D = C_i$ for some i. Therefore $r' \in \Phi_\rho^\sim$. □

We can now prove Lemma 1.6 stated in Introduction.

Proof of Lemma 1.6. Since $\pi^* S_Y$ (resp. $\pi'^* S_{Y'}$) is the invariant sublattice of the Enriques involution on X (resp. on X'), the strong equivalence isometry μ is compatible with the action of the Enriques involutions on $\overline{M}_\rho$ and $\overline{M}_{\rho'}$. Since μ_Y maps Φ_ρ to $\Phi_{\rho'}$ bijectively, Lemma 4.8 implies that μ maps $\Phi_\rho^\sim$ to $\Phi_{\rho'}^\sim$ bijectively. □

Proposition 4.9. *The set* $\{r \in \operatorname{Roots}(\overline{R}_\rho) \mid r \text{ is liftable}\}$ *is equal to* $\operatorname{Roots}(R_\rho)$.

Proof. Let $h_\rho \in S_Y$ be the class of the pull-back by $\rho\colon Y \to \overline{Y}$ of an ample divisor of $\overline{Y}$. Since Φ_ρ is orthogonal to h_ρ, the sublattices R_ρ and $\overline{R}_\rho$ of S_Y are orthogonal to h_ρ. Suppose that $r \in \operatorname{Roots}(\overline{R}_\rho)$ is liftable. Replacing r by $-r$ if necessary, we can assume that $\langle r, a\rangle > 0$. By Lemma 4.4, the root r is the class of a splitting effective divisor D. Since $\langle r, h_\rho\rangle = 0$, the divisor D is contracted by ρ. Therefore D is a linear combination of $C_1, \dots, C_n$. Thus we have $r \in \operatorname{Roots}(R_\rho)$. Suppose that $r \in \operatorname{Roots}(R_\rho)$. Replacing r by $-r$ if necessary, we can assume that $\langle r, a\rangle > 0$, and hence there exists an effective divisor $D = \sum a_i C_i$ such that $r = [D]$. Then D is splitting by Lemma 4.6, and hence r is liftable by Lemma 4.4. □

4.3. **A criterion for geometric realizability.** For an embedding $f\colon \Phi \hookrightarrow L_{10}$ of an ADE-configuration $\Phi = \{r_1, \dots, r_n\}$ into L_{10}, we will construct a sequence (4.7), which is a lattice-theoretic counterpart of (4.5). For $r_i \in \Phi$, we denote by $r_i^+ \in \Phi_f$ the image of r_i by f, so that $\Phi_f = \{r_1^+, \dots, r_n^+\}$. Let $\Phi^- = \{r_1^-, \dots, r_n^-\}$ be a copy of Φ with a fixed isomorphism $\Phi \cong \Phi^-$ denoted by $r_i \mapsto r_i^-$. We put

$$B_\Phi := L_{10}(2) \oplus \langle \Phi^- \rangle(2).$$

We denote by

$$\varpi^*\colon L_{10} \xrightarrow{\sim} L_{10}(2), \quad \varphi\colon \langle\Phi^-\rangle \xrightarrow{\sim} \langle\Phi^-\rangle(2),$$

the identity maps on the underlying $\mathbb{Z}$-modules. Hence we have

$$\text{(4.6)} \qquad \langle \varpi^* r_i^+, \varpi^* r_j^+ \rangle = \langle \varphi(r_i^-), \varphi(r_j^-)\rangle = 2\langle r_i, r_j\rangle.$$

We define vectors r_i' and r_i'' of $B_\Phi^\vee \subset B_\Phi \otimes \mathbb{Q}$ by

$$r_i' := (\varpi^* r_i^+ + \varphi(r_i^-))/2, \quad r_i'' := (\varpi^* r_i^+ - \varphi(r_i^-))/2.$$

Let M_f denote the submodule of $B_\Phi^\vee$ generated by B_Φ and $r_1', \dots, r_n'$. Then M_f is an even overlattice of B_Φ. Note that r_i'' also belongs to M_f. Let $\overline{M}_f$ be an even overlattice of M_f. Recall that R_f is the sublattice of L_{10} generated by Φ_f, and that $\overline{R}_f$ is the primitive closure of R_f in L_{10}. Then we have a sequence of inclusions

$$\varpi^*\Phi_f \subset \varpi^*R_f \subset \varpi^*\overline{R}_f \subset \varpi^*L_{10} \subset B_\Phi \subset M_f \subset \overline{M}_f. \tag{4.7}$$

Let $N(\overline{M}_f)$ denote the orthogonal complement of ϖ^*L_{10} in $\overline{M}_f$. We put

$$\mathcal{T}(\overline{M}_f) := \{ \, t \in N(\overline{M}_f) \;\mid\; \langle t, t\rangle = -4 \, \}.$$

A root r of $\overline{R}_f$ is said to be *$\overline{M}_f$-liftable* if there exists an element $t \in \mathcal{T}(\overline{M}_f)$ such that $r' := (\varpi^*r + t)/2 \in \overline{M}_f \otimes \mathbb{Q}$ is contained in $\overline{M}_f$, and when this is the case, we say that r' is an *$\overline{M}_f$-lift* of r.

Lemma 4.10. *Every root of R_f is $\overline{M}_f$-liftable for any even overlattice $\overline{M}_f$ of M_f.*

Proof. Let r^+ be a root of R_f. We can write r^+ as $\sum a_i r_i^+$ with $a_i \in \mathbb{Z}$. Then, putting $r^- := \sum a_i r_i^- \in \langle \Phi^-\rangle$, we have $\varphi(r^-) \in \mathcal{T}(M_f) \subset \mathcal{T}(\overline{M}_f)$, and hence r^+ has an $\overline{M}_f$-lift $(\varpi^*r^+ + \varphi(r^-))/2 = \sum a_i r_i'$. □

Definition 4.11. We define the following conditions (C1)-(C4) on $\overline{M}_f$. Let L_{K3} denote the $K3$ lattice, that is, L_{K3} is an even unimodular lattice of signature $(3, 19)$, which is unique up to isomorphism.

- (C1) The lattice $\overline{M}_f$ can be embedded primitively into L_{K3}.
- (C2) The sublattice ϖ^*L_{10} is primitive in $\overline{M}_f$.
- (C3) The negative-definite even lattice $N(\overline{M}_f)$ contains no roots.
- (C4) The set $\{r \in \mathrm{Roots}(\overline{R}_f) \,|\, r \text{ is } \overline{M}_f\text{-liftable}\}$ is equal to $\mathrm{Roots}(R_f)$.

Definition 4.12. We say that an even overlattice $\overline{M}_f$ of M_f is *strongly realized* by an RDP-Enriques surface (Y, ρ) if there exists an isometry

$$m \colon \overline{M}_f \xrightarrow{\sim} \overline{M}_\rho$$

with the following properties; the isometry m maps ϖ^*L_{10} to π^*S_Y isomorphically, and the isometry $m_Y \colon L_{10} \xrightarrow{\sim} S_Y$ induced by m maps $\mathcal{P}_{10}$ to $\mathcal{P}_Y$ and Φ_f to Φ_ρ bijectively. An isometry $m \colon \overline{M}_f \xrightarrow{\sim} \overline{M}_\rho$ satisfying these conditions is called a *strong-realization isometry.*

Remark 4.13. If an even overlattice $\overline{M}_f$ of M_f is strongly realized by an RDP-Enriques surface (Y, ρ), then $f \colon \Phi \hookrightarrow L_{10}$ is geometrically realized by (Y, ρ).

Theorem 4.14. *If an embedding $f \colon \Phi \hookrightarrow L_{10}$ is geometrically realized by an* RDP-*Enriques surface (Y, ρ), then there exist an even overlattice $\overline{M}_f$ of M_f strongly realized by (Y, ρ).*

Proof. By the assumption, we have an isometry $m_Y \colon L_{10} \xrightarrow{\sim} S_Y$ that maps $\mathcal{P}_{10}$ to $\mathcal{P}_Y$ and Φ_f to Φ_ρ bijectively. We use the notation about (Y, ρ) fixed in Section 4.2, and index the elements $r_1, \dots, r_n$ of Φ in such a way that $m_Y(r_i^+) = [C_i]$ holds for $i = 1, \dots, n$. Then m_Y induces an isometry

$$m^+ \;:\; \varpi^*L_{10} \xrightarrow{\sim} \pi^*S_Y$$

that maps $\varpi^*r_i^+$ to $[\pi^*(C_i)]$. By (4.3), we also have an isometry

$$m^- \;:\; \langle \Phi^-\rangle(2) \xrightarrow{\sim} \mathrm{Im}(\phi \colon \langle \Phi_\rho\rangle(2) \hookrightarrow N_Y)$$

that maps $\varphi(r_i^-)$ to $\phi([C_i]) = [C_i'] - [C_i'']$. Thus we have an isometry

$$m := m^+ \oplus m^- \;:\; B_\Phi \xrightarrow{\sim} B_\rho.$$

Then $m \otimes \mathbb{Q}$ maps r_i' to $[C_i']$ and r_i'' to $[C_i'']$. Therefore we obtain an isometry $m\colon M_f \xrightarrow{\sim} M_\rho$. Let $\overline{M}_f$ be the even overlattice of M_f corresponding to the even overlattice $\overline{M}_\rho$ of M_ρ via m, so that $m\colon M_f \xrightarrow{\sim} M_\rho$ extends to $m\colon \overline{M}_f \xrightarrow{\sim} \overline{M}_\rho$. Then m induces an isomorphism from the sequence (4.7) to the sequence (4.5). In particular, m is a strong-realization isometry. □

Theorem 4.15. *If an even overlattice $\overline{M}_f$ of M_f is strongly realized by an* RDP-*Enriques surface (Y,ρ), then $\overline{M}_f$ satisfies* (C1)-(C4).

Proof. Let $m\colon \overline{M}_f \xrightarrow{\sim} \overline{M}_\rho$ be a strong-realization isometry. Since $\overline{M}_\rho$ is a primitive sublattice of the primitive sublattice S_X of the $K3$-lattice $H^2(X,\mathbb{Z}) \cong L_{\mathrm{K3}}$, the lattice $\overline{M}_f \cong \overline{M}_\rho$ satisfies (C1). Since $\pi^* S_Y$ is primitive in S_X and hence in $\overline{M}_\rho$, the lattice $\overline{M}_f$ satisfies (C2). Since the isometry m maps the lattice $N(\overline{M}_f)$ isomorphically to a sublattice of N_Y, Lemma 4.2 implies that $\overline{M}_f$ satisfies (C3). If $r \in \mathrm{Roots}(\overline{R}_f)$ is $\overline{M}_f$-liftable, then $m(\varpi^* r) \in \mathrm{Roots}(\overline{R}_\rho)$ is liftable, because m maps $\mathcal{T}(\overline{M}_f)$ to a subset of $\mathcal{T}_Y$. Hence the isometry $m_Y\colon L_{10} \xrightarrow{\sim} S_Y$ induced by m induces the horizontal injection in the commutative diagram below:

$$\begin{array}{ccc} \{r \in \mathrm{Roots}(\overline{R}_f) \mid r \text{ is } \overline{M}_f\text{-liftable}\} & \hookrightarrow & \{r \in \mathrm{Roots}(\overline{R}_\rho) \mid r \text{ is liftable}\} \\ {\scriptstyle \text{by Lemma 4.10}} \uparrow & & \wr\| {\scriptstyle \text{ by Proposition 4.9}} \\ \mathrm{Roots}(R_f) & \underset{\text{by } m_Y}{\xrightarrow{\sim}} & \mathrm{Roots}(R_\rho). \end{array}$$

By Proposition 4.9 and Lemma 4.10, we see that the upward injection in the left column of the diagram above is a bijection. Hence $\overline{M}_f$ satisfies (C4). □

Theorem 4.16. *Let $\overline{M}_f$ be an even overlattice of M_f satisfying* (C1)-(C4). *Then there exists an* RDP-*Enriques surface (Y,ρ) that strongly realizes $\overline{M}_f$.*

Proof. The main tool is the Torelli theorem for $K3$ surfaces and the surjectivity of the period map of $K3$ surfaces (see [4, Chapter VIII]). By (C1) for $\overline{M}_f$ and the surjectivity of the period map, there exists a $K3$ surface X equipped with a marking

$$\mu\colon \overline{M}_f \cong S_X.$$

Our task is to construct an Enriques involution ε on X, and an RDP-Enriques surface $\rho\colon Y := X/\langle\varepsilon\rangle \to \overline{Y}$ that strongly realizes $\overline{M}_f$.

Replacing μ by $-\mu$ if necessary, we can assume that μ induces an embedding

$$\mu \circ \varpi^*|_{\mathcal{P}} \ \ : \ \ \mathcal{P}_{10} \hookrightarrow \mathcal{P}_X.$$

We put $\mathcal{P}(\Phi_f) := (\Phi_f)^\perp = ([\Phi_f]^\perp \otimes \mathbb{R}) \cap \mathcal{P}_{10}$, which is a positive cone of the hyperbolic sublattice $[\Phi_f]^\perp = R_f^\perp = \overline{R}_f^\perp$ of L_{10}. There exists a Vinberg chamber Δ of L_{10} such that $\mathcal{P}(\Phi_f) \cap \Delta$ contains a non-empty open subset of $\mathcal{P}(\Phi_f)$. Moving Δ by an element of the subgroup $W(\Phi_f, L_{10}) \subset \mathrm{O}^+(L_{10})$ generated by the reflections s_r with respect to the roots $r \in \Phi_f$, we can assume that Δ satisfies the following:

$$\langle r, v\rangle \ge 0 \quad \text{holds for all } r \in \Phi_f \text{ and } v \in \Delta. \tag{4.8}$$

Let r' be a root of $\overline{M}_f$. By (C3) for $\overline{M}_f$, the locus

$$(r')^\perp := \{\, x \in \mathcal{P}_{10} \ \mid \ \langle \varpi^* x, r'\rangle = 0 \,\}$$

is a hyperplane of $\mathcal{P}_{10}$. Note that the family $\{(r')^\perp \mid r' \in \mathrm{Roots}(\overline{M}_f)\}$ of hyperplanes of $\mathcal{P}_{10}$ is locally finite. Let η' be a general point of $\mathcal{P}(\Phi_f) \cap \Delta \cap (L_{10} \otimes \mathbb{Q})$, let $\mathcal{U}$ be a sufficiently small neighborhood of η' in $\mathcal{P}_{10}$, and let α' be a general point of $\mathcal{U} \cap \Delta \cap (L_{10} \otimes \mathbb{Q})$. We have a positive integer c such that $\alpha := c\alpha' \in L_{10}$ and $\eta := c\eta' \in L_{10}$. Then we have the following:

(i) there exist no roots r' in $\overline{M}_f$ such that $\varpi^*\alpha \in (r')^\perp$,
(ii) there exist no roots r' in $\overline{M}_f$ such that $\langle \varpi^*\alpha, r'\rangle > 0$ and $\langle \varpi^*\eta, r'\rangle < 0$, and
(iii) the set of $r \in \mathrm{Roots}(L_{10})$ satisfying $\langle \eta, r\rangle = 0$ is equal to $\mathrm{Roots}(\overline{R}_f)$.

We then put

$$a_X := \mu \circ \varpi^*|_{\mathcal{P}}(\alpha) \in \mathcal{P}_X \cap S_X, \quad h_X := \mu \circ \varpi^*|_{\mathcal{P}}(\eta) \in \mathcal{P}_X \cap S_X.$$

By (i), composing the marking μ by an element of $W(S_X)$, we can assume that a_X is ample. Then (ii) implies that h_X is nef.

By (C2), the sublattice $\mu(\varpi^* L_{10})$ of S_X is primitive in S_X and hence is primitive in the even unimodular lattice $H^2(X, \mathbb{Z})$. Let K denote the orthogonal complement of $\mu(\varpi^* L_{10})$ in $H^2(X, \mathbb{Z})$. Then we have an isomorphism

$$A_K \cong A_{\mu(\varpi^* L_{10})} \cong A_{L_{10}(2)} \cong \mathbb{F}_2^{10}$$

of discriminant groups by [14]. Since A_K is 2-elementary, the scalar multiplication by -1 on K acts on A_K trivially. By [14] again, we obtain an isometry ε' of $H^2(X, \mathbb{Z})$ that preserves each of $\mu(\varpi^* L_{10})$ and K, induces the identity on $\mu(\varpi^* L_{10})$, and induces the scalar multiplication by -1 on K. Since ε' acts on $H^0(X, \Omega_X^2)$ as the scalar multiplication by -1, it preserves the Hodge structure of $H^2(X, \mathbb{Z})$. In particular, the action of ε' preserves S_X. Since ε' fixes the ample class a_X, the action of ε' preserves the nef cone of X. Hence the Torelli theorem implies that ε' comes from an involution $\varepsilon\colon X \to X$. By Proposition 4.1, we see that ε is an Enriques involution. Let $\pi\colon X \to Y := X/\langle\varepsilon\rangle$ be the quotient morphism. Since $\mu(\varpi^* L_{10})$ is the invariant sublattice of ε' in $H^2(X, \mathbb{Z})$, we have $\mu(\varpi^* L_{10}) = \pi^* S_Y$. Therefore μ induces an isometry

$$\mu_Y\colon L_{10} \xrightarrow{\sim} S_Y.$$

Note that $a := \mu_Y(\alpha)$ is ample on Y and $h := \mu_Y(\eta)$ is nef on Y, because $\pi^* a = a_X$ and $\pi^*(h) = h_X$. Let H be the nef divisor of Y whose class is h. We construct a birational morphism $\rho_h\colon Y \to \overline{Y}$ by H as in Section 4.1. It remains to show that μ_Y maps Φ_f to $\Phi_h := \Phi_{\rho_h}$ bijectively. Recall the definition (4.1) of Π_h^+. The property (iii) of η above implies that μ_Y maps $\mathrm{Roots}(\overline{R}_f)$ to $\Pi_h^+ \cup (-1)\Pi_h^+$ bijectively. Hence μ_Y identifies $\overline{R}_f$ and the sublattice $\langle \Pi_h^+\rangle$ of S_Y generated by Π_h^+. Since μ is an isometry from $\overline{M}_f$ to S_X, the map μ induces a bijection from $\mathcal{T}(\overline{M}_f)$ to $\mathcal{T}_Y$. Therefore the isometry μ_Y induces a bijection from the set of $\overline{M}_f$-liftable roots of $\overline{R}_f$ to the set of liftable root of $\langle \Pi_h^+\rangle$. Recall that α is in the interior of the Vinberg chamber Δ. By (C4) for $\overline{M}_f$ and (4.8), the ADE-basis of the root sublattice generated by $\{r \in \mathrm{Roots}(\overline{R}_f) \mid r \text{ is } \overline{M}_f\text{-liftable}\}$ associated with the linear form $\langle -, \alpha\rangle$ is Φ_f. On the other hand, Proposition 4.5 implies that the ADE-basis of the root sublattice generated by $\{r \in \mathrm{Roots}(\langle \Pi_h^+\rangle) \mid r \text{ is liftable}\}$ associated with $\langle -, a\rangle$ is Φ_h. Therefore μ_Y maps Φ_f to Φ_h bijectively. □

Corollary 4.17. *An embedding $f\colon \Phi \hookrightarrow L_{10}$ is geometrically realizable if and only if there exists an even overlattice $\overline{M}_f$ of M_f satisfying the conditions* (C1)-(C4).

Corollary 4.18. *There exists a bijection between the set of strong equivalence classes of* RDP*-Enriques surfaces geometrically realizing* $f\colon \Phi \hookrightarrow L_{10}$ *and the set of orbits of the action of the group*

$$U(M_f) := \{\ g \in \mathrm{O}(M_f) \ \mid\ \varpi^* {\Phi_f}^g = \varpi^*\Phi_f,\ \ \varpi^* {L_{10}}^g = \varpi^* L_{10}\ \}.$$

on the set of even overlattices of M_f *satisfying* (C1)-(C4).

Remark 4.19. Let $\overline{M}_f$ and $\overline{M}'_f$ be even overlattices of M_f such that $\overline{M}_f \subset \overline{M}'_f$. If $\overline{M}'_f$ satisfies (C2), (C3), and (C4), then so does $\overline{M}_f$. In particular, if an even overlattice $\overline{M}_f$ satisfies (C2), (C3), and (C4), then so does M_f.

A finite generating set of the group $U(M_f)$ is calculated as follows. First we define a homomorphism

$$\mathrm{Stab}(\Phi_f, L_{10}) \to \mathrm{O}^+(M_f), \qquad g \mapsto \tilde{g}.$$

Let g be an element of $\mathrm{Stab}(\Phi_f, L_{10})$. Then g induces an automorphism of the ADE-configuration Φ_f. Since Φ_f and Φ^- are canonically isomorphic by $r_i^+ \mapsto r_i^-$, we obtain an automorphism $g^- \in \mathrm{Aut}(\Phi^-)$ and hence an isometry $g^- \in \mathrm{O}(\langle \Phi^- \rangle)$. The action of $g \oplus g^- \in \mathrm{O}(B_\Phi)$ indices a permutation of $r'_1, \dots, r'_n \in B_\Phi^\vee$, and hence preserves the even overlattice M_f of B_Φ. Therefore we obtain $\tilde{g} \in \mathrm{O}^+(M_f)$.

Let c be the number of connected components of the Dynkin diagram of Φ^-, and let

$$\langle \Phi^- \rangle(2) = \langle \Phi_1^- \rangle(2) \oplus \cdots \oplus \langle \Phi_c^- \rangle(2)$$

be the orthogonal direct-sum decomposition of $\langle \Phi^- \rangle(2)$ according to the connected components of the Dynkin diagram of Φ^-. For $k = 1, \dots, c$, let $u'_k \in \mathrm{O}(\langle \Phi^- \rangle(2))$ denote the isometry that acts on the direct-summand $\langle \Phi_j^- \rangle(2)$ as the identity for $j \neq k$ and as the multiplication by -1 for $j = k$. We then define $u_k \in \mathrm{O}(B_\Phi)$ to be the direct sum of $\mathrm{id}_{L_{10}(2)}$ and u'_k. Then u_k acts on each of the subsets $\{r'_j, r''_j\}$ of $B_\Phi^\vee$, and hence we can regard u_k as an element of $\mathrm{O}^+(M_f)$.

Proposition 4.20. *Suppose that there exists an* RDP*-Enriques surfaces geometrically realizing* $f\colon \Phi \hookrightarrow L_{10}$*. Then the group* $U(M_f)$ *is generated by the image of the homomorphism* $\mathrm{Stab}(\Phi_f, L_{10}) \to \mathrm{O}^+(M_f)$ *above and the isometries* $u_1, \dots, u_c$*.*

Proof. Let $U' \subset \mathrm{O}^+(M_f)$ be the group generated by the image of the homomorphism $\mathrm{Stab}(\Phi_f, L_{10}) \to \mathrm{O}^+(M_f)$ and the isometries $u_1, \dots, u_c$. By construction, we have $U' \subset U(M_f)$. We prove $U' \supset U(M_f)$. Since $\varpi^* L_{10} = L_{10}(2)$, we have a natural restriction map

$$\xi\colon U(M_f) \to \mathrm{O}^+(\varpi^* L_{10}) = \mathrm{O}^+(L_{10}), \quad g \mapsto g|_{\varpi^* L_{10}}.$$

The image of ξ is contained in $\mathrm{Stab}(\Phi_f, L_{10})$ by definition, and ξ has a section over $\mathrm{Stab}(\Phi_f, L_{10}) \subset \mathrm{O}^+(L_{10})$. Therefore it suffices to show that an arbitrary element g of $\mathrm{Ker}\,\xi$ belongs to the subgroup generated by $u_1, \dots, u_c$. By the assumption and Remark 4.19, the orthogonal complement $N(M_f)$ of $\varpi^* L_{10}$ in M_f contains no roots. Hence the same argument as in the proof of Lemma 4.3 implies that each $r_i^+ \in \Phi_f$ has exactly two $N(M_f)$-lifts r'_i, r''_i. Since $r_i^{+g} = r_i^+$, we have $\{r'_i, r''_i\}^g = \{r'_i, r''_i\}$. Since $\varphi(r_i^-) = r'_i - r''_i$, we have $\varphi(r_i^-)^g = \pm\varphi(r_i^-)$. By (4.6), if $\langle r_i, r_j \rangle \neq 0$, then $\varphi(r_i^-)^g$ determines $\varphi(r_j^-)^g$. Therefore the action of g on $\langle \Phi^- \rangle(2)$ is equal to the action of an element of $\langle u_1, \dots, u_c \rangle$ on $\langle \Phi^- \rangle(2)$. □

4.4. **Complete list of RDP-Enriques surfaces.** In Section 3, we have calculated the complete list of equivalence classes of embeddings $f\colon \Phi \hookrightarrow L_{10}$. From each equivalence class, we choose a representative $f\colon \Phi \hookrightarrow L_{10}$, calculate a finite generating set of the group $\mathrm{Stab}(\Phi_f, L_{10})$ by the method given in Section 3.4, and calculate the lattice M_f and a finite generating set of the group $U(M_f) \subset \mathrm{O}^+(M_f)$. Then we calculate the finite set $\mathcal{L}'(M_f)$ of even overlattices of M_f that satisfy (C2), (C3), (C4) by Proposition 2.5.

Remark 4.21. We enumerate even overlattices of M_f by enlarging successively the corresponding totally isotropic subgroups of the discriminant form q_{M_f} of M_f. By Remark 4.19, if $\overline{M}_f$ fails to satisfy (C2), (C3), or (C4), then we do not have to enlarge the totally isotropic subgroup $\overline{M}_f/M_f$ any more.

We then decompose $\mathcal{L}'(M_f)$ into the union of the orbits under the action of $U(M_f)$. Note that the image of $U(M_f)$ by the natural homomorphism $\mathrm{O}(M_f) \to \mathrm{O}(q_{M_f})$ is of course finite, and is calculated from the finite generating set of $U(M_f)$. For each orbit o, we choose a representative element $\overline{M}_f \in o$, and check whether $\overline{M}_f$ satisfies (C1) or not by Proposition 2.6 and Remark 2.7. If $\overline{M}_f$ satisfies (C1), then o corresponds to a strong equivalence class. Thus we obtain the complete list of strong equivalence classes of RDP-Enriques surfaces. The result is given in Table 1.1, and Theorem 1.7 is proved.

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