RATIONAL DOUBLE POINTS ON ENRIQUES SURFACES:
COMPUTATIONAL DATA

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1. Introduction

The computational data for the results of the paper [2] is contained in the folder RDPEnriquesFolder. These data is presented in GAP format [1]. The zip-file of this folder is available from the webpage http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3.html. The total data is large (more than 20 MB when unzipped). The data Table, which is less than 2 MB and gives more detailed information of Table 1.1 of [2], may be enough for geometric investigation of Enriques surfaces. In the following, we use the notation fixed in the paper [2].

2. General rules on lattices and discriminant forms

2.1. Lattice. Let $L$ be an even lattice of rank $n$ with a Gram matrix $\text{Gram}_L$ with respect to a basis $b_1, \ldots, b_n$. An element of $L$ is written as a row vector with respect to $b_1, \ldots, b_n$, and hence we have

$$\langle x, y \rangle = x \cdot \text{Gram}_L \cdot y.$$

An element of $O(L)$, which acts on $L$ from the right, is expressed by an $n \times n$ matrix $M$ that satisfies

$$M \cdot \text{Gram}_L \cdot M = \text{Gram}_L.$$

2.2. Discriminant form. Suppose that a basis $b_1, \ldots, b_n$ of an even lattice $L$ is fixed. The discriminant form $q_L$ of $L$ is then expressed by the record $\text{disc}_L$ with the following items.

- $\text{disc}_L.\text{discg} = [a_1, \ldots, a_l]$ indicates that the discriminant group $A_L = L^\vee/L$ of $L$ is isomorphic to

$$Z/a_1Z \times \cdots \times Z/a_lZ. \quad (2.1)$$

When $A_L = L^\vee/L$ is trivial, $\text{disc}_L.\text{discg}$ is the empty list [ ].
\[ \text{discL.reps} \text{ is an } l \times n \text{ matrix with components in } \mathbb{Q} \text{ such that the } i\text{th row vector } \lambda_i \text{ of } \text{discL.reps} \text{ is an element of } L^\vee \subset L \otimes \mathbb{Q}, \text{ expressed in terms of the basis } b_1, \ldots, b_n \text{ of } L \otimes \mathbb{Q}, \text{ whose class } \bar{\lambda}_i := \lambda_i \text{ mod } L \in A_L \text{ generates the } i\text{th cyclic factor } \mathbb{Z}/a_i\mathbb{Z} \text{ of (2.1).} \]

\[ \text{discL.discf} \text{ is an } l \times l \text{ matrix whose } (i, j)-\text{component is a rational number } \alpha_{ij} \text{ such that} \]
\[
\begin{align*}
\alpha_{ii} &\equiv q_L(\bar{\lambda}_i) \text{ mod } 2\mathbb{Z} \quad \text{if } i = j, \\
\alpha_{ij} &\equiv b_L(\bar{\lambda}_i, \bar{\lambda}_j) \text{ mod } \mathbb{Z} \quad \text{if } i \neq j,
\end{align*}
\]
where \( b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z} \) is the bilinear form associated with \( q_L \). Hence we have
\[ \text{discL.reps} \cdot \text{GramL} \cdot \text{discL.reps} \equiv \text{discL.discf}, \]
where \( M \equiv M' \) means that all components of \( M - M' \) are integers and that the diagonal components of \( M - M' \) are even.

\[ \text{discL.proj} \text{ is an } n \times l \text{ matrix such that the natural projection } L^\vee \to A_L \text{ is given by} \]
\[ v \mapsto v \cdot \text{discL.proj} \text{ mod } [ a_1, \ldots, a_l ] \]
with respect to the basis \( b_1, \ldots, b_n \) of \( L \otimes \mathbb{Q} \) and the generators \( \bar{\lambda}_1, \ldots, \bar{\lambda}_l \) of \( A_L \). Here we use the notation
\[ (v_1, \ldots, v_l) \text{ mod } [ a_1, \ldots, a_l ] := (v_1 \text{ mod } a_1, \ldots, v_l \text{ mod } a_l) \]
for \( (v_1, \ldots, v_l) \in \mathbb{Z}^l \).

### 2.2.1. Subspaces of a discriminant form.

Suppose that a record \( \text{discL} \) of \( q_L \) is fixed, and that \( \text{discL.discg} \) is \( [ a_1, \ldots, a_l ] \), so that \( A_L = \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_l\mathbb{Z} \). Then an element \( x_1\bar{\lambda}_1 + \cdots + x_l\bar{\lambda}_l \) of \( A_L \) is expressed as a row vector
\[ [x_1, \ldots, x_l] \text{ mod } \mathbb{Z}^l. \]

A subgroup \( H \) of \( A_L \) is expressed by a list \( [\xi_1, \ldots, \xi_k] \) of vectors \( \xi_i \in \mathbb{Z}^l \) such that the elements
\[ \tilde{\xi}_1 := \xi_1 \text{ mod } [ a_1, \ldots, a_l ], \ldots, \tilde{\xi}_k := \xi_k \text{ mod } [ a_1, \ldots, a_l ] \]
of \( A_L \) generate \( H \).

Each subspace \( H \) of \( A_L \) has a unique standard generating list of elements defined as follows:

**Definition 2.1.** A **standard generating matrix** of a subspace \( H \) of \( A_L = \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_l\mathbb{Z} \) is an \( l \times l \) matrix \( M \) with integer components such that

- \( M \) is in a Hermite normal form, and
- the row vectors of \( M \) form a basis of the inverse image \( \tilde{H} \) of \( H \) by the natural projection \( \mathbb{Z}^l \to A_L \).
For example, if \( H = 0 \), then its standard generating matrix is the diagonal matrix with diagonal components \( a_1, \ldots, a_l \).

2.2.2. Automorphisms of a discriminant form. Let \( A_L \) and \( \lambda_1, \ldots, \lambda_l \) be as above. An automorphism \( \gamma \) of the finite abelian group \( A_L \) is expressed by an \( l \times l \) matrix whose \( i \)th row vector modulo \( [a_1, \ldots, a_l] \) expresses \( \lambda_i^j \). Suppose that \( g \in O(L) \) is given by a matrix \( M \) as in Section 2.1. Then \( g \) induces an automorphism of \( q_L \) given by the matrix

\[
\text{discL.reps} \cdot M \cdot \text{discL.proj}.
\]

2.3. Overlattices. Let \( L' \) be an even overlattice of \( L \). Then a basis \( b'_1, \ldots, b'_n \) of \( L' \) is specified by an \( n \times n \) matrix \( \text{emb} \) such that \( v \mapsto v \cdot \text{emb} \) is the canonical embedding \( L \hookrightarrow L' \) with respect to the basis \( b_1, \ldots, b_n \) of \( L \) and the basis \( b'_1, \ldots, b'_n \) of \( L' \). Namely, the row vectors of \( \text{emb}^{-1} \) are the vector representations of \( b'_1, \ldots, b'_n \) with respect to the basis \( b_1, \ldots, b_n \) of \( L \otimes \mathbb{Q} \).

Let \( H \) be an isotropic subspace of \( q_L \), and let \( L' \) be the corresponding even overlattice of \( L \). Then the matrix \( \text{emb} \) that describes \( L \hookrightarrow L' \) can be easily calculated from \( \text{discL.reps} \) and the generating matrix of \( H \).

2.4. Negative-definite root lattices. An ADE-type is expressed as a list of indecomposable ADE-types

\[
"A1", "A2", \ldots, "D4", "D5", \ldots, "E6", "E7", "E8".
\]

Each ADE-type is sorted by the ordering

\[
"A1" < "A2" < \ldots < "D4" < "D5" < \ldots < "E6" < "E7" < "E8".
\]

For example, an ADE-type \( t = E_6 + 3A_4 + A_1 + D_7 \) is expressed as

\[
\]

The following data are available.
The record $\text{GramADE}$ contains the following data. For each indecomposable ADE-type $y$, the Gram matrix $\text{GramADE}.y$ of the negative-definite root lattice $R(y)$ of type $y$ is given with respect to the basis given in Figure 2.1. For example, we have

$$\text{GramADE}.D4 = \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$ 

The record $\text{GramADE}$ is given only for indecomposable ADE-types of rank $\leq 9$; that is, for $A_l (l \leq 9), D_m (4 \leq m \leq 9), E_6, E_7, E_8$.

- The record $\text{discADE}$ is the record such that, for an indecomposable ADE-type $y$ of rank $\leq 9$, $\text{discADE}.y$ is the record that describes the discriminant form of the negative-definite root lattice of type $y$.

**Definition 2.2.** Let $t = [y_1, \ldots, y_m]$ be an ADE-type, where $y_1, \ldots, y_m$ are indecomposable ADE-types sorted as above. Then an ADE-basis of the negative-definite root lattice $R(t)$ of type $t$ is an ordered basis such that the Gram matrix with respect to this basis is equal to the block-diagonal matrix whose diagonal blocks are $\text{GramADE}.y_1, \ldots, \text{GramADE}.y_m$.

For example, if $t = 2A_1 + A_2 = ["A1", "A1", "A2"]$, then the Gram matrix of the negative-definite root lattice of type $t$ with respect to an ADE-basis is

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$ 

The Gram matrix of the negative-definite root lattice $R(t)$ of type $t$ is constructed from $\text{GramADE}$, and the discriminant form of $R(t)$ is constructed from $\text{discADE}$.

3. **The lattice $L_{10}$**

$\text{GramL10}$ is the Gram matrix of $L_{10}$ with respect to the fixed basis $e_1, \ldots, e_{10}$. In the following, every element of $O^+(L_{10})$ is expressed by a $10 \times 10$ matrix with respect to the basis $e_1, \ldots, e_{10}$.

- Each element $\Sigma = \{e_{i_1}, \ldots, e_{i_n}\}$ of $S$ is expressed by the list $[i_1, \ldots, i_n]$ of indices such that $1 \leq i_1 < \cdots < i_n \leq 10$.

$\text{SigmasList}$ is the list of all $\Sigma \in S$.

$\text{Sigmas}$ is the list of 1021 records. Each record $\text{Sigma}$ in this list contains the following data of $\Sigma \in S$. The list $\text{Sigmas}$ is sorted according to $\text{SigmasList}$. 
• Sigma.vects is the list \([i_1, \ldots, i_n]\) such that \(1 \leq i_1 < \cdots < i_n \leq 10\) and 
\(\Sigma = \{e_{i_1}, \ldots, e_{i_n}\}\).

• Sigma.type is the ADE-type \(\tau(\Sigma)\) of \(\Sigma\).

• Sigma.ADEbasis is a permutation \([j_1, \ldots, j_n]\) of \([i_1, \ldots, i_n]\) such that the 
vectors \(e_{j_1}, \ldots, e_{j_n}\) form an ADE-basis of the negative-definite root lattice 
\(\langle \Sigma \rangle\).

• Sigma.embSigmaL10 is the matrix of the embedding \(\langle \Sigma \rangle \hookrightarrow L_{10}\) with re-
spect to the ADE-basis of \(\langle \Sigma \rangle\) given by Sigma.ADEbasis and the basis 
e_1, \ldots, e_{10} of \(L_{10}\).

• Sigma.Gram is the Gram matrix of \(\langle \Sigma \rangle\) with respect to the ADE-basis 
e_{j_1}, \ldots, e_{j_n}.

• Sigma.AutGenerators is a generating set of \(\text{Aut}(\Sigma)\), which is a list of matrices 
in \(O(\langle \Sigma \rangle)\) with respect to the ADE-basis of \(\langle \Sigma \rangle\) fixed by Sigma.ADEbasis.
When \(\text{Aut}(\Sigma)\) is trivial, this list contains only the identity matrix of size \(n\).

• Sigma.opposite is the isometry \(\xi \in O^+(L_{10})\) such that \(\Delta_0^\xi\) is the Vinberg 
chamber opposite to \(\Delta_0\) with respect to the face \(F_\Sigma\) of \(\Delta_0\).

• Sigma.partners is the list \([\nu_1, \ldots, \nu_K]\) with \(1 \leq \nu_1 < \cdots < \nu_K \leq 1021\) such that the set of indices \(\{i | \text{SigmasList[i].type} = \tau(\Sigma)\}\) is equal to 
\([\nu_1, \ldots, \nu_K]\).

• Sigma.isomto is the first index \(\nu_1\) of Sigma.partners, which indicates the 
representative element \(\Sigma_0\) of the set \(\{\Sigma' \in \mathcal{S} | \tau(\Sigma') = \tau(\Sigma)\}\), which can be 
identified with the orbit of the action of \(O^+(L_{10})\) on \(\mathcal{N}\) containing \(\Sigma\).

• Sigma.isomby is an isometry \(g \in O^+(L_{10})\) such that \(\langle \Sigma \rangle^g = \langle \Sigma_0 \rangle\).

• Sigma.kappatildeGSigma is the list of elements of \(\tilde{\kappa}(\mathcal{G}_\Sigma)\), which generates 
\(\text{Stab}(\Sigma, L_{10})\). Each element of Sigma.kappatildeGSigma is a matrix in 
\(O^+(L_{10})\). When \(\text{Stab}(\Sigma, L_{10})\) is trivial, this list contains only the identity 
matrix of size 10.

• Sigma.HSigmaGenerators is the list of elements of \(\kappa(\text{res}(\mathcal{G}_\Sigma)) = \text{res}(\tilde{\kappa}(\mathcal{G}_\Sigma))\), 
which generates the subgroup \(H_\Sigma\) of \(\text{Aut}(\Sigma)\). Each element of this list is 
a matrix in \(O(\langle \Sigma \rangle)\) with respect to the ADE-basis of \(\langle \Sigma \rangle\). When \(H_\Sigma\) is 
trivial, this list contains only the identity matrix of size \(n\).

4. Even overlattices of \(\langle \Phi \rangle\) 

\textbf{PhisList} is the list of ADE-types \(\tau(\Phi)\) of ADE-configurations \(\Phi\) with \(|\Phi| < 10\).

\textbf{Phis} is a list of records. Each record \textbf{Phi} in \textbf{Phis} contains the following data 
of an ADE-configuration \(\Phi\) whose type is in \textbf{PhisList}. The list \textbf{Phis} is sorted 
according to \textbf{PhisList}. We put \(n := |\Phi| < 10\).

• Phi.type is the ADE-type \(\tau(\Phi)\) of \(\Phi\).

• Phi.Gram is the Gram matrix of the negative-definite root lattice \(\langle \Phi \rangle\) with 
respect to the ADE-basis \(\Phi\) of \(\langle \Phi \rangle\).
• **Phi.disc** is the record of the discriminant form $q_{\Phi}$ of $\langle \Phi \rangle$. Let $[a_1, \ldots, a_l]$ be the item **Phi.disc.discg**, so that $A_{\langle \Phi \rangle} \cong \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_l\mathbb{Z}$.

• **Phi.AutGenerators** is a generating set of $\text{Aut}(\Phi)$, which is a list of matrices in $O(\langle \Phi \rangle)$ with respect to the ADE-basis of $\langle \Phi \rangle$. When $\text{Aut}(\Phi)$ is trivial, this list contains only the identity matrix of size $n$.

• **Phi.discAutGenerators** is the set of images of elements of the generating set **Phi.AutGenerators** of $\text{Aut}(\Phi)$ by the natural homomorphism $O(\langle \Phi \rangle) \rightarrow O(q_{\langle \Phi \rangle})$. Each automorphism in **Phi.discAutGenerators** is expressed by an $l \times l$ matrix with respect to the ADE-basis of $\langle \Phi \rangle$. When $\text{Aut}(\Phi)$ is trivial, this list contains only the identity matrix of size $l$. (When **Phi.type** is "$E8"" (that is, **Phi.disc.discg** is the empty list $[ ]$), **Phi.discAutGenerators** is the empty list.)

• **Phi.overlattices** is the list of representatives of orbits of the action of $\text{Aut}(\Phi)$ on the set $L(\Phi)$ of even overlattices of $\langle \Phi \rangle$. Each element of this list is a record **Rbar** that describes the following data of an even overlattice $R$ of $\langle \Phi \rangle$:
  - **Rbar.isotropicspace** is the standard generating matrix of the isotropic subspace $R/\langle \Phi \rangle \subset A_L$ of $q_{\Phi}$.
  - **Rbar.torsion** $=[b_1, \ldots, b_m]$ indicates that the finite abelian group $R/\langle \Phi \rangle$ is isomorphic to $\mathbb{Z}/b_1\mathbb{Z} \times \cdots \times \mathbb{Z}/b_m\mathbb{Z}$. When $R = \langle \Phi \rangle$, we have $Rbar.torsion=[ ]$.
  - **Rbar.embRRbar** is an $n \times n$ matrix that describes the embedding $\langle \Phi \rangle \hookrightarrow R$ with respect to the ADE-basis of $\langle \Phi \rangle$. This matrix **Rbar.embRRbar** is used to fix a basis $v_1, \ldots, v_n$ of $\overline{R}$ by the remark in Section 2.3. (See also Remark 4.1 below.)
  - **Rbar.Gram** is the Gram matrix of $\overline{R}$ with respect to the basis of $\overline{R}$ fixed by **Rbar.embRRbar**.
  - **Rbar.StabGenerators** is a generating set of the stabilizer subgroup $\text{Stab}(\overline{R}, \Phi)$ of $\overline{R}$ in $\text{Aut}(\Phi)$. Each element of **Rbar.StabGenerators** is expressed as a matrix in $O(\langle \Phi \rangle)$ with respect to the ADE-basis of $\langle \Phi \rangle$. If $\text{Stab}(\overline{R}, \Phi)$ is trivial, this list contains only the identity matrix of size $n$.
  - **Rbar.IsADE** indicates whether $\overline{R}$ is a negative-definite root lattice or not. If $\overline{R}$ is not a negative-definite root lattice, then **Rbar.IsADE** is false. If $\overline{R}$ is a negative-definite root lattice, then **Rbar.IsADE** is the ADE-type $\tau(\overline{R})$ of $\overline{R}$.

**Remark 4.1.** Let $b_1, \ldots, b_n$ be the ADE-basis of $\langle \Phi \rangle$, and let $v_1, \ldots, v_n$ be the basis of $\overline{R}$ fixed by the matrix **Rbar.embRRbar**. Suppose that $\overline{R}$ is a negative-definite root lattice. Then the basis $v_1, \ldots, v_n$ of $\overline{R}$ is chosen so that $v_1, \ldots, v_n$ form an
ADE-basis of \( \overline{\mathcal{R}} \), and that the image of the connected component
\[
\{ x \in (\Phi) \otimes \mathbb{R} \mid \langle x, b_i \rangle > 0 \text{ for } i = 1, \ldots, n \}
\]
of \( ((\Phi) \otimes \mathbb{R})^\circ \) by the embedding \( v \mapsto v \cdot \overline{\text{emb}R_{\overline{\mathcal{R}}}} \) contains the connected component
\[
\{ y \in \overline{\mathcal{R}} \otimes \mathbb{R} \mid \langle y, v_i \rangle > 0 \text{ for } i = 1, \ldots, n \}
\]
of \( (\overline{\mathcal{R}} \otimes \mathbb{R})^\circ \).

Remark 4.2. The natural homomorphism \( \text{Stab}(\overline{\mathcal{R}}, \Phi) \to \text{O}(\overline{\mathcal{R}}) \) can be easily calculated by \( \overline{\text{emb}R_{\overline{\mathcal{R}}}} \).

5. The equivalence classes of embeddings

The data \( \text{EmbsList} \) is the list of pairs \( (\tau(\Phi), \Sigma) \), where \( \tau(\Phi) \) is the ADE-type of an ADE-configuration \( \Phi \) with \( n := |\Phi| < 10 \), and \( \Sigma = [i_1, \ldots, i_n] \) is an element of \( \mathcal{S} \) with \( 1 \leq i_1 < \cdots < i_n \leq 10 \) such that \( (\Sigma) \) is isomorphic to an even overlattice \( \overline{\mathcal{R}} \) of \( \langle \Phi \rangle \). Each pair gives a representative embedding \( f: \Phi \hookrightarrow L_{10} \) of the single element of
\[
\text{emb}(\overline{\mathcal{R}}; [\Sigma]) = \text{Stab}(\overline{\mathcal{R}}, \Phi) \backslash \text{O}(\langle \Sigma \rangle)/\text{Stab}(\langle \Sigma \rangle, L_{10}) = H_{\Phi} \backslash \text{Aut}(\Sigma)/H_{\Sigma}.
\]
Therefore \( \text{EmbsList} \) gives the set \( \text{Aut}(\Phi) \backslash \text{Embs}(\Phi)/\text{O}^+(L_{10}) \).

The data \( \text{Embs} \) is a list of records. A member \( f \) of \( \text{Embs} \) gives the following data of an embedding \( f: \Phi \hookrightarrow L_{10} \) such that \( \overline{\mathcal{R}}_f = (\Sigma) \). The list \( \text{Embs} \) is sorted according to \( \text{EmbsList} \). Let \( \Phi_i \) be the member of \( \text{Phis} \) that describes \( \Phi = \{r_1, \ldots, r_n\} \), and let \( \Sigma \) be the member of \( \text{Sigmas} \) that describes \( \Sigma = \{e_{i_1}, \ldots, e_{i_n}\} \). Recall that \( \Phi_f = \{r_1^+, \ldots, r_n^+\} \).

- \( f.\Phi \) is the ADE-type \( \tau(\Phi) \) of \( \Phi = \{r_1, \ldots, r_n\} \).
- \( f.\Sigma \) is a copy of the record \( \Sigma \). Hence, for example, the ADE-type \( \tau(\Sigma) \) of \( \Sigma \) is given by \( f.\Sigma \).type.
- \( f.\text{GramPhi} \) is the Gram matrix of \( \langle \Phi \rangle \) with respect to the ADE-basis \( \Phi \).
- \( f.\text{GramSigma} \) is the Gram matrix of \( \langle \Sigma \rangle \) with respect to the ADE-basis given by \( f.\Sigma \).ADEbasis.
- \( f.\text{Rbar} \) is a record that describes the even overlattice \( \overline{\mathcal{R}} \) of \( \langle \Phi \rangle \) corresponding to the even overlattice \( \overline{\mathcal{R}}_f \) of \( R_f \) via \( f: \langle \Phi \rangle \cong R_f \). This record is a copy of the member of \( \text{Phi.overlattices} \) that describes \( \overline{\mathcal{R}} \). By Remark 4.1, the row vectors of \( (f.\text{Rbar.embRBar})^{-1} \) are the vector representations of an ADE-basis of \( \overline{\mathcal{R}} \) with respect to the ADE-basis \( \Phi \) of \( \langle \Phi \rangle \otimes \mathbb{Q} \). Identifying this ADE-basis of \( \overline{\mathcal{R}} \) with the ADE-basis \( f.\Sigma \).ADEbasis of \( \langle \Sigma \rangle \) gives an element \( g_0 \in \text{Isom}(\overline{\mathcal{R}}, \langle \Sigma \rangle) \) explicitly. Using \( g_0 \) as a reference point, we identify \( \text{Isom}(\overline{\mathcal{R}}, \langle \Sigma \rangle) \) with \( \text{O}(\langle \Sigma \rangle) \), and regard \( \text{Stab}(\overline{\mathcal{R}}, \Phi) \) as a subgroup of \( \text{O}(\langle \Sigma \rangle) \).
f.embPhiSigma is the $n \times n$ matrix whose $i$th row vector is the vector representation of $r_i^+ \in \langle \Sigma \rangle$ with respect to the ADE-basis $e_{j_1}, \ldots, e_{j_n}$ of $\langle \Sigma \rangle$. By the identification $g_0$, we have $f.embPhiSigma$ equal to $f.Rbar.embRRbar$.

f.embSigmaL10 is the $n \times 10$ matrix whose row vectors are $e_{j_1}; \ldots; e_{j_n}$.

f.embPhiL10 is the $n \times 10$ matrix whose $i$th row vectors is the vector representation of $r_i^+$ with respect to the basis $e_{j_1}; \ldots; e_{j_{10}}$ of $L_{10}$. Hence $f.embPhiL10$ is equal to $(f.embPhiSigma) \cdot (f.embSigmaL10)$.

f.HPhiGenerators is the image of elements of the generating set $f.Rbar.StabGenerators$ of $Stab(\mathcal{R}, \Phi)$ by the homomorphism $Stab(\mathcal{R}, \Phi) \to O(\mathcal{R}) \cong O(\langle \Sigma \rangle) \to Aut(\Sigma)$, where $O(\mathcal{R}) \cong O(\langle \Sigma \rangle)$ is induced by $g_0: \mathcal{R} \cong \langle \Sigma \rangle$ and $O(\langle \Sigma \rangle) \to Aut(\Sigma)$ is the quotient homomorphism by $W(\langle \Sigma \rangle)$. Each element of $f.HPhiGenerators$ is a matrix in $O(\langle \Sigma \rangle)$ with respect to the ADE-basis $e_{j_1}; \ldots; e_{j_n}$ of $\langle \Sigma \rangle$.

The fact that $emb(\mathcal{R}, [\Sigma])$ consists of a single element (Theorem 3.22 of [2]) is proved by confirming the following fact: Every element of the finite group $Aut(\Sigma)$ generated by $f.SigmaRecord.AutGenerators$ is written as $h_\Phi h_\Sigma$, where $h_\Phi$ is an element of $H_\Phi$ generated by the elements of $f.HPhiGenerators$ and $h_\Sigma$ is an element of $H_\Sigma$ generated by the elements of $f.SigmaRecord.HSigmaGenerators$.

6. Geometric realizability

Let $f$ be a member of $Embs$ as in the previous section. The record $f$ also contains the following data of geometric realizability of $f: \Phi \mapsto L_{10}$.

- $f.StabPhiL10Generators$ is a generating set of $Stab(\Phi_f, L_{10})$. Each element of $f.StabPhiL10Generators$ is a matrix in $O^+(L_{10})$ with respect to the basis $e_{1j}; \ldots; e_{10j}$. When $Stab(\Phi_f, L_{10})$ is trivial, this list contains only the identity matrix $I_{10}$ of size 10.

- $f.varpiB$ is the matrix of the embedding $\varpi*: L_{10} \mapsto \Phi_f$ with respect to the basis $e_{1j}; \ldots; e_{10j}$ of $L_{10}$ and the basis $\varpi^*(e_1), \ldots, \varpi^*(e_{10}), \varphi(r_{1j}), \ldots, \varphi(r_{nj})$ (6.1) of $\Phi_f$; that is, the $10 \times (10 + n)$ matrix $[I_{10} \mid O]$.

- $f.GramB$ is the Gram matrix of $\Phi_f$ with respect to the basis (6.1) of $\Phi_f$; that is, the block-diagonal matrix with diagonal blocks $2 \cdot GramL10$ and $2 \cdot f.GramPhi$.

- $f.involB$ is the matrix representation of the involution $\varepsilon$ of $\Phi_f$ that acts as the identity on $\varpi^*L_{10}$ and as the scalar multiplication by $-1$ on its orthogonal complement; that is, $f.involB$ is the block-diagonal matrix with diagonal blocks $I_{10}$ and $-I_n$.

- $f.discB$ is the record that describes the discriminant form of $\Phi_f$. 
\[ f.\text{Lifts} \] is the list of lifts \( r'_1, \ldots, r'_n \in B_\Phi \otimes \mathbb{Q} \) written with respect to the basis (6.1) of \( B_\Phi \).

\[ f.\text{embBMf} \] is a matrix of the embedding \( B_\Phi \hookrightarrow M_f \) with respect to the basis (6.1) of \( B_\Phi \). This matrix is used to fix a basis of \( M_f \) by the remark in Section 2.3.

\[ f.\text{GramMf} \] is the Gram matrix of \( M_f \) with respect to the basis fixed by \( f.\text{embBMf} \).

\[ f.\text{UGenerators} \] is a generating set of the subgroup \( U(M_f) \subset O^+(M_f) \), each element of which is written as a matrix with respect the basis (6.1) of \( B_\Phi \) (not with respect the basis of \( M_f \) fixed by \( f.\text{embBMf} \)).

\[ f.\text{UGeneratorsDisc} \] is a generating set of the image of \( U(M_f) \subset O(B_\Phi) \) by the natural homomorphism \( O(B_\Phi) \to O(q_{B_\Phi}) \). Each element of this list is written with respect to the basis of the discriminant group of \( B_\Phi \) fixed by \( f.\text{discB} \).

\[ f.\text{Mfisotropicspace} \] is the standard generating matrix of the isotropic subspace \( M_f/B_\Phi \) of \( q_{B_\Phi} \).

\[ f.\text{varpiMf} \] is the matrix of \( \varpi^*: L_{10} \hookrightarrow M_f \) with respect to the basis \( e_1, \ldots, e_{10} \) of \( L_{10} \) and the basis of \( M_f \) fixed by \( f.\text{embBMf} \).

\[ f.\text{involMf} \] is the matrix representation of the involution \( \varepsilon \) of \( M_f \) with respect the basis of \( M_f \) fixed by \( f.\text{embBMf} \).

Let \( \mathcal{L}'(M_f) \) be the set of even overlattices of \( M_f \) that satisfy the conditions (C2), (C3), (C4). This set is calculated as the set of isotropic subspaces of \( q_{B_\Phi} \) containing \( M_f/B_\Phi \). The orbits of the action of \( U(M_f) \) on \( \mathcal{L}'(M_f) \) is calculated by means of the action of the finite group generated by \( f.\text{UGeneratorsDisc} \) on the set of all isotropic subspaces of \( q_{B_\Phi} \) containing \( M_f/B_\Phi \).

\[ f.\text{Mfbars} \] is the list of records \( \text{Mfbar} \) that describe the representatives of orbits of the action of \( U(M_f) \) on \( \mathcal{L}'(M_f) \). Each record \( \text{Mfbar} \) contains the following data of an even overlattice \( \overline{M_f} \) of \( M_f \) that satisfies the conditions (C2), (C3), (C4).

- \( \text{Mfbar.embBMfbar} \) is a matrix of the embedding \( B_\Phi \hookrightarrow \overline{M_f} \) with respect to the basis (6.1) of \( B_\Phi \). This matrix is used to fix a basis of \( \overline{M_f} \) by the remark in Section 2.3.

- \( \text{Mfbar.embMfMfbar} \) is a matrix of the embedding \( M_f \hookrightarrow \overline{M_f} \) with respect to the basis of \( M_f \) fixed by \( f.\text{embBMf} \) and the basis of \( \overline{M_f} \) fixed by \( f.\text{embBMfbar} \).

- \( \text{Mfbar.Gram} \) is the Gram matrix of \( \overline{M_f} \) with respect to the basis fixed by \( f.\text{embBMfbar} \).

- \( \text{Mfbar.varpi} \) is the matrix of \( \varpi^*: L_{10} \hookrightarrow \overline{M_f} \) with respect to the basis \( e_1, \ldots, e_{10} \) of \( L_{10} \) and the basis of \( \overline{M_f} \) fixed by \( \text{Mfbar.embBMfbar} \).
- \texttt{Mfbar.invol} is the matrix representation of the involution \( \varepsilon \) of \( \overline{M_f} \) respect the basis of \( \overline{M_f} \) fixed by \texttt{Mfbar.embMfbar}.
- \texttt{Mfbar.isotropicspace} is the standard generating matrix of the isotropic subspace \( \overline{M_f}/B_\Phi \).
- \texttt{Mfbar.Q} is the finite abelian group \( Q := \overline{M_f}/M_f \). \texttt{Mfbar.Q} = \{[b_1, \ldots, b_m]\} means that \( Q \) is isomorphic to \( \mathbb{Z}/b_1\mathbb{Z} \times \cdots \times \mathbb{Z}/b_m\mathbb{Z} \). When \( Q \) is trivial, we have \texttt{Mfbar.Q} = \{\}.
- \texttt{Mfbar.disc} is the record that describes the discriminant form of \( \overline{M_f} \), which is used to determine whether \( \overline{M_f} \) satisfies (C1) or not.
- \texttt{Mfbar.C1} tells whether \( M_f \) satisfies the condition (C1). This data is true if \( \overline{M_f} \) satisfies (C1), whereas it is false otherwise.

7. Table

The data Table is a list of records. This list gives more detailed information of Table 1.1 of [2]. Each record \texttt{tablerow} in Table contains the following data of each row of Table 1.1 of [2].

- \texttt{tablerow.No} is the number.
- \texttt{tablerow.Phi} is the ADE-type \( \tau(\Phi_f) \) of \( \Phi_f \).
- \texttt{tablerow.SigmaType} is the ADE-type \( \tau(\overline{R_f}) = \langle \Sigma \rangle \).
- \texttt{tablerow.SigmaVects} is the vector representations of elements of \( \Sigma \) with respect to \( e_1, \ldots, e_{10} \). These vectors are sorted in such a way that they form an ADE-basis of \( \overline{R_f} = \langle \Sigma \rangle \).
- \texttt{tablerow.embPhiSigma} is the matrix of the embedding \( f: \langle \Phi_f \rangle \hookrightarrow \langle \overline{R_f} = \langle \Sigma \rangle \) with respect to the ADE-bases \( \Phi_f \) and \( \Sigma \).
- \texttt{tablerow.embSigmaL10} is the matrix of the embedding \( \langle \Sigma \rangle \hookrightarrow L_{10} \) with respect to the ADE-basis \( \Sigma \) sorted as \texttt{tablerow.SigmaVects} and the basis \( e_1, \ldots, e_{10} \) of \( L_{10} \). This matrix is identical with \texttt{tablerow.SigmaType}.
- \texttt{tablerow.embPhiL10} is the matrix of the embedding \( f: \langle \Phi_f \rangle \hookrightarrow L_{10} \) with respect to the ADE-basis \( \Phi_f \) and the basis \( e_1, \ldots, e_{10} \) of \( L_{10} \). Hence this matrix is equal to \( (\texttt{tablerow.embPhiSigma}) \cdot (\texttt{tablerow.embSigmaL10}) \).
- \texttt{tablerow.varpiB} is the matrix of the embedding \( \varpi^*: L_{10} \hookrightarrow B_\Phi \) with respect to the basis \( e_1, \ldots, e_{10} \) of \( L_{10} \) and the basis (6.1) of \( B_\Phi \); that is, \texttt{tablerow.varpiB} is equal to \([I_{10} | O]\).
- \texttt{tablerow.embBMf} is the matrix of the embedding \( B_\Phi \hookrightarrow M_f \) with respect to the basis (6.1) of \( B_\Phi \). This matrix fixes a basis of \( M_f \) by the remark in Section 2.3.
- \texttt{tablerow.GramPhi} is the Gram matrix of \( \langle \Phi \rangle \) with respect to the ADE-basis \( \Phi \).
- \texttt{tablerow.GramSigma} is the Gram matrix of \( \overline{R_f} = \langle \Sigma \rangle \) with respect to the ADE-basis \texttt{tablerow.SigmaVects}.
• `tablerow.GramMf` is the Gram matrix of $M_f$ with respect to the basis of $M_f$ fixed by `tablerow.embBMf`.

• `tablerow.StrongEquivClasses` is the list of representatives of strong equivalence classes of RDP-Enriques surfaces that geometrically realize the embedding $f : \Phi \hookrightarrow L_{10}$; that is, `tablerow.StrongEquivClasses` is the list of representatives $\overline{M}_f$ of the orbits of the action of $U(M_f)$ on the set of even overlattices of $M_f$ that satisfy the conditions (C1), ..., (C4). Each record `strongequiv` of `tablerow.StrongEquivClasses` contains the following data of an even overlattice $\overline{M}_f$.
  - `strongequiv.embMfMfbar` is a matrix of the embedding $M_f \hookrightarrow \overline{M}_f$ with respect to the basis of $M_f$ fixed by `tablerow.embBMf`. This matrix fixes a basis of $\overline{M}_f$ by the remark in Section 2.3.
  - `strongequiv.GramMfbar` is the Gram matrix of $\overline{M}_f$ with respect to the basis of $\overline{M}_f \cong S_X$ fixed by `tablerow.embBMfMfbar`.
  - `strongequiv.involMfbar` is the matrix representation of the Enriques involution $\epsilon$ on $\overline{M}_f \cong S_X$.
  - `strongequiv.Q` is the finite abelian group $Q := \overline{M}_f/M_f$.
    `strongequiv.Q` $= [b_1, \ldots, b_m]$ means that $Q = \overline{M}_f/M_f$ is isomorphic to $\mathbb{Z}/b_1\mathbb{Z} \times \cdots \times \mathbb{Z}/b_m\mathbb{Z}$. When $Q$ is trivial, `strongequiv.Q` is an empty list `[]`.

References


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