RATIONAL DOUBLE POINTS ON ENRIQUES SURFACES: COMPUTATIONAL DATA

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1. Introduction

The computational data for the results of the paper [2] is contained in the folder

`RDPEnriquesFolder`.

These data is presented in `GAP` format [1]. The `zip`-file of this folder is available from the webpage

`http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3.html`.

The total data is large (more than 20 MB when unzipped). The data `Table`, which is less than 2 MB and gives more detailed information of Table 1.1 of [2], may be enough for geometric investigation of Enriques surfaces. In the following, we use the notation fixed in the paper [2].

2. General rules on lattices and discriminant forms

2.1. **Lattice.** Let L be an even lattice of rank n with a Gram matrix `GramL` with respect to a basis $b_1, \dots, b_n$. An element of L is written as a row vector with respect to $b_1, \dots, b_n$, and hence we have

$$\langle x, y \rangle = x \cdot \texttt{GramL} \cdot {}^t y.$$

An element of $\mathrm{O}(L)$, which acts on L from the right, is expressed by an $n \times n$ matrix M that satisfies

$$M \cdot \texttt{GramL} \cdot {}^t M = \texttt{GramL}.$$

2.2. **Discriminant form.** Suppose that a basis $b_1, \dots, b_n$ of an even lattice L is fixed. The discriminant form q_L of L is then expressed by the record `discL` with the following items.

- `discL.discg` $= [a_1, \dots, a_l]$ indicates that the discriminant group $A_L = L^\vee/L$ of L is isomorphic to

$$\mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_l\mathbb{Z}. \tag{2.1}$$

  When $A_L = L^\vee/L$ is trivial, `discL.discg` is the empty list [ ].

- `discL.reps` is an $l \times n$ matrix with components in $\mathbb{Q}$ such that the ith row vector λ_i of `discL.reps` is an element of $L^\vee \subset L \otimes \mathbb{Q}$, expressed in terms of the basis $b_1, \dots, b_n$ of $L \otimes \mathbb{Q}$, whose class $\bar{\lambda}_i := \lambda_i \bmod L \in A_L$ generates the ith cyclic factor $\mathbb{Z}/a_i\mathbb{Z}$ of (2.1).
- `discL.discf` is an $l \times l$ matrix whose (i, j)-component is a rational number α_{ij} such that

$$\begin{cases} \alpha_{ii} \equiv q_L(\bar{\lambda}_i) \mod 2\mathbb{Z} & \text{if } i = j, \\ \alpha_{ij} \equiv b_L(\bar{\lambda}_i, \bar{\lambda}_j) \mod \mathbb{Z} & \text{if } i \neq j, \end{cases}$$

  where $b_L \colon A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$ is the bilinear form associated with q_L. Hence we have

$$\texttt{discL.reps} \cdot \texttt{GramL} \cdot {}^t\texttt{discL.reps} \;\equiv\; \texttt{discL.discf},$$

  where $M \equiv M'$ means that all components of $M - M'$ are integers and that the diagonal components of $M - M'$ are even.
- `discL.proj` is an $n \times l$ matrix such that the natural projection $L^\vee \to A_L$ is given by

$$v \mapsto v \cdot \texttt{discL.proj} \mod [a_1, \dots, a_l]$$

  with respect to the basis $b_1, \dots, b_n$ of $L \otimes \mathbb{Q}$ and the generators $\bar{\lambda}_1, \dots, \bar{\lambda}_l$ of A_L. Here we use the notation

$$(v_1, \dots, v_l) \mod [a_1, \dots, a_l] \;:=\; (v_1 \bmod a_1, \dots, v_l \bmod a_l)$$

  for $(v_1, \dots, v_l) \in \mathbb{Z}^l$.

2.2.1. *Subspaces of a discriminant form.* Suppose that a record `discL` of q_L is fixed, and that `discL.discg` is $[a_1, \dots, a_l]$, so that $A_L = \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_l\mathbb{Z}$. Then an element $x_1\bar{\lambda}_1 + \cdots + x_l\bar{\lambda}_l$ of A_L is expressed as a row vector

$$[x_1, \dots, x_l] in \mathbb{Z}^l.$$

A subgroup H of A_L is expressed by a list $[\xi_1, \dots, \xi_k]$ of vectors $\xi_i \in \mathbb{Z}^l$ such that the elements

$$\bar{\xi}_1 := \xi_1 \bmod [a_1, \dots, a_l], \;\; \dots, \;\; \bar{\xi}_k := \xi_k \bmod [a_1, \dots, a_l]$$

of A_L generate H.

Each subspace H of A_L has a unique standard generating list of elements defined as follows:

Definition 2.1. A *standard generating matrix* of a subspace H of $A_L = \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_l\mathbb{Z}$ is an $l \times l$ matrix M with integer components such that

- M is in a Hermite normal form, and
- the row vectors of M form a basis of the inverse image $\widetilde{H}$ of H by the natural projection $\mathbb{Z}^l \to A_L$.

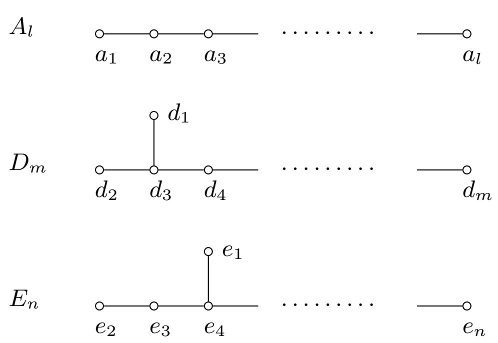

FIGURE 2.1. Dynkin diagrams of type ADE

For example, if $H = 0$, then its standard generating matrix is the diagonal matrix with diagonal components $a_1, \dots, a_l$.

2.2.2. *Automorphisms of a discriminant form.* Let A_L and $\bar{\lambda}_1, \dots, \bar{\lambda}_l$ be as above. An automorphism γ of the finite abelian group A_L is expressed by an $l \times l$ matrix whose ith row vector modulo $[a_1, \dots, a_l]$ expresses $\bar{\lambda}_i^\gamma$. Suppose that $g \in \mathrm{O}(L)$ is given by a matrix M as in Section 2.1. Then g induces an automorphism of q_L given by the matrix

$$\texttt{discL.reps} \cdot M \cdot \texttt{discL.proj}.$$

2.3. **Overlattices.** Let L' be an even overlattice of L. Then a basis $b'_1, \dots, b'_n$ of L' is specified by an $n \times n$ matrix `emb` such that $v \mapsto v \cdot \texttt{emb}$ is the canonical embedding $L \hookrightarrow L'$ with respect to the basis $b_1, \dots, b_n$ of L and the basis $b'_1, \dots, b'_n$ of L'. Namely, the row vectors of $\texttt{emb}^{-1}$ are the vector representations of $b'_1, \dots, b'_n$ with respect to the basis $b_1, \dots, b_n$ of $L \otimes \mathbb{Q}$.

Let H be an isotropic subspace of q_L, and let L' be the corresponding even overlattice of L. Then the matrix `emb` that describes $L \hookrightarrow L'$ can be easily calculated from `discL.reps` and the generating matrix of H.

2.4. **Negative-definite root lattices.** An ADE-type is expressed as a list of indecomposable ADE-types

`"A1", "A2", ..., , "D4", "D5", ..., "E6", "E7", "E8".`

Each ADE-type is sorted by the ordering

`"A1" < "A2" < ... < "D4" <"D5" < ... < "E6" < "E7" < "E8".`

For example, an ADE-type $t = E_6 + 3A_4 + A_1 + D_7$ is expressed as

`["A1", "A4", "A4", "A4", "D7", "E6"].`

The following data are available.

- The record `GramADE` contains the following data. For each indecomposable ADE-type y, the Gram matrix `GramADE.y` of the negative-definite root lattice $R(y)$ of type y is given with respect to the basis given in Figure 2.1. For example, we have

$$\texttt{GramADE.D4} = \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 1 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

  The record `GramADE` is given only for indecomposable ADE-types of rank ≤ 9; that is, for A_l ($l \le 9$), D_m ($4 \le m \le 9$), and E_6, E_7, E_8.
- The record `discADE` is the record such that, for an indecomposable ADE-type y of rank ≤ 9, `discADE.y` is the record that describes the discriminant form of the negative-definite root lattice of type y.

Definition 2.2. Let $t = [y_1, \dots, y_m]$ be an ADE-type, where $y_1, \dots, y_m$ are indecomposable ADE-types sorted as above. Then an ADE-*basis* of the negative-definite root lattice $R(t)$ of type t is an *ordered* basis such that the Gram matrix with respect to this basis is equal to the block-diagonal matrix whose diagonal blocks are

$$\texttt{GramADE.}y_1\texttt{, ..., GramADE.}y_m.$$

For example, if $t = 2A_1 + A_2 =$ `["A1", "A1", "A2"]`, then the Gram matrix of the negative-definite root lattice of type t with respect to an ADE-basis is

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

The Gram matrix of the negative-definite root lattice $R(t)$ of type t is constructed from `GramADE`, and the discriminant form of $R(t)$ is constructed from `discADE`.

3. The lattice L_{10}

`GramL10` is the Gram matrix of L_{10} with respect to the fixed basis $e_1, \dots, e_{10}$. In the following, every element of $\mathrm{O}^+(L_{10})$ is expressed by a 10×10 matrix with respect to the basis $e_1, \dots, e_{10}$.

Each element $\Sigma = \{e_{i_1}, \dots, e_{i_n}\}$ of $\mathcal{S}$ is expressed by the list $[i_1, \dots, i_n]$ of indices such that $1 \le i_1 < \cdots < i_n \le 10$.

`SigmasList` is the list of all $\Sigma \in \mathcal{S}$.

`Sigmas` is the list of 1021 records. Each record `Sigma` in this list contains the following data of $\Sigma \in \mathcal{S}$. The list `Sigmas` is sorted according to `SigmasList`.

- `Sigma.vects` is the list $[i_1, \dots, i_n]$ such that $1 \le i_1 < \cdots < i_n \le 10$ and $\Sigma = \{e_{i_1}, \dots, e_{i_n}\}$.
- `Sigma.type` is the ADE-type $\tau(\Sigma)$ of Σ.
- `Sigma.ADEbasis` is a permutation $[j_1, \dots, j_n]$ of $[i_1, \dots, i_n]$ such that the vectors $e_{j_1}, \dots, e_{j_n}$ form an ADE-basis of the negative-definite root lattice $\langle\Sigma\rangle$.
- `Sigma.embSigmaL10` is the matrix of the embedding $\langle\Sigma\rangle \hookrightarrow L_{10}$ with respect to the ADE-basis of $\langle\Sigma\rangle$ given by `Sigma.ADEbasis` and the basis $e_1, \dots, e_{10}$ of L_{10}.
- `Sigma.Gram` is the Gram matrix of $\langle\Sigma\rangle$ with respect to the ADE-basis $e_{j_1}, \dots, e_{j_n}$.
- `Sigma.AutGenerators` is a generating set of $\mathrm{Aut}(\Sigma)$, which is a list of matrices in $\mathrm{O}(\langle\Sigma\rangle)$ with respect to the ADE-basis of $\langle\Sigma\rangle$ fixed by `Sigma.ADEbasis`. When $\mathrm{Aut}(\Sigma)$ is trivial, this list contains only the identity matrix of size n.
- `Sigma.opposite` is the isometry $\xi \in \mathrm{O}^+(L_{10})$ such that Δ_0^ξ is the Vinberg chamber opposite to Δ_0 with respect to the face F_Σ of Δ_0.
- `Sigma.partners` is the list $[\nu_1, \dots, \nu_K]$ with $1 \le \nu_1 < \cdots < \nu_K \le 1021$ such that the set of indices $\{\mathtt{i} \mid \mathtt{SigmasList[i].type} = \tau(\Sigma)\}$ is equal to $\{\nu_1, \dots, \nu_K\}$.
- `Sigma.isomto` is the first index ν_1 of `Sigma.partners`, which indicates the representative element Σ_0 of the set $\{\Sigma' \in \mathcal{S} \mid \tau(\Sigma') = \tau(\Sigma)\}$, which can be identified with the orbit of the action of $\mathrm{O}^+(L_{10})$ on $\mathcal{N}$ containing Σ.
- `Sigma.isomby` is an isometry $g \in \mathrm{O}^+(L_{10})$ such that $\langle\Sigma\rangle^g = \langle\Sigma_0\rangle$.
- `Sigma.kappatildeGSigma` is the list of elements of $\tilde{\kappa}(\mathcal{G}_\Sigma)$, which generates $\mathrm{Stab}(\Sigma, L_{10})$. Each element of `Sigma.kappatildeGSigma` is a matrix in $\mathrm{O}^+(L_{10})$. When $\mathrm{Stab}(\Sigma, L_{10})$ is trivial, this list contains only the identity matrix of size 10.
- `Sigma.HSigmaGenerators` is the list of elements of $\kappa(\mathrm{res}(\mathcal{G}_\Sigma)) = \mathrm{res}(\tilde{\kappa}(\mathcal{G}_\Sigma))$, which generates the subgroup H_Σ of $\mathrm{Aut}(\Sigma)$. Each element of this list is a matrix in $\mathrm{O}(\langle\Sigma\rangle)$ with respect to the ADE-basis of $\langle\Sigma\rangle$. When H_Σ is trivial, this list contains only the identity matrix of size n.

4. Even overlattices of $\langle\Phi\rangle$

`PhisList` is the list of ADE-types $\tau(\Phi)$ of ADE-configurations Φ with $|\Phi| < 10$.

`Phis` is a list of records. Each record `Phi` in `Phis` contains the following data of an ADE-configuration Φ whose type is in `PhisList`. The list `Phis` is sorted according to `PhisList`. We put $n := |\Phi| < 10$.

- `Phi.type` is the ADE-type $\tau(\Phi)$ of Φ.
- `Phi.Gram` is the Gram matrix of the negative-definite root lattice $\langle\Phi\rangle$ with respect to the ADE-basis Φ of $\langle\Phi\rangle$.

- `Phi.disc` is the record of the discriminant form $q_{\langle\Phi\rangle}$ of $\langle\Phi\rangle$. Let $[a_1, \dots, a_l]$ be the item `Phi.disc.discg`, so that $A_{\langle\Phi\rangle} \cong \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_l\mathbb{Z}$.
- `Phi.AutGenerators` is a generating set of $\mathrm{Aut}(\Phi)$, which is a list of matrices in $\mathrm{O}(\langle\Phi\rangle)$ with respect to the ADE-basis of $\langle\Phi\rangle$. When $\mathrm{Aut}(\Phi)$ is trivial, this list contains only the identity matrix of size n.
- `Phi.discAutGenerators` is the set of images of elements of the generating set `Phi.AutGenerators` of $\mathrm{Aut}(\Phi)$ by the natural homomorphism $\mathrm{O}(\langle\Phi\rangle) \to \mathrm{O}(q_{\langle\Phi\rangle})$. Each automorphism in `Phi.discAutGenerators` is expressed by an $l \times l$ matrix with respect to the generators of $A_{\langle\Phi\rangle}$ fixed by `Phi.disc`. When the image of $\mathrm{Aut}(\Phi)$ by $\mathrm{O}(\langle\Phi\rangle) \to \mathrm{O}(q_{\langle\Phi\rangle})$ is trivial, this list contains only the identity matrix of size l. (When `Phi.type` is ["E8"] (that is, `Phi.disc.discg` is the empty list [ ]), `Phi.discAutGenerators` is the empty list.)
- `Phi.overlattices` is the list of representatives of orbits of the action of $\mathrm{Aut}(\Phi)$ on the set $\mathcal{L}(\Phi)$ of even overlattices of $\langle\Phi\rangle$. Each element of this list is a record `Rbar` that describes the following data of an even overlattice $\overline{R}$ of $\langle\Phi\rangle$:
  - `Rbar.isotropicspace` is the standard generating matrix of the isotropic subspace $\overline{R}/\langle\Phi\rangle \subset A_L$ of $q_{\langle\Phi\rangle}$.
  - `Rbar.torsion` $= [b_1, \dots, b_m]$ indicates that the finite abelian group $\overline{R}/\langle\Phi\rangle$ is isomorphic to $\mathbb{Z}/b_1\mathbb{Z} \times \cdots \times \mathbb{Z}/b_m\mathbb{Z}$. When $\overline{R} = \langle\Phi\rangle$, we have `Rbar.torsion`= [ ].
  - `Rbar.embRRbar` is an $n \times n$ matrix that describes the embedding $\langle\Phi\rangle \hookrightarrow \overline{R}$ with respect to the ADE-basis Φ of $\langle\Phi\rangle$. This matrix `Rbar.embRRbar` is used to fix a basis $v_1, \dots, v_n$ of $\overline{R}$ by the remark in Section 2.3. (See also Remark 4.1 below.)
  - `Rbar.Gram` is the Gram matrix of $\overline{R}$ with respect to the basis of $\overline{R}$ fixed by `Rbar.embRRbar`.
  - `Rbar.StabGenerators` is a generating set of the stabilizer subgroup $\mathrm{Stab}(\overline{R}, \Phi)$ of $\overline{R}$ in $\mathrm{Aut}(\Phi)$. Each element of `Rbar.StabGenerators` is expressed as a matrix in $\mathrm{O}(\langle\Phi\rangle)$ with respect to the ADE-basis Φ of $\langle\Phi\rangle$. If $\mathrm{Stab}(\overline{R}, \Phi)$ is trivial, this list contains only the identity matrix of size n.
  - `Rbar.IsADE` indicates whether $\overline{R}$ is a negative-definite root lattice or not. If $\overline{R}$ is not a negative-definite root lattice, then `Rbar.IsADE` is `false`. If $\overline{R}$ is a negative-definite root lattice, then `Rbar.IsADE` is the ADE-type $\tau(\overline{R})$ of $\overline{R}$.

Remark 4.1. Let $b_1, \dots, b_n$ be the ADE-basis Φ of $\langle\Phi\rangle$, and let $v_1, \dots, v_n$ be the basis of $\overline{R}$ fixed by the matrix `Rbar.embRRbar`. Suppose that $\overline{R}$ is a negative-definite root lattice. Then the basis $v_1, \dots, v_n$ of $\overline{R}$ is chosen so that $v_1, \dots, v_n$ form an

ADE-basis of $\overline{R}$, and that the image of the connected component

$$\{ \, x \in \langle \Phi \rangle \otimes \mathbb{R} \;\mid\; \langle x, b_i \rangle > 0 \;\text{ for } i = 1, \dots, n \; \}$$

of $(\langle \Phi \rangle \otimes \mathbb{R})^\circ$ by the embedding $v \mapsto v \cdot$ `Rbar.embRRbar` contains the connected component

$$\{ \, y \in \overline{R} \otimes \mathbb{R} \;\mid\; \langle y, v_i \rangle > 0 \;\text{ for } i = 1, \dots, n \; \}$$

of $(\overline{R} \otimes \mathbb{R})^\circ$.

Remark 4.2. The natural homomorphism $\mathrm{Stab}(\overline{R}, \Phi) \to \mathrm{O}(\overline{R})$ can be easily calculated by `Rbar.embRRbar`.

5. The equivalence classes of embeddings

The data `EmbsList` is the list of pairs $(\tau(\Phi), \Sigma)$, where $\tau(\Phi)$ is the ADE-type of an ADE-configuration Φ with $n := |\Phi| < 10$, and $\Sigma = [i_1, \dots, i_n]$ is an element of $\mathcal{S}$ with $1 \le i_1 < \dots < i_n \le 10$ such that $\langle \Sigma \rangle$ is isomorphic to an even overlattice $\overline{R}$ of $\langle \Phi \rangle$. Each pair gives a representative embedding $f \colon \Phi \hookrightarrow L_{10}$ of the single element of

$$\overline{\mathrm{emb}}([\overline{R}], [\Sigma]) = \mathrm{Stab}(\overline{R}, \Phi) \backslash \mathrm{O}(\langle \Sigma \rangle) / \mathrm{Stab}(\langle \Sigma \rangle, L_{10}) = H_\Phi \backslash \mathrm{Aut}(\Sigma) / H_\Sigma.$$

Therefore `EmbsList` gives the set $\mathrm{Aut}(\Phi) \backslash \mathrm{Emb}(\Phi) / \mathrm{O}^+(L_{10})$.

The data `Embs` is a list of records. A member `f` of `Embs` gives the following data of an embedding $f \colon \Phi \hookrightarrow L_{10}$ such that $\overline{R}_f = \langle \Sigma \rangle$. The list `Embs` is sorted according to `EmbsList`. Let `Phi` be the member of `Phis` that describes $\Phi = \{r_1, \dots, r_n\}$, and let `Sigma` be the member of `Sigmas` that describes $\Sigma = \{e_{i_1}, \dots, e_{i_n}\}$. Recall that $\Phi_f = \{r_1^+, \dots, r_n^+\}$.

- `f.Phi` is the ADE-type $\tau(\Phi)$ of $\Phi = \{r_1, \dots, r_n\}$.
- `f.SigmaRecord` is a copy of the record `Sigma`. Hence, for example, the ADE-type $\tau(\Sigma)$ of Σ is given by `f.SigmaRecord.type`.
- `f.GramPhi` is the Gram matrix of $\langle \Phi \rangle$ with respect to the ADE-basis Φ.
- `f.GramSigma` is the Gram matrix of $\langle \Sigma \rangle$ with respect to the ADE-basis given by `f.SigmaRecord.ADEbasis`.
- `f.Rbar` is a record that describes the even overlattice $\overline{R}$ of $\langle \Phi \rangle$ corresponding to the even overlattice $\overline{R}_f$ of R_f via $f \colon \langle \Phi \rangle \cong R_f$. This record is a copy of the member of `Phi.overlattices` that describes $\overline{R}$. By Remark 4.1, the row vectors of $($`f.Rbar.embRRbar`$)^{-1}$ are the vector representations of an ADE-basis of $\overline{R}$ with respect to the ADE-basis Φ of $\langle \Phi \rangle \otimes \mathbb{Q}$. Identifying this ADE-basis of $\overline{R}$ with the ADE-basis `f.SigmaRecord.ADEbasis` of $\langle \Sigma \rangle$ gives an element $g_0 \in \mathrm{Isom}(\overline{R}, \langle \Sigma \rangle)$ explicitly. Using g_0 as a reference point, we identify $\mathrm{Isom}(\overline{R}, \langle \Sigma \rangle)$ with $\mathrm{O}(\langle \Sigma \rangle)$, and regard $\mathrm{Stab}(\overline{R}, \Phi)$ as a subgroup of $\mathrm{O}(\langle \Sigma \rangle)$.

- `f.embPhiSigma` is the $n\times n$ matrix whose ith row vector is the vector representation of $r_i^+ \in \langle\Sigma\rangle$ with respect to the ADE-basis $e_{j_1},\dots,e_{j_n}$ of $\langle\Sigma\rangle$. By the identification g_0, we have `f.embPhiSigma` is equal to `f.Rbar.embRRbar`.
- `f.embSigmaL10` is the $n\times 10$ matrix whose row vectors are $e_{j_1},\dots,e_{j_n}$.
- `f.embPhiL10` is the $n\times 10$ matrix whose ith row vectors is the vector representation of r_i^+ with respect to the basis $e_1,\dots,e_{10}$ of L_{10}. Hence `f.embPhiL10` is equal to (`f.embPhiSigma`) $\cdot$ (`f.embSigmaL10`).
- `f.HPhiGenerators` is the image of elements of the generating set `f.Rbar.StabGenerators` of $\mathrm{Stab}(\overline{R},\Phi)$ by the homomorphism

$$\mathrm{Stab}(\overline{R},\Phi)\to \mathrm{O}(\overline{R})\cong \mathrm{O}(\langle\Sigma\rangle)\to \mathrm{Aut}(\Sigma),$$

  where $\mathrm{O}(\overline{R})\cong \mathrm{O}(\langle\Sigma\rangle)$ is induced by $g_0\colon \overline{R}\cong\langle\Sigma\rangle$ and $\mathrm{O}(\langle\Sigma\rangle)\to\mathrm{Aut}(\Sigma)$ is the quotient homomorphism by $W(\langle\Sigma\rangle)$. Each element of `f.HPhiGenerators` is a matrix in $\mathrm{O}(\langle\Sigma\rangle)$ with respect to the ADE-basis $e_{j_1},\dots,e_{j_n}$ of $\langle\Sigma\rangle$.

The fact that $\overline{\mathrm{emb}}([\overline{R}],[\Sigma])$ consists of a single element (Theorem 3.22 of [2]) is proved by confirming the following fact: Every element of the finite group $\mathrm{Aut}(\Sigma)$ generated by `f.SigmaRecord.AutGenerators` is written as $h_\Phi h_\Sigma$, where h_Φ is an element of H_Φ generated by the elements of `f.HPhiGenerators` and h_Σ is an element of H_Σ generated by the elements of `f.SigmaRecord.HSigmaGenerators`.

6. Geometric realizability

Let `f` be a member of `Embs` as in the previous section. The record `f` also contains the following data of geometric realizability of $f\colon\Phi\hookrightarrow L_{10}$.

- `f.StabPhiL10Generators` is a generating set of $\mathrm{Stab}(\Phi_f,L_{10})$. Each element of `f.StabPhiL10Generators` is a matrix in $\mathrm{O}^+(L_{10})$ with respect to the basis $e_1,\dots,e_{10}$. When $\mathrm{Stab}(\Phi_f,L_{10})$ is trivial, this list contains only the identity matrix I_{10} of size 10.
- `f.varpiB` is the matrix of the embedding $\varpi^*\colon L_{10}\hookrightarrow B_\Phi$ with respect to the basis $e_1,\dots,e_{10}$ of L_{10} and the basis

$$\varpi^*(e_1),\dots,\varpi^*(e_{10}),\varphi(r_1^-),\dots,\varphi(r_n^-) \tag{6.1}$$

  of B_Φ; that is, the $10\times(10+n)$ matrix $[\,I_{10}\,|\,O\,]$.
- `f.GramB` is the Gram matrix of B_Φ with respect to the basis (6.1) of B_Φ; that is, the block-diagonal matrix with diagonal blocks $2\cdot$ `GramL10` and $2\cdot$ `f.GramPhi`.
- `f.involB` is the matrix representation of the involution ε of B_Φ that acts as the identity on ϖ^*L_{10} and as the scalar multiplication by -1 on its orthogonal complement; that is, `f.involB` is the block-diagonal matrix with diagonal blocks I_{10} and $-I_n$.
- `f.discB` is the record that describes the discriminant form of B_Φ.

- `f.Lifts` is the list of lifts $r'_1, \dots, r'_n \in B_\Phi \otimes \mathbb{Q}$ written with respect to the basis (6.1) of B_Φ.
- `f.embBMf` is a matrix of the embedding $B_\Phi \hookrightarrow M_f$ with respect to the basis (6.1) of B_Φ. This matrix is used to fix a basis of M_f by the remark in Section 2.3.
- `f.GramMf` is the Gram matrix of M_f with respect to the basis fixed by `f.embBMf`.
- `f.UGenerators` is a generating set of the subgroup $U(M_f) \subset \mathrm{O}^+(M_f)$, each element of which is written as a matrix with respect the basis (6.1) of B_Φ (*not* with respect the basis of M_f fixed by `f.embBMf`).
- `f.UGeneratorsDisc` is a generating set of the image of $U(M_f) \subset \mathrm{O}(B_\Phi)$ by the natural homomorphism $\mathrm{O}(B_\Phi) \to \mathrm{O}(q_{B_\Phi})$. Each element of this list is written with respect to the basis of the discriminant group of B_Φ fixed by `f.discB`.
- `f.Mfisotropicspace` is the standard generating matrix of the isotropic subspace M_f/B_Φ of q_{B_Φ}.
- `f.varpiMf` is the matrix of $\varpi^* \colon L_{10} \hookrightarrow M_f$ with respect to the basis $e_1, \dots, e_{10}$ of L_{10} and the basis of M_f fixed by `f.embBMf`
- `f.involMf` is the matrix representation of the involution ε of M_f with respect the basis of M_f fixed by `f.embBMf`.

Let $\mathcal{L}'(M_f)$ be the set of even overlattices of M_f that satisfy the conditions (C2), (C3), (C4). This set is calculated as the set of isotropic subspaces of q_{B_Φ} containing M_f/B_Φ. The orbits of the action of $U(M_f)$ on $\mathcal{L}'(M_f)$ is calculated by means of the action of the finite group generated by `f.UGeneratorsDisc` on the set of all isotropic subspaces of q_{B_Φ} containing M_f/B_Φ.

- `f.Mfbars` is the list of records `Mfbar` that describe the representatives of orbits of the action of $U(M_f)$ on $\mathcal{L}'(M_f)$. Each record `Mfbar` contains the following data of an even overlattice $\overline{M}_f$ of M_f that satisfies the conditions (C2), (C3), (C4).
  - `Mfbar.embBMfbar` is a matrix of the embedding $B_\Phi \hookrightarrow \overline{M}_f$ with respect to the basis (6.1) of B_Φ. This matrix is used to fix a basis of $\overline{M}_f$ by the remark in Section 2.3.
  - `Mfbar.embMfMfbar` is a matrix of the embedding $M_f \hookrightarrow \overline{M}_f$ with respect to the basis of M_f fixed by `f.embBMf` and the basis of $\overline{M}_f$ fixed by `f.embBMfbar`.
  - `Mfbar.Gram` is the Gram matrix of $\overline{M}_f$ with respect to the basis fixed by `f.embBMfbar`.
  - `Mfbar.varpi` is the matrix of $\varpi^* \colon L_{10} \hookrightarrow \overline{M}_f$ with respect to the basis $e_1, \dots, e_{10}$ of L_{10} and the basis of $\overline{M}_f$ fixed by `Mbar.embBMfbar`

– `Mfbar.invol` is the matrix representation of the involution ε of $\overline{M}_f$ respect the basis of $\overline{M}_f$ fixed by `Mfbar.embBMfbar`.
– `Mfbar.isotropicspace` is the standard generating matrix of the isotropic subspace $\overline{M}_f/B_\Phi$ of the discriminant form of B_Φ.
– `Mfbar.Q` is the finite abelian group $Q := \overline{M}_f/M_f$. `Mfbar.Q`$= [b_1, \dots, b_m]$ means that Q is isomorphic to $\mathbb{Z}/b_1\mathbb{Z}\times\cdots\times\mathbb{Z}/b_m\mathbb{Z}$. When Q is trivial, we have `Mfbar.Q`$=[\ ]$.
– `Mfbar.disc` is the record that describes the discriminant form of $\overline{M}_f$, which is used to determine whether $\overline{M}_f$ satisfies (C1) or not.
– `Mfbar.C1` tells whether $\overline{M}_f$ satisfies the condition (C1). This data is `true` if $\overline{M}_f$ satisfies (C1), whereas it is `false` otherwise.

7. Table

The data `Table` is a list of records. This list gives more detailed information of Table 1.1 of [2]. Each record `tablerow` in `Table` contains the following data of each row of Table 1.1 of [2].

- `tablerow.No` is the number.
- `tablerow.Phi` is the ADE-type $\tau(\Phi_f)$ of Φ_f.
- `tablerow.SigmaType` is is the ADE-type $\tau(\overline{R}_f)$ of $\overline{R}_f = \langle\Sigma\rangle$.
- `tablerow.SigmaVects` is the vector representations of elements of Σ with respect to $e_1, \dots, e_{10}$. These vectors are sorted in such a way that they form an ADE-basis of $\overline{R}_f = \langle\Sigma\rangle$.
- `tablerow.embPhiSigma` is the matrix of the embedding $f\colon \langle\Phi_f\rangle \hookrightarrow \overline{R}_f = \langle\Sigma\rangle$ with respect to the ADE-bases Φ_f and Σ.
- `tablerow.embSigmaL10` is the matrix of the embedding $\langle\Sigma\rangle \hookrightarrow L_{10}$ with respect to the ADE-basis Σ sorted as `tablerow.SigmaVects` and the basis $e_1, \dots, e_{10}$ of L_{10}. This matrix is identical with `tablerow.SigmaType`.
- `tablerow.embPhiL10` is the matrix of the embedding $f\colon \langle\Phi_f\rangle \hookrightarrow L_{10}$ with respect to the ADE-basis Φ_f and the basis $e_1, \dots, e_{10}$ of L_{10}. Hence this matrix is equal to (`tablerow.embPhiSigma`) $\cdot$ (`tablerow.embSigmaL10`).
- `tablerow.varpiB` is the matrix of the embedding $\varpi^*\colon L_{10} \hookrightarrow B_\Phi$ with respect to the basis $e_1, \dots, e_{10}$ of L_{10} and the basis (6.1) of B_Φ; that is, `tablerow.varpiB` is equal to $[\,I_{10}\,|\,O\,]$.
- `tablerow.embBMf` is the matrix of the embedding $B_\Phi \hookrightarrow M_f$ with respect to the basis (6.1) of B_Φ. This matrix fixes a basis of M_f by the remark in Section 2.3.
- `tablerow.GramPhi` is the Gram matrix of $\langle\Phi\rangle$ with respect to the ADE-basis Φ.
- `tablerow.GramSigma` is the Gram matrix of $\overline{R}_f = \langle\Sigma\rangle$ with respect to the ADE-basis `tablerow.SigmaVects`.

- `tablerow.GramMf` is the Gram matrix of M_f with respect to the basis of M_f fixed by `tablerow.embBMf`.
- `tablerow.StrongEquivClasses` is the list of representatives of strong equivalence classes of RDP-Enriques surfaces that geometrically realize the embedding $f\colon \Phi \hookrightarrow L_{10}$; that is, `tablerow.StrongEquivClasses` is the list of representatives $\overline{M}_f$ of the orbits of the action of $U(M_f)$ on the set of even overlattices of M_f that satisfy the conditions (C1), ..., (C4). Each record `strongequiv` of `tablerow.StrongEquivClasses` contains the following data of an even overlattice $\overline{M}_f$.
  - `strongequiv.embMfMfbar` is a matrix of the embedding $M_f \hookrightarrow \overline{M}_f$ with respect to the basis of M_f fixed by `tablerow.embBMf`. This matrix fixes a basis of $\overline{M}_f$ by the remark in Section 2.3.
  - `strongequiv.GramMfbar` is the Gram matrix of $\overline{M}_f$ with respect to the basis of $\overline{M}_f \cong S_X$ fixed by `tablerow.embMfMfbar`.
  - `strongequiv.involMfbar` is the matrix representation of the Enriques involution ε on $\overline{M}_f \cong S_X$.
  - `strongequiv.Q` is the finite abelian group $Q := \overline{M}_f/M_f$. `strongequiv.Q`$= [b_1, \dots, b_m]$ means that $Q = \overline{M}_f/M_f$ is isomorphic to $\mathbb{Z}/b_1\mathbb{Z} \times \cdots \times \mathbb{Z}/b_m\mathbb{Z}$. When Q is trivial, `strongequiv.Q` is an empty list [ ].

References

[1] The GAP Group. GAP - Groups, Algorithms, and Programming. Version 4.7.9; 2015 (http://www.gap-system.org).

[2] Ichiro Shimada. Rational double points on Enriques surfaces, preprint, 2017, http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3.html.

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