THE ELLIPTIC MODULAR SURFACE OF LEVEL 4 AND ITS REDUCTION MODULO 3: COMPUTATIONAL DATA

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1. Introduction

This note is an explanation of the computational data obtained in the paper [1]. The data are available from the author's webpage:

`http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3andEnriques.html`

The data are written in four files:

- `SOS3.txt` (Section 2 of the paper [1]),
- `PGU.txt` (the elements of $\mathrm{PGU}_4(\mathbb{F}_9)$),
- `Borcherds.txt` (Sections 4 and 5 of the paper [1]),
- `Enriques.txt` (Section 6 of the paper [1]).

A zipped folder (`XOX3compdata.zip`) containing these files is also available from the site above.

We use the notation of [1] in the following.

2. Conventions

(1) Every finite set is sorted in a certain way, and is expressed as a list

$$[\mathtt{elm}_1, \dots, \mathtt{elm}_M].$$

A map f from a finite set $X = [\mathtt{elm}_1, \dots, \mathtt{elm}_M]$ to a finite set $Y = [\mathtt{elm}'_1, \dots, \mathtt{elm}'_N]$ is expressed as a list of indices $[i_1, \dots, i_M]$ such that f maps $\mathtt{elm}_\nu \in X$ to $\mathtt{elm}'_{i_\nu} \in Y$ for $\nu = 1, \dots, M$. In particular, a permutation of X is expressed by a permutation of $1, \dots, M$.

(2) Every lattice has a fixed basis. Let L be a lattice of rank m. Then L is expressed by the Gram matrix with respect to the fixed basis. Each element of $L \otimes \mathbb{Q}$ is expressed by a row vector of length m with respect to the fixed basis, and every isometry $g \in \mathrm{O}(L)$ is expressed by an $m \times m$ matrix with respect to the fixed basis. (Recall that $\mathrm{O}(L)$

acts on L from the right.) A map f from a finite set $X = [\mathtt{elm}_1, \dots, \mathtt{elm}_M]$ to $L \otimes \mathbb{Q}$ is given by the list of row vectors expressing $f(\mathtt{elm}_\nu)$ for $\nu = 1, \dots, M$. Let L' be a lattice of rank n. A linear map from $L \otimes \mathbb{Q}$ to $L' \otimes \mathbb{Q}$ is expressed by an $m \times n$ matrix with respect to the fixed bases of L and L'.

(3) The discriminant form (A, q) of an even lattice L of rank n is expressed by a list

$$[\,\mathtt{discg},\ \mathtt{discf},\ \mathtt{proj},\ \mathtt{lift}\,].$$

The list $\mathtt{discg} = [a_1, \dots, a_k]$ of integers $a_i > 1$ indicates that the discriminant group $A = L^\vee/L$ is isomorphic to

$$\mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_k\mathbb{Z}.$$

Let $\bar{v}_1, \dots, \bar{v}_k$ be generators of A such that $\bar{v}_i$ generates the ith factor $\mathbb{Z}/a_i\mathbb{Z}$ of A. The (i, j)-component of the $k \times k$ matrix $\mathtt{discf}$ is a rational number that expresses

$$\begin{cases} q(\bar{v}_i) \in \mathbb{Q}/2\mathbb{Z} & \text{if } i = j, \\ b(\bar{v}_i, \bar{v}_j) \in \mathbb{Q}/\mathbb{Z} & \text{if } i \neq j, \end{cases}$$

where $b(x, y) = (q(x + y) - q(x) - q(y))/2$. The third item $\mathtt{proj}$ is the $n \times k$ integer matrix M such that $v \mapsto \bar{v} = vM$ is the canonical projection $L^\vee \to A$, where $v \in L^\vee$ is expressed by a row vector with respect to the fixed basis of $L \otimes \mathbb{Q}$ (*not* with respect to the canonical dual basis of $L^\vee$). Hence the (i, j)-component of M should be regarded as an element of $\mathbb{Z}/a_j\mathbb{Z}$. The item $\mathtt{lift}$ is the list $[v_1, \dots, v_k]$ of elements of $L^\vee \subset L \otimes \mathbb{Q}$ that are mapped to $[\bar{v}_1, \dots, \bar{v}_k]$ by the canonical projection. (Again, the vector v_i is written with respect to the fixed basis of $L \otimes \mathbb{Q}$.) An automorphism $\bar{g}$ of (A, q) is expressed by a $k \times k$ integer matrix $M_{\bar{g}}$ with respect to the basis $\bar{v}_1, \dots, \bar{v}_k$ of A. Hence the (i, j)-component of $M_{\bar{g}}$ should be regarded as an element of $\mathbb{Z}/a_j\mathbb{Z}$. Note that, by $\mathtt{proj}$ and $\mathtt{lift}$, we can easily calculate the image $\eta_L(g) \in \mathrm{O}(q)$ of $g \in \mathrm{O}(L)$ by the natural homomorphism $\eta_L \colon \mathrm{O}(L) \to \mathrm{O}(q)$.

(4) Let $\Gamma = (V, \eta)$ be a finite graph. The set $V = \{v_1, \dots, v_n\}$ of vertices is sorted in a certain way. The graph Γ is expressed by an $n \times n$ matrix whose (i, j)-component is

$$\begin{cases} -2 & \text{if } i = j, \\ \eta(\{v_i, v_j\}) & \text{if } i \neq j. \end{cases}$$

A map from a finite simple graph (V, E) to a finite simple graph (V', E') is expressed by a map $V \to V'$ of sets of vertices (see Convention (1) above). A map from a finite graph (V, E) to a lattice L is expressed by a map from V to L (see Convention (2) above).

(5) Let $\bar{Z}$ be a normal $K3$ surface, that is, a normal surface whose minimal resolution Z is a $K3$ surface. Then $\bar{Z}$ has only rational double points as its singularities. The singularities of $\bar{Z}$ are described by a list of pairs [`type`, `rs`]. Each pair gives the ADE-type `type` (expressed by a string such as `"A1"`, `"A2"`, ...) of a singular point P of $\bar{Z}$ and the list `rs` $= [r_1, \dots, r_m]$ of classes $r_i = [C_i] \in S_Z$ of smooth rational curves C_i on Z that are contracted to the singular point P.

3. The file `SOS3.txt`

In the file `SOS3.txt`, we have the following data, which are related to the materials in Section 2 of the paper [1].

3.1. QP-**graphs.** The set of vertices of the Petersen graph $\mathcal{P}$ is the list

$$V_{\mathcal{P}} = [1, \dots, 10].$$

The set of vertices of a QP-graph $\mathcal{Q}$ is the list

$$V_{\mathcal{Q}} = [1, \dots, 40].$$

- `PG` is the Petersen graph $\mathcal{P}$.
- `GraphQP0` is the graph $\mathcal{Q}_0$.
- `GraphQP1` is the graph $\mathcal{Q}_1$.
- `QPgamma0` is the QP-covering map $\gamma_0 \colon \mathcal{Q}_0 \to \mathcal{P}$.
- `QPgamma1` is the QP-covering map $\gamma_1 \colon \mathcal{Q}_1 \to \mathcal{P}$.
- `GramQP0` is the Gram matrix of $\langle \mathcal{Q}_0 \rangle$.
- `GramQP1` is the Gram matrix of $\langle \mathcal{Q}_1 \rangle$.
- `embQP0` is the canonical map $\mathcal{Q}_0 \hookrightarrow \langle \mathcal{Q}_0 \rangle$. (See Convention (4).) From `embQP0`, we can recover the basis of $\langle \mathcal{Q}_0 \rangle$ with respect to which `GramQP0` is written. We do not have a simple description of this basis.

- `embQP1` is the canonical map $\mathcal{Q}_1 \hookrightarrow \langle \mathcal{Q}_1 \rangle$. From `embQP1`, we see that $\langle \mathcal{Q}_1 \rangle$ has a basis consisting of the classes of the vertices

$$1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 21, 23, 25, 29, 33.$$

- `discQP0` is the discriminant form of $\langle \mathcal{Q}_0 \rangle$.
- `discQP1` is the discriminant form of $\langle \mathcal{Q}_1 \rangle$.
- `AutPG` is $\mathrm{Aut}(\mathcal{P})$, which is a list of permutations of $V_{\mathcal{P}} = [1, \dots, 10]$.
- `AutQP0` is $\mathrm{Aut}(\mathcal{Q}_0)$, which is a list of permutations of $V_{\mathcal{Q}_0} = [1, \dots, 40]$.
- `AutQP1` is $\mathrm{Aut}(\mathcal{Q}_1)$, which is a list of permutations of $V_{\mathcal{Q}_1} = [1, \dots, 40]$.

3.2. **The line configuration $\mathcal{L}_{112}$ on X_3 and the lattice S_3.** We denote by `I` the element $\sqrt{-1} \in \mathbb{F}_9$. An element of $\mathbb{F}_9$ is written as $a + b\,$`I`, where $a, b \in \{0, 1, -1\}$.

- `L112eqs` is the list of equations of lines on $X_3 \cong F_3$. An equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = 0$$

of a line with $a_{ij} \in \mathbb{F}_9$ is expressed by the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}.$$

The set $\mathcal{L}_{112}$ is sorted according to the list `L112eqs`. We denote by ℓ_i the ith element of $\mathcal{L}_{112}$.

- `GraphL112` is the dual graph of $\mathcal{L}_{112}$.
- `GramS3` is the Gram matrix of S_3.
- `discS3` is the discriminant form of S_3.
- `L112vs` expresses the embedding $\mathcal{L}_{112} \hookrightarrow S_3$ given by $\ell \mapsto [\ell]$, that is, `L112vs` is the list [$[\ell_1], \dots, [\ell_{112}]$] of vectors representing the classes of lines. Looking at `L112vs`, we see that the set of classes of lines

$$\ell_1,\ \ell_2,\ \ell_3,\ \ell_4,\ \ell_5,\ \ell_6,\ \ell_7,\ \ell_9,\ \ell_{10},\ \ell_{11},\ \ell_{17},\ \ell_{18},\ \ell_{19},$$
$$\ell_{21},\ \ell_{22},\ \ell_{23},\ \ell_{25},\ \ell_{26},\ \ell_{27},\ \ell_{33},\ \ell_{35},\ \ell_{49}$$

is the basis of S_3.

- `h3` is the ample class $h_3 \in S_3$.

3.3. The configuration $\mathcal{L}_{40}$ on X_0 and the lattice S_0.

- `GraphL40` is the dual graph of $\mathcal{L}_{40}$. We identify $\mathcal{L}_{40}$ with the set of vertices of $\mathcal{Q}_1$. Hence `GraphL40` is identical to `GraphQP1`.
- The matrix `GramS0` is the Gram matrix of S_0. We fix a basis of S_0 so that `GramS0` is identical with `GramQP1`.
- `discS0` is the discriminant form of S_0. Note that `discS0` is identical with `discQP1`.
- `L40vs` expresses the canonical embedding $\mathcal{L}_{40} \hookrightarrow S_0$ given by $\ell \mapsto [\ell]$. Since we have identified $\mathcal{L}_{40}$ with the set of vertices of $\mathcal{Q}_1$, the item `L40vs` is identical to `embQP1`.
- `h0` is the ample class $h_0 \in S_0$.

3.4. The embeddings $\rho_{\mathcal{L}} \colon \mathcal{L}_{40} \hookrightarrow \mathcal{L}_{112}$ and $\rho \colon S_0 \hookrightarrow S_3$.

- `rhoL` is the embedding $\rho_{\mathcal{L}} \colon \mathcal{L}_{40} \hookrightarrow \mathcal{L}_{112}$.
- `rho` is the embedding $\rho \colon S_0 \hookrightarrow S_3$.

4. The file `PGU.txt`

The group $\mathrm{PGU}_4(\mathbb{F}_9)$ is very large (of order 13063680). Hence this group is recorded in the following way in the file `PGU.txt`. For each line $\ell_k \in \mathcal{L}_{112}$, we choose an element $\tau_k \in \mathrm{PGU}_4(\mathbb{F}_9)$ such that

$$\ell_1^{\tau_k} = \ell_k.$$

Let D be the set of lines ℓ_j such that $\langle \ell_1, \ell_j \rangle = 0$. Then we have $\ell_6 \in D$ and $|D| = 81$. For each $\ell_\nu \in D$, we choose an element $\sigma(\ell_\nu) \in \mathrm{PGU}_4(\mathbb{F}_9)$ such that

$$\ell_1^{\sigma(\ell_\nu)} = \ell_1, \quad \ell_6^{\sigma(\ell_\nu)} = \ell_\nu.$$

We define the following subsets of $\mathrm{PGU}_4(\mathbb{F}_9)$:

$$\begin{aligned}
\mathrm{PGU}_R &:= \{\, g \in \mathrm{PGU}_4(\mathbb{F}_9) \mid \ell_1^g = \ell_1, \ \ \ell_6^g = \ell_6 \,\} = \{\rho_1, \dots, \rho_{1440}\}, \\
\mathrm{PGU}_S &:= \{\, \sigma(\ell_\nu) \in \mathrm{PGU}_4(\mathbb{F}_9) \mid \ell_\nu \in D \,\} = \{\sigma_1, \dots, \sigma_{81}\}, \\
\mathrm{PGU}_T &:= \{\tau_1, \dots, \tau_{112}\}.
\end{aligned}$$

Since $\mathrm{PGU}_4(\mathbb{F}_9)$ acts transitively on the set of ordered pairs of disjoint lines on X_3, every element of $\mathrm{PGU}_4(\mathbb{F}_9)$ is uniquely written in the form

$$\rho_i \sigma_j \tau_k \quad (\ 1 \le i \le 1440, \ \ 1 \le j \le 81, \ \ 1 \le k \le 112\ ).$$

Therefore, as a *set*, $\mathrm{PGU}_4(\mathbb{F}_9)$ can be obtained as the Cartesian product

$$\mathrm{PGU}_R \times \mathrm{PGU}_S \times \mathrm{PGU}_T.$$

Caution. The group $\mathrm{PGU}_4(\mathbb{F}_9)$ acts on $\mathbb{P}^3$ from the *left*, and acts on $\mathcal{L}_{112}$ and S_3 from the *right* by the pull-back. Therefore, if a line $\ell \in \mathcal{L}_{112}$ is defined by an equation $Ax = 0$, where A is a 2×4 matrix, then, for $g \in \mathrm{PGU}_4(\mathbb{F}_9)$, the line $\ell^g = g^{-1}(\ell)$ is defined by the equation $(Ag)x = 0$.

- The three lists `PGUR`, `PGUS`, `PGUT` are the lists of matrices in $\mathrm{PGU}_4(\mathbb{F}_9)$ representing the elements of $\mathrm{PGU}_R, \mathrm{PGU}_S, \mathrm{PGU}_T$, respectively. An item of each of these lists is a 4×4 matrix g with components in $\mathbb{F}_9$ such that ${}^T g \cdot \bar{g}$ is a scalar matrix, where $\bar{g}$ is the matrix obtained from g by applying $a \mapsto a^3$ to each component.
- The three lists `PGURperm`, `PGUSperm`, `PGUTperm` are the lists of permutations on the set $\mathcal{L}_{112}$ induced by the elements of $\mathrm{PGU}_R, \mathrm{PGU}_S, \mathrm{PGU}_T$, respectively. The set `PGURperm` is sorted according to `PGUR`, and the same for `PGUSperm` and `PGUTperm`.
- The three lists `PGUROS3`, `PGUSOS3`, `PGUTOS3` are the lists of isometries of S_3 induced by the elements of $\mathrm{PGU}_R, \mathrm{PGU}_S, \mathrm{PGU}_T$, respectively. The set `PGUROS3` is sorted according to `PGUR`, and the same for `PGUSOS3` and `PGUTOS3`.

5. The file `Borcherds`

This file contains the computational data related to Borcherds' method for X_0 and X_3 (Sections 4.1 and 4.2 of [1]), the data related to $\mathrm{Aut}(X_0, h_0)$ (Section 4.3 of [1]), and the data related to the proof of Theorems 1.7 and 1.8 (Section 5 of [1]).

Let S be an even hyperbolic lattice. Let $i\colon S \hookrightarrow L_{26}$ be a primitive embedding inducing $i_{\mathcal{P}}\colon \mathcal{P}(S) \hookrightarrow \mathcal{P}(L_{26})$, and let $\mathrm{pr}_S\colon L_{26} \otimes \mathbb{Q} \to S \otimes \mathbb{Q}$ be the orthogonal projection. Let $w \in L_{26}$ be a Weyl vector. A wall $(v)^{\perp}$ of a $\mathcal{V}(i)$-chamber $D = i_{\mathcal{P}}^{-1}(\mathcal{C}(w))$ is expressed by a pair $[v, r]$ of the primitive vector v of $S^{\vee}$ defining the wall $(v)^{\perp} \cap D$ of D and a Leech root $r \in \mathcal{R}(L_{26})$ with respect to w such that $(\mathrm{pr}_S(r))^{\perp} = (v)^{\perp}$.

5.1. The lattice L_{26}.

- `GramL26` is the Gram matrix of L_{26}.
- `w0` is the Weyl vector $w_0 \in L_{26}$.
- `w0prime` is a Weyl vector $w_0' \in L_{26}$ such that $\langle w_0, w_0' \rangle = 1$. We can confirm that w_0 is a Weyl vector by showing that the orthogonal complement in L_{26} of the

lattice $\langle w_0, w_0' \rangle$ of rank 2 is an even negative-definite unimodular lattice with no roots.

5.2. **Borcherds' method for X_3.**

- `i3` is the primitive embedding $i_3 \colon S_3 \hookrightarrow L_{26}$.
- `pr3` is the orthogonal projection $\mathrm{pr}_3 \colon L_{26} \otimes \mathbb{Q} \to S_3 \otimes \mathbb{Q}$.
- `OqS3` is the group $\mathrm{O}(q(S_3))$. Each element is expressed by a matrix with respect to the generators of $A(S_3)$ fixed in `discS3`.
- `OqS3period` is the group $\mathrm{O}(q(S_3), \omega)$. Each element is expressed by a matrix with respect to the generators of $A(S_3)$ fixed in `discS3`.
- `h3` is the ample class $h_3 \in S_3$. (This is identical to `h3` given in `SOS3.txt`.)
- `Wout3` is the list of outer-walls of the initial $\mathcal{V}(i_3)$-chamber D_3. The projection $[v, r] \mapsto v$ gives a bijection from `Wout3` to `L112vs`.
- `O648` is the orbit O'_{648} of inner-walls of D_3.
- `O5184` is the orbit O'_{5184} of inner-walls of D_3.

The group $\mathrm{Aut}(X_3, h_3)$ is equal to $\mathrm{PGU}_4(\mathbb{F}_9)$, which is recorded in the file `PGU.txt`. Hence we omit it.

The double-plane involution $g(b'_{10})$.

- `gdpp10` is the double-plane involution $g(b'_{10}) \in \mathrm{Aut}(X_3)$ expressed by a 22×22-matrix acting on S_3.
- `innwall10` is the primitive vector of $S_3^\vee$ (written with respect to the fixed basis of $S_3 \otimes \mathbb{Q}$) that defines the inner-wall of D_3 in the orbit O'_{648} across which $D_3^{g(b'_{10})}$ is adjacent to D_3.
- `dpp10` is a double-plane polarization $b'_{10} \in S_3$ that induces the involution $g(b'_{10})$.
- `Singdpp10` is the singularities of the normal $K3$ surface that is the finite double coverer of $\mathbb{P}^2$ in the Stein factorization of the morphism $X_3 \to \mathbb{P}^2$ induced by b'_{10}. (See Convention (5).)

The double-plane involution $g(b'_{31})$.

- `gdpp31` is the double-plane involution $g(b'_{31}) \in \mathrm{Aut}(X_3)$ expressed by a 22×22-matrix acting on S_3.

- `innwall31` is the primitive vector of $S_3^\vee$ (written with respect to the fixed basis of $S_3 \otimes \mathbb{Q}$) that defines the inner-wall of D_3 in the orbit O'_{5184} across which $D_3^{g(b'_{31})}$ is adjacent to D_3.
- `dpp31` is a double-plane polarization $b'_{31} \in S_3$ that induces the involution $g(b'_{31})$.
- `Singdpp31` is the singularities of the normal $K3$ surface that is the finite double coverer of $\mathbb{P}^2$ in the Stein factorization of the morphism $X_3 \to \mathbb{P}^2$ induced by b'_{31}.

5.3. Borcherds' method for X_0.

- `i0` is the primitive embedding $i_0 \colon S_0 \hookrightarrow L_{26}$.
- `pr0` is the orthogonal projection $\mathrm{pr}_0 \colon L_{26} \otimes \mathbb{Q} \to S_0 \otimes \mathbb{Q}$.
- `OqS0` is the group $\mathrm{O}(q(S_0))$. Each element is expressed by a matrix with respect to the generators of $A(S_0)$ fixed in `discS0`.
- `OqS0period` is the group $\mathrm{O}(q(S_0), \omega)$. Each element is expressed by a matrix with respect to the generators of $A(S_0)$ fixed in `discS0`.
- `h0` is the ample class $h_0 \in S_0$. (This is identical to `h0` given in `S0S3.txt`.)
- `Wout0` is the list of outer-walls of the initial $\mathcal{V}(i_0)$-chamber D_0.
- `O64` is the orbit O_{64} of inner-walls of D_0.
- `O40` is the orbit O_{40} of inner-walls of D_0.
- `O160` is the orbit O_{160} of inner-walls of D_0.
- `O320` is the orbit O_{320} of inner-walls of D_0.
- `AutX0h0` is the group $\mathrm{Aut}(X_0, h_0)$. The order is 3840. Each element of this list is a 20×20 matrix acting on S_0.

The double-plane involution $g(b_{80})$.

- `gdpp80` is the double-plane involution $g(b_{80}) \in \mathrm{Aut}(X_0)$ expressed by a 20×20-matrix acting on S_0.
- `innwall80` is the primitive vector of $S_0^\vee$ (written with respect to the fixed basis of $S_0 \otimes \mathbb{Q}$) that defines the inner-wall of D_0 in the orbit O_{64} across which $D_0^{g(b_{80})}$ is adjacent to D_0.
- `dpp80` is a double-plane polarization $b_{80} \in S_0$ that induces the involution $g(b_{80})$.
- `Singdpp80` is the singularities of the normal $K3$ surface that is the finite double coverer of $\mathbb{P}^2$ in the Stein factorization of the morphism $X_0 \to \mathbb{P}^2$ induced by b_{80}. (See Convention (5).)

The double-plane involution $g(b_{112})$.

- gdpp112 is the double-plane involution $g(b_{112}) \in \mathrm{Aut}(X_0)$ expressed by a 20×20-matrix acting on S_0.
- innwall112 is the primitive vector of $S_0^\vee$ (written with respect to the fixed basis of $S_0 \otimes \mathbb{Q}$) that defines the inner-wall of D_0 in the orbit O_{40} across which $D_0^{g(b_{112})}$ is adjacent to D_0.
- dpp112 is a double-plane polarization $b_{112} \in S_0$ that induces the involution $g(b_{112})$.
- Singdpp112 is the singularities of the normal $K3$ surface that is the finite double coverer of $\mathbb{P}^2$ in the Stein factorization of the morphism $X_0 \to \mathbb{P}^2$ induced by b_{112}.

The double-plane involution $g(b_{296})$.

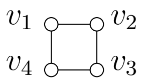
- gdpp296 is the double-plane involution $g(b_{296}) \in \mathrm{Aut}(X_0)$ expressed by a 20×20-matrix acting on S_0.
- innwall296 is the primitive vector of $S_0^\vee$ (written with respect to the fixed basis of $S_0 \otimes \mathbb{Q}$) that defines the inner-wall of D_0 in the orbit O_{160} across which $D_0^{g(b_{296})}$ is adjacent to D_0.
- dpp296 is a double-plane polarization $b_{296} \in S_0$ that induces the involution $g(b_{296})$.
- Singdpp296 is the singularities of the normal $K3$ surface that is the finite double coverer of $\mathbb{P}^2$ in the Stein factorization of the morphism $X_0 \to \mathbb{P}^2$ induced by b_{296}.

The double-plane involution $g(b_{688})$.

- gdpp688 is the double-plane involution $g(b_{688}) \in \mathrm{Aut}(X_0)$ expressed by a 20×20-matrix acting on S_0.
- innwall688 is the primitive vector of $S_0^\vee$ (written with respect to the fixed basis of $S_0 \otimes \mathbb{Q}$) that defines the inner-wall of D_0 in the orbit O_{320} across which $D_0^{g(b_{688})}$ is adjacent to D_0.
- dpp688 is a double-plane polarization $b_{688} \in S_0$ that induces the involution $g(b_{688})$.
- Singdpp688 is the singularities of the normal $K3$ surface that is the finite double coverer of $\mathbb{P}^2$ in the Stein factorization of the morphism $X_0 \to \mathbb{P}^2$ induced by b_{688}.

5.4. The finite group $\mathrm{Aut}(X_0, h_0)$ (Section 4.3 of [1]).

- SixFs is the list of 6 quadrangles $F_c = [v_1, v_2, v_3, v_4]$ of singular fibers of the Jacobian fibration $\sigma\colon X_0 \to \mathbb{P}^1$, where v_1, v_2, v_3, v_4 are sorted in such a way that they form the dual graph



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and the six quadrangles F_c are sorted according to the critical values c sorted as $\mathrm{Cr}(\sigma) = [0, \infty, 1, -1, i, -i]$.

- `fsigma` is the class $f \in S_0$ of a fiber of $\sigma\colon X_0 \to \mathbb{P}^1$.
- `zsigma` is the class $z \in S_0$ of the zero section of $\sigma\colon X_0 \to \mathbb{P}^1$.
- `AutX0f` is the group $\mathrm{Aut}(X_0, f)$. The order is 768. Each element of this list is a 20×20 matrix acting on S_0.
- `iotasigmaz` is the inversion $\iota_\sigma \in \mathrm{Aut}(X_0, f)$ of the Jacobian fibration (σ, z). This automorphism is expressed by a 20×20 matrix acting on S_0.
- `MWtorsigmaz` is the list of 16 pairs $[v, [a, b]]$, where $v \in \mathcal{L}_{40}$ is the class of a section of $\sigma\colon X_0 \to \mathbb{P}^1$ that defines $[a, b] \in (\mathbb{Z}/4\mathbb{Z})^2$ under a fixed isomorphism between the Mordell-Weil group $\mathrm{MW}(\sigma, z)$ and $(\mathbb{Z}/4\mathbb{Z})^2$.
- `Tsigma` is the list of translations by sections of $\sigma\colon X_0 \to \mathbb{P}^1$. Each element of this list is a 20×20 matrix acting on S_0, and the elements are sorted according to `MWtorsigmaz`.
- `Galmu` is the Galois group $\mathrm{Gal}(\mu)$. The order is 32. Each element of this list is a 20×20 matrix acting on S_0.

5.5. **Proof of Theorems 1.7 and 1.8 of** [1]**.**

- `pr30` is the orthogonal projection $\mathrm{pr}_{30}\colon S_3 \otimes \mathbb{Q} \to S_0 \otimes \mathbb{Q}$.
- `GramQ` is the Gram matrix of Q.
- `embQS3` is the embedding $Q \hookrightarrow S_3$.
- `prQ` is the orthogonal projection $\mathrm{pr}_Q\colon S_3 \otimes \mathbb{Q} \to Q \otimes \mathbb{Q}$.
- `v1v2` is the pair $[v_1, v_2]$ of primitive vectors of $S_3^\vee$ that define the hyperplanes $(v_1)^\perp, (v_2)^\perp$ in Lemma 5.4 of [1].
- `FourD3s` is the list $[\mathrm{id}, \gamma_1, \gamma_2, \varepsilon]$ such that $D_3 = D_3^{\mathrm{id}}, D_3^{\gamma_1}, D_3^{\gamma_2}, D_3^{\varepsilon}$ are the $\mathcal{V}(i_3)$-chambers containing the face D_0 of D_3.
- `CCC4` is the list $\mathcal{C}_4$.
- `CCC7` is the list of the two orbits of the action of $\mathrm{PGU}_4(\mathbb{F}_9)$ on $\mathcal{C}_7$.

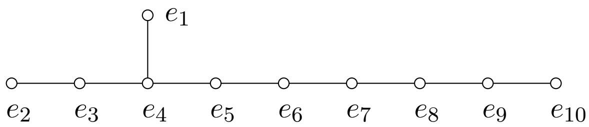

FIGURE 6.1. Basis of L_{10}

- `liftAutX0h0` is the list of 4 lists of 960 pairs $[\tilde{g}, g]$ such that $\tilde{g}$ is an element of $\mathrm{PGU}_4(\mathbb{F}_9) \cdot \gamma \subset \mathrm{Aut}(X_3)$ preserving $S_0 \subset S_3$, where $\gamma \in [\mathrm{id}, \gamma_1, \gamma_2, \varepsilon]$, and g is the restriction of $\tilde{g}$ to S_0.
- `liftgdpp112` is the element of $\mathrm{O}^+(S_3, S_0) \cap \mathrm{Aut}(X_3)$ that is mapped to the double-plane involution $g(b_{112})$ of X_0 by $\tilde{\rho}|_{\mathrm{Aut}}$. This is a double-plane involution of X_3 given by the double-plane polarization $\rho(b_{112}) \in S_3$, and the classes of smooth rational curves contracted by $\Phi_{\rho(b_{112})}\colon X_3 \to \mathbb{P}^2$ are the image by ρ of those contracted by $\Phi_{b_{112}}\colon X_0 \to \mathbb{P}^2$.
- `liftgdpp688` is the element of $\mathrm{O}^+(S_3, S_0) \cap \mathrm{Aut}(X_3)$ that is mapped to the double-plane involution $g(b_{688})$ of X_0 by $\tilde{\rho}|_{\mathrm{Aut}}$. This is a double-plane involution of X_3 given by the double-plane polarization $\rho(b_{688}) \in S_3$, and the classes of smooth rational curves contracted by $\Phi_{\rho(b_{688})}\colon X_3 \to \mathbb{P}^2$ are the image by ρ of those contracted by $\Phi_{b_{688}}\colon X_0 \to \mathbb{P}^2$.

6. THE FILE `Enriques.txt`

- `ConfigIV` is the dual graph of the smooth rational curves on $Y_{\mathrm{IV},p}$ (Figure 1.2 of [1]). The set of vertices is $[1, \dots, 20]$.
- `GramL10` is the Gram matrix of the even unimodular hyperbolic lattice L_{10} with the basis given by the 10 roots $e_1, \dots, e_{10}$ forming the dual graph in Figure 6.1 above.
- `SixEnriques` is the list of the data of the six Enriques involutions $\varepsilon^{(1)}, \dots, \varepsilon^{(6)}$ in $\mathrm{Aut}(X_0, h_0)$. Each data is the list

$$[\mathtt{e0}, \mathtt{emb}, \mathtt{proj}, \mathtt{e3}, \mathtt{Zen}]$$

of the following items. Let ε_0 be one of $\varepsilon^{(1)}, \dots, \varepsilon^{(6)}$. The item `e0` is the matrix representation of the action of ε_0 on S_0. Let $\pi\colon X_0 \to Y_0 := X_0/\langle \varepsilon_0 \rangle$ be the quotient morphism, and let S_Y be the Néron-Severi lattice of Y_0. We fix an identification $L_{10} \cong S_Y$ (see Remark below). The item `emb` is the embedding $L_{10}(2) \cong S_Y(2) \hookrightarrow S_0$ induced by $\pi^*\colon S_Y \hookrightarrow S_0$, and the item `proj` is the orthogonal projection $S_0 \otimes \mathbb{Q} \to S_Y \otimes \mathbb{Q}$ to the image of $\pi^* \otimes \mathbb{Q}$. Let ε_3 be the Enriques involution in $\mathrm{O}^+(S_3, S_0) \cap \mathrm{Aut}(X_3)$ that is mapped to ε_0 by $\tilde{\rho}|_{\mathrm{Aut}}$. The item `e3` is the matrix representation of the action of ε_3 on S_3. The item `Zen` is the list of 4 lists of 160 triples $[\tilde{g}, g, g|S_Y]$, where

- $\tilde{g} \in \mathrm{O}^+(S_3, S_0) \cap \mathrm{Aut}(X_3)$ is an element of $Z_{\mathrm{Aut}(X_3)}(\varepsilon_3) \cap \mathrm{PGU}_4(\mathbb{F}_9) \cdot \gamma$, where γ is an element of `FourD3s` $= [\mathrm{id}, \gamma_1, \gamma_2, \varepsilon]$,
- $g \in \mathrm{Aut}(X_0)$ is the restriction $\tilde{g}|_{S_0}$, which is an element of $Z_{\mathrm{Aut}(X_0)}(\varepsilon_0)$, and
- $g|S_Y$ is the restriction of g to $S_Y \subset S_0$, which is an element of $\mathrm{Aut}(Y_0)$, and is expressed by a 10×10 matrix acting on S_Y.

Remark 6.1. The identification $L_{10} \cong S_Y$ is chosen so that the image of h_0 by the orthogonal projection $S_0 \otimes \mathbb{Q} \to S_Y \otimes \mathbb{Q} = L_{10} \otimes \mathbb{Q}$ generates the 1-dimensional subspace

$$(e_1)^\perp \cap \cdots \cap (e_5)^\perp \cap (e_7)^\perp \cap \cdots \cap (e_{10})^\perp$$

of $\mathcal{P}(L_{10})$.

References

[1] Ichiro Shimada. The elliptic modular surface of level 4 and its reduction modulo 3, 2018. preprint, `http://www.math.sci.hiroshima-u.ac.jp/~shimada/preprints.html`.

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