# ON A SMOOTH QUARTIC SURFACE CONTAINING 56 LINES WHICH IS ISOMORPHIC AS A $K3$ SURFACE TO THE FERMAT QUARTIC: COMPUTATIONAL DATA

ICHIRO SHIMADA

## 1. Introduction

We give an explanation of the computational data that are used in the paper

[X56 ] I. Shimada and T. Shioda: On a smooth quartic surface containing 56 lines which is isomorphic as a $K3$ surface to the Fermat quartic,
`http://www.math.sci.hiroshima-u.ac.jp/~shimada/preprints.html`

and available from the author's webpage

`http://www.math.sci.hiroshima-u.ac.jp/~shimada/K3.html.`

All data is written on a single text file `X56compdata.txt` (13.6 MB). The complex number

$$\zeta = \exp(2\pi\sqrt{-1}/8)$$

is expressed as `zeta`. We use the same notation as [X56].

## 2. Explanation of the computational data

### 2.1. The lines on $X_{48}$.

- `LineTags48` The list of tags of the 48 lines on $X_{48}$.
- `LineEqs48` The list of equations of the 48 lines on $X_{48}$, which is sorted according to `LineTags48`. The $k$th equation is given as a $2 \times 4$ matrix $M_k$ with components in $\mathbb{Q}(\zeta)$ in the row-reduced echelon form, and the $k$th line is defined by $M_k\, x = 0$, where $x = [x_1, x_2, x_3, x_4]^T$.
- `LineIntNumbs48` The $48 \times 48$ matrix of the intersection numbers of lines on $X_{48}$. The rows and columns are sorted according to `LineTags48`.
- `BasisSX` The tags of the 20 lines $[l_1], \dots, [l_{20}]$ that form a basis of $S_X$.
- `GramSX` The Gram matrix $\mathrm{G}_{S_X}$ of $S_X$ with respect to the basis $[l_1], \dots, [l_{20}]$ of $S_X$.
- `h48` The class $h_{48}$.
- `LineClasses48` The list $\mathcal{F}_{48}$ of classes of lines on $X_{48}$, which is sorted according to `LineTags48`.
- `taupoints` The list of $\tau$-points.

In the following, the $i$th line $\ell_i$ on $X_{48}$ means the line labelled by the $i$th tag in `LineTags48`.

- `basisnumbs48` A list $[i_1, \dots, i_{20}]$ such that the classes $[\ell_{i_j}]$ of the $i_j$th line on $X_{48}$ form a basis of $S_X$.

**2.2. The lattice $S_X$.**

- `discSXreps` The vectors $s_1$ and $s_2$ of $S_X^\vee$ in the dual representation.
- `discSX` The discriminant form with respect to the basis $\sigma_1, \sigma_2$ of $\mathrm{disc}(S_X)$.
- `discSXproj` The matrix $P$ that expresses the projection $S_X^\vee \to \mathrm{disc}(S_X)$ with respect to the dual basis of $S_X^\vee$ and the basis $\sigma_1, \sigma_2$ of $\mathrm{disc}(S_X)$.
- `OqSX` The group $\mathrm{O}(q_{S_X})$.
- `GramT` The Gram matrix of $T$ with respect to the basis $t_1, t_2$.
- `discTreps` The vectors $t_1^\vee$ and $t_2^\vee$ of $T^\vee$ in the dual representation, that is, $[1, 0]$ and $[0, 1]$.
- `discT` The discriminant form with respect to the basis $\tau_1, \tau_2$ of $\mathrm{disc}(T)$.
- `discTproj` The matrix that expresses the projection $T^\vee \to \mathrm{disc}(T)$ with respect to the dual basis $t_1^\vee, t_2^\vee$ of $T^\vee$ and the basis $\tau_1, \tau_2$ of $\mathrm{disc}(T)$.
- `OT` The group $\mathrm{O}(T)$.
- `OqT` The group $\mathrm{O}(q_T)$.
- `GammatildeT` The group $\tilde{\Gamma}_T$.
- `etaTGammatildeT` The subgroup $\eta_T(\tilde{\Gamma}_T)$ of $\mathrm{O}(q_T)$.
- `Isoms` The list of isomorphisms from $q_{S_X}$ to $-q_T$.
- `GammaSX` The subgroup $\Gamma_{S_X}$ of $\mathrm{O}(q_{S_X})$.

**2.3. The groups associated with $X_{48}$.**

- `Gtilde48` The group $\tilde{G}_{48}$ as a list of $20 \times 20$ matrices in $\mathrm{O}(S_X)$.
- `Gtilde48Perms` The group $\tilde{G}_{48}$ as a list of permutations on $\mathcal{F}_{48}$. An element $[i_1, \dots, i_{48}]$ of this list maps the class $[\ell_j]$ to the class of $[\ell_{i_j}]$ for $j = 1, \dots, 48$. The list `Gtilde48Perms` is sorted according to the list `Gtilde48`.
- `G48` The group $G_{48}$ as a list of $20 \times 20$ matrices in $\mathrm{O}(S_X)$.
- `G48Perms` The group $G_{48}$ as a list of permutations on $\mathcal{F}_{48}$. The list `G48Perms` is sorted according to the list `G48`.
- `Gal` The group $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ as a subgroup of `Gtilde48`.
- `GalPerms` The group $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ as a subgroup of `Gtilde48Perms`. The list `GalPerms` is sorted according to the list `Gal`.

**2.4. The orbit decomposition of the set of pairs of lines on $X_{48}$ by $G_{48}$.**

- `OrbitsOfPairs` The list of the orbits $o_1, \dots, o_8$. Each orbit is a list of pairs $[i, j]$, which denote the pairs $\{\ell_i, \ell_j\}$.
- `OrbitsMat` The $48 \times 48$ matrix whose $(i, j)$ component is $k$ if $\{\ell_i, \ell_j\} \in o_k$ and 0 if $i = j$.
- `Aos` The list of the matrices $A(o_1), \dots, A(o_8)$.

**2.5. The sets $\mathcal{H}_5$ and $\mathcal{H}_6$.** The lists `H5reps` and `H6reps` are the list of orbits in the sets $\mathcal{H}_5$ and $\mathcal{H}_6$ under the action of $G_{48}$. Each list consists of the data

$$[v, \texttt{size}, \texttt{flags}],$$

where $v$ is a representative of the orbit, `size` is the size of the orbit, and `flags` has the following meaning:

- if `flags` is [`false`], then $v$ is not nef,
- if `flags` is [`true`, `false`], then $v$ is nef, but $|\mathcal{L}_v|$ has a fixed-component,
- if `flags` is [`true`, `true`, `false`], then $v$ is nef, $|\mathcal{L}_v|$ is fixed-component free, but the morphism $\Phi_v \colon X_{48} \to \mathbb{P}^3$ is hyperelliptic,

- if `flags` is [$\texttt{true}, \texttt{true}, \texttt{true}, \texttt{false}, R$], then $v$ is nef, $|\mathcal{L}_v|$ is fixed-component free, $\Phi_v \colon X_{48} \to \mathbb{P}^3$ is non-hyperelliptic, but the image $X_v$ of $\Phi_v$ has singularities of $ADE$-type $R$ (for example, $R =$ `["A1", "A1"]` means that $X_v$ has two ordinary nodes as its only singularities), and
- if `flags` is [$\texttt{true}, \texttt{true}, \texttt{true}, \texttt{true}, n$], then $v$ is very ample and $X_v$ is contains exactly $n$ lines.

The representative element $v$ of an orbit is chosen so that the vector representation of $v$ with respect to the basis $[l_1], \dots, [l_{20}]$ of $S_X$ is minimal with respect to the lexicographic order.

2.6. **The surface $X_{56}$.**

- `X56Pols` The list of $X_{56}$-polarizations. This list consists of two lists of size 192, each of which is an orbit under the action of $G_{48}$.
- `X56ConfigOrbitsMat` The $7 \times 7$ matrix that defines the $X_{56}$-configuration; that is, a 7-tuple $(\ell'_1, \dots, \ell'_7)$ of lines on $X_{48}$ is an $X_{56}$-configuration if $\{\ell'_i, \ell'_j\}$ is in $o_k$ for $i, j = 1, \dots, 7$ with $i \neq j$, where $k$ is the $(i, j)$ component of `X56ConfigOrbitsMat`. (The diagonal components of `X56ConfigOrbitsMat` are set to be 0.)
- `X56Configs` The list of $X_{56}$-configurations. Each member is an ordered list $[i_1, \dots, i_7]$, which indicates the list $(\ell_{i_1}, \dots, \ell_{i_7})$ of seven lines on $X_{48}$ that form an $X_{56}$-configuration.
- `TheX56Config` The fixed $X_{56}$-configuration $(\ell_1, \ell_{11}, \ell_3, \ell_{17}, \ell_{22}, \ell_{36}, \ell_{18})$ that gives the $X_{56}$-polarization $h_{56}$, given as $[1, 11, 3, 17, 22, 36, 18]$. See (4.2) of [X56].
- `h56` The class $h_{56}$.
- `A` $A = -1 - 2\zeta - 2\zeta^3$.
- `B` $B = 2 - 2\zeta - 2\zeta^3$.
- `Psi` The equation $\Psi$.
- `fPolys` The homogeneous cubic polynomials $f_1, \dots, f_4$ in `x1`, `x2`, `x3`, `x4`.

The following are the data used in the proof of Theorem 4.6 in [X56].

- `BasisGamma3` The basis of the 20-dimensional linear space $\Gamma_3$, consisting of monomials in `x1`, `x2`, `x3`, `x4` of degree 3.
- `M56` The $24 \times 20$ matrix $\mathcal{M}_{56}$ such that the solution space of the linear equation $\mathcal{M}_{56} z = 0$ defines the 4-dimensional subspace $H^0(X, \mathcal{L}_{h_{56}})$ of $\Gamma_3$ with respect to the basis `BasisGamma3`.
- `fVects` The basis $f_1, \dots, f_4$ of the solution space of $\mathcal{M}_{56} z = 0$, written as row vectors with respect to the basis `BasisGamma3`.
- `BasisbarGamma12` A basis of the 290-dimensional linear space $\bar{\Gamma}_{12}$, consisting of monomials in `x1`, `x2`, `x3`, `x4` of degree 12 with the degree with respect to `x1` is $\leq 3$.
- `BasisSigma4` A basis of the 35-dimensional linear space $\Sigma_4$, consisting of monomials in `y1`, `y2`, `y3`, `y4` of degree 4.
- `rhosigma` The $290 \times 35$ matrix representing the linear map $\rho \circ \sigma \colon \Sigma_4 \to \bar{\Gamma}_{12}$ with respect to the bases `BasisSigma4` and `BasisbarGamma12`.
- `PsiVect` The equation $\Psi \in \Sigma_4$ represented as a row vector with respect to the basis `BasisSigma4` of $\Sigma_4$.

2.7. **The lines on $X_{56}$.**

- `LineClasses56` The list $\mathcal{F}_{56}$ of classes of lines on $X_{56}$. The lines on $X_{56}$ are sorted according to this list, that is, the $k$th line on $X_{56}$ means the line on $X_{56}$ whose class is the $k$th vector of `LineClasses56`.
- `LineIntNumbs56` The $56 \times 56$ matrix of the intersection numbers of lines on $X_{56}$. The rows and columns are sorted according to `LineClasses56`.
- `LineEqs56` The list of equations of the 56 lines on $X_{56}$. The $k$th equation is given as a $2 \times 4$ matrix $M_k$ with components in $\mathbb{Q}(\zeta)$ in the row-reduced echelon form, and the $k$th line is defined by $M_k y = 0$, where $y = [y_1, y_2, y_3, y_4]^T$.

2.8. **The groups associated with $X_{56}$.**

- `basisnumbs56` A list $[i_1, \dots, i_{20}]$ such that the classes of the $i_j$th line on $X_{56}$ form a basis of $S_X$.
- `Gtilde56` The group $\tilde{G}_{56}$ as a list of $20 \times 20$ matrices in $\mathrm{O}(S_X)$.
- `Gtilde56Perms` The group $\tilde{G}_{56}$ as a list of permutations on $\mathcal{F}_{56}$. An element $[i_1, \dots, i_{56}]$ of this list maps the class of the $j$th line on $X_{56}$ (that is, the $j$th vector of `LineClasses56`) to the class of the $i_j$th line on $X_{56}$ (that is, the $i_j$th vector of `LineClasses56`) for $j = 1, \dots, 56$. The list `Gtilde56Perms` is sorted according to the list `Gtilde56`.
- `G56` The group $G_{56}$ as a list of $20 \times 20$ matrices in $\mathrm{O}(S_X)$.
- `G56Perms` The group $G_{56}$ as a list of permutations on $\mathcal{F}_{56}$. The list `G56Perms` is sorted according to the list `G56`.
- `G56PGL` The group $G_{56}$ as a list of elements of $\mathrm{PGL}_3(\mathbb{C})$. The list `G56PGL` is sorted according to the list `G56Perms`, in the sense that, for the $i$th element $g$ of `G56PGL`, the mapping $[\lambda] \mapsto [g(\lambda)]$ induces the $i$th permutation in `G56Perms`. CAUTION: Hence the representation of the $i$th element $g$ of `G56PGL` on $S_X$ is *not* the $i$th element $f$ of `G56`, but its inverse $f^{-1}$, because the representation is defined by the pull-back.
- `GeneratorsG56` A set of generators of `G56` consisting of two elements.
- `GeneratorsG56Perms` The generators of `G56Perms` corresponding the two elements in `GeneratorsG56`.
- `GeneratorsG56PGL` The generators of `G56PGL` corresponding the two elements in `GeneratorsG56`. These two matrices are $\gamma_1$ and $\gamma_2$ in Theorem 4.9 of [X56].
- `OrbitsLines56` The orbit decomposition of lines on $X_{56}$ under the action of $G_{56}$. Each orbit is a list $[i_1, \dots, i_t]$, which denote the orbit consisting of the $i_j$th lines for $j = 1, \dots, t$.

2.9. **The reductions.**

- `PsivarA` The equation $\Psi$ with $A$ written as `varA` and $B$ written as `varA+3`.
- `F56prime` The subset $\mathcal{F}'_{56}$ of $S_X^\vee$. Each element is given as a row vector with respect to the basis $[l_1], \dots, [l_{20}]$ of $S_X$.

2.10. **The reduction at $P_3$.**

- `LineEqs56P3` The list of $[M_\lambda, U_\lambda, \overline{U_\lambda M_\lambda}]$, where $M_\lambda$ is the defining matrix of a line $\lambda$ on $X_{56}$ given in `LineEqs56`, and $U_\lambda$ is a $2 \times 2$ invertible matrix with components in $\mathbb{Q}(\zeta)$ such that

$$\overline{U_\lambda M_\lambda} := U_\lambda M_\lambda \bmod P_3$$

is a matrix of rank 2 in the row-reduced echelon form with components in $\kappa_{P_3} \cong \mathbb{F}_9$. The $\gamma =$ `gamma` that appears in $\overline{U_\lambda M_\lambda}$ is a 8th root of unity in $\mathbb{F}_9$ such that $A \bmod P_3 = -1 - 2\gamma - 2\gamma^3$ is zero, that is, $\gamma \in \mathbb{F}_9$ is a root of $\gamma^2 + 2\gamma + 2 = 0$.

- `LineEqs112P3` The list of defining equations of the 112 lines on $X_{56}(P_3)$. The equations in the second half of this list come from the reduction (that is, they appear as the third members of elements of `LineEqs56P3`), while those in the first half are new (that is, they define the unique common intersecting lines of $\Lambda(r') \otimes \kappa_{P_3}$ of $r' \in \mathcal{F}'_{56}$).

Department of Mathematics, Graduate School of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526 JAPAN
*E-mail address*: `shimada@math.sci.hiroshima-u.ac.jp`