A note on Zariski pairs

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§0. Introduction

In [1], Artal Bartolo defined the notion of Zariski pairs as follows:

Definition. A couple of complex reduced projective plane curves $C_1$ and $C_2$ of the same degree is said to make a Zariski pair, if there exist tubular neighborhoods $T(C_i) \subset \mathbb{P}^2$ of $C_i$ for $i = 1, 2$ such that $(T(C_1), C_1)$ and $(T(C_2), C_2)$ are diffeomorphic, while the pairs $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ are not homeomorphic; that is, the singularities of $C_1$ and $C_2$ are topologically equivalent, but the embeddings of $C_1$ and $C_2$ into $\mathbb{P}^2$ are not topologically equivalent.

The first example of Zariski pair was discovered and studied by Zariski in [9] and [11]. He showed that there exist projective plane curves $C_1$ and $C_2$ of degree 6 with 6 cusps and no other singularities such that $\pi_1(\mathbb{P}^2 \setminus C_1)$ and $\pi_1(\mathbb{P}^2 \setminus C_2)$ are not isomorphic. Indeed, the placement of the 6 cusps on the sextic curve has a crucial effect on the fundamental group of the complement. Let $C_1$ be a sextic curve defined by an equation $f^2 + g^3 = 0$, where $f$ and $g$ are general homogeneous polynomials of degree 3 and 2, respectively. Then $C_1$ has 6 cusps lying on a conic defined by $g = 0$. In [9], it was shown that $\pi_1(\mathbb{P}^2 \setminus C_1)$ is isomorphic to the free product $\mathbb{Z}/(2) * \mathbb{Z}/(3)$ of cyclic groups of order 2 and 3. On the other hand, in [11], it was proved that there exists a sextic curve $C_2$ with 6 cusps which are not lying on any conic, and that the fundamental group $\pi_1(\mathbb{P}^2 \setminus C_2)$ is cyclic of order 6. In [4], Oka gave an explicit defining equation of $C_2$. In [1], Artal Bartolo presented a simple way to construct $(C_1, C_2)$ from a cubic curve $C$ by means of a Kummer covering of $\mathbb{P}^2$ of exponent 2 branched along three lines tangent to $C$ at its points of inflection.

Except for this example, very few Zariski pairs are known ([1], [8]). In [5], and independently in [7], infinite series of Zariski pairs have been constructed from the above example of Zariski by means of covering tricks of the plane.

In this paper, we present a method to construct Zariski pairs, which yields two infinite series of new examples of Zariski pairs as special cases.

A germ of curve singularity is called of type $(p, q)$ if it is locally defined by $x^p + y^q = 0$.

Series I. This series consists of pairs $(C_1(q), C_2(q))$ of curves of degree $3q$, where $q$ runs through the set of integers $\geq 2$ prime to 3. Each of $C_1(q)$ and $C_2(q)$ has $3q$ singular points of type $(3, q)$ and no other singularities. The fundamental group $\pi_1(\mathbb{P}^2 \setminus C_1(q))$ is non-abelian, while $\pi_1(\mathbb{P}^2 \setminus C_2(q))$ is abelian. When $q = 2$, this example is nothing but the classical one of the sextic curves due to Zariski.

Series II. This series consists of pairs $(D_1(q), D_2(q))$ of curves of degree $4q$, where $q$ runs through the set of odd integers $> 2$. Each of $D_1(q)$ and $D_2(q)$ has $8q$ singular points of type $(2, q)$ that is, rational double points of type $A_{q-1}$ and no other singularities. The fundamental group $\pi_1(\mathbb{P}^2 \setminus D_1(q))$ is non-abelian, while $\pi_1(\mathbb{P}^2 \setminus D_2(q))$ is abelian.
Our method is a generalization of Artal Bartolo’s method for re-construction of the classical example of Zariski to higher dimensions and arbitrary exponents of the Kummer covering. Indeed, when \( q = 2 \) in Series I, our construction coincides with his.

Instead of the computation of the first Betti number of the cyclic branched covering of \( \mathbb{P}^2 \), which was employed in [1], we use the fundamental groups of the complements in order to distinguish two embeddings of curves in \( \mathbb{P}^2 \). For the calculation of the fundamental groups, we use [6; Theorem 1] and a result of [3] and [7].

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§1. A method of constructing Zariski pairs

1.1. Non-abelian members. Let \( p \) and \( q \) be integers \( \geq 2 \) prime to each other. We choose homogeneous polynomials \( f \in H^0(\mathbb{P}^2, \mathcal{O}(pk)) \) and \( g \in H^0(\mathbb{P}^2, \mathcal{O}(qk)) \), where \( k \) is an integer \( \geq 1 \). Suppose that \( f \) and \( g \) are generally chosen. Consider the projective plane curve

\[
C_{p,q,k} : f^q + g^p = 0
\]

of degree \( pqk \) (cf. [2]). It is easy to see that the singular locus of this curve consists of \( pqk^2 \) points of type \( (p, q) \). In [7; Example (3) in §0], the following is shown.

**Proposition 1.** The fundamental group \( \pi_1(\mathbb{P}^2 \setminus C_{p,q,k}) \) is isomorphic to the group \( \langle a, b, c | a^p = b^q = c, c^k = 1 \rangle \). In particular, it is non-abelian.

See also [3], in which the fundamental groups of the complements of curves of this type are calculated. There the groups are presented in a different way.

This curve \( C_{p,q,k} \) will be a member \( C_1 \) of a Zariski pair.

1.2. Abelian partners. We shall construct the other member \( C_2 \) of the Zariski pair such that \( \pi_1(\mathbb{P}^2 \setminus C_2) \) is abelian.

Let \( p, q \) and \( k \) be integers as above. We put \( n = pk \). Interchanging \( p \) and \( q \) if necessary, we may assume that \( n \geq 3 \). Let \( S_0 \subset \mathbb{P}^{n-1} \) be a hypersurface of degree \( n \) defined by \( F_0(X_1, \ldots, X_n) = 0 \). We consider a linear pencil of hypersurfaces

\[
S_t : F_0(X_1, \ldots, X_n) + t \cdot X_1 \cdots X_n = 0,
\]

which is spanned by \( S_0 \) and \( S_\infty := \{X_1 \cdots X_n = 0\} \). We put \( H_i = \{X_i = 0\} \) \((i = 1, \ldots, n)\).

We consider the morphism \( \phi_q : \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1} \) given by

\[
(Y_1 : \ldots : Y_n) \quad \mapsto \quad (X_1 : \ldots : X_n) = (Y_1^q : \ldots : Y_n^q),
\]

which is a covering of degree \( q^{n-1} \) branched along \( S_\infty \).

**Proposition 2.** Suppose that (1) every member \( S_t \) is reduced, and that (2) \( S_0 \) contains none of the hyperplanes \( H_i \). Then \( \pi_1(\mathbb{P}^{n-1} \setminus \phi_q^{-1}(S_t)) \) is abelian for a general member \( S_t \).
Proof. Let \( \mathbb{P}^1 \) be the t-line, and we put \( \mathbb{A}^1 := \mathbb{P}^1 \setminus \{\infty\} \). Let \( \mathcal{W} \subset \mathbb{P}^{n-1} \times \mathbb{A}^1 \) be the divisor defined by

\[
X_1 \cdots X_n \cdot ( F_0(X_1, \ldots, X_n) + t \cdot X_1 \cdots X_n ) = 0,
\]

which is the union of \( S_\infty \times \mathbb{A}^1 \) and the universal family of the affine part \( \{ S_t ; t \in \mathbb{A}^1 \} \) of the pencil. For \( t \in \mathbb{A}^1 \), we denote by \( W_t \subset \mathbb{P}^{n-1} \) the divisor obtained from the scheme theoretic intersection \( (\{t\} \times \mathbb{P}^{n-1}) \cap \mathcal{W} \), which is equal with the divisor \( S_t + S_\infty \).

First, we shall show that \( \pi_1(\mathbb{P}^{n-1} \setminus W_t) \) is abelian for a general \( t \). Remark that the assumption (2) implies that \( S_t \) contains none of \( H_i \) unless \( t = \infty \). Combining this with the assumption (1), we see that \( W_t \) is reduced for all \( t \in \mathbb{A}^1 \). Hence, by [6; Theorem 1], the inclusion \( \mathbb{P}^{n-1} \setminus W_t \leftrightarrow (\mathbb{P}^{n-1} \times \mathbb{A}^1) \setminus \mathcal{W} \) induces an isomorphism on the fundamental groups for a general \( t \). Therefore, it is enough to show that \( \pi_1((\mathbb{P}^{n-1} \times \mathbb{A}^1) \setminus \mathcal{W}) \) is abelian.

In order to prove this, we consider the first projection

\[
p : (\mathbb{P}^{n-1} \times \mathbb{A}^1) \setminus \mathcal{W} \longrightarrow \mathbb{P}^{n-1} \setminus S_\infty.
\]

Since \( \{ S_t ; t \in \mathbb{P}^1 \} \) is a pencil whose base locus is contained in \( S_\infty \), there is a unique member \( S_{t(P)} \) \((t(P) \neq \infty)\) containing \( P \) for each point \( P \in \mathbb{P}^{n-1} \setminus S_\infty \). Therefore \( p^{-1}(P) \) is a punctured affine line \( \mathbb{A}^1 \setminus \{t(P)\} \) for every \( P \in \mathbb{P}^{n-1} \setminus S_\infty \). Consequently, \( p \) has a section

\[
s : \mathbb{P}^{n-1} \setminus S_\infty \longrightarrow (\mathbb{P}^{n-1} \times \mathbb{A}^1) \setminus \mathcal{W},
\]

which is given by, for example, \( s(P) = (P, \ t(P) + 1) \). Hence the homotopy exact sequence of \( p \) splits. Combining this with the fact that the image of the injection \( \pi_1(\mathbb{A}^1 \setminus \{t(P)\}) \rightarrow \pi_1((\mathbb{P}^{n-1} \times \mathbb{A}^1) \setminus \mathcal{W}) \) is contained in the center, we see that

\[
\pi_1((\mathbb{P}^{n-1} \times \mathbb{A}^1) \setminus \mathcal{W}) \cong \pi_1(\mathbb{P}^{n-1} \setminus S_\infty) \times \pi_1(\mathbb{A}^1 \setminus \{ \text{a point} \}).
\]

This shows that \( \pi_1((\mathbb{P}^{n-1} \times \mathbb{A}^1) \setminus \mathcal{W}) \) is abelian.

Note that \( \phi_q : \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1} \) is étale over \( \mathbb{P}^{n-1} \setminus W_t \) for every \( t \). Hence the natural homomorphism

\[
\phi_{q*} : \pi_1(\mathbb{P}^{n-1} \setminus \phi_q^{-1}(W_t)) \longrightarrow \pi_1(\mathbb{P}^{n-1} \setminus W_t)
\]

is injective. This implies that \( \pi_1(\mathbb{P}^{n-1} \setminus \phi_q^{-1}(W_t)) \) is abelian for a general \( t \). On the other hand, since \( \mathbb{P}^{n-1} \setminus \phi_q^{-1}(W_t) \) is a Zariski open dense subset of \( \mathbb{P}^{n-1} \setminus \phi_q^{-1}(S_t) \), the inclusion induces a surjective homomorphism

\[
\pi_1(\mathbb{P}^{n-1} \setminus \phi_q^{-1}(W_t)) \twoheadrightarrow \pi_1(\mathbb{P}^{n-1} \setminus \phi_q^{-1}(S_t)).
\]

Thus \( \pi_1(\mathbb{P}^{n-1} \setminus \phi_q^{-1}(S_t)) \) is also abelian for a general \( t \). \( \square \)

**Proposition 3.** Suppose the following: (3) \( S_0 \cap H_i \) is a non-reduced divisor \( pD_i \) of \( H_i \) of multiplicity \( p \), where \( D_i \) is a reduced divisor of \( H_i \), none of whose irreducible components
is contained in $H_i \cap (\cup_{j \neq i} H_j)$, and (4) the singular locus of $S_t$ is of codimension $\geq 2$ in $S_t$ for a general $t$. Then the general plane section $\mathbb{P}^2 \cap \phi_q^{-1}(S_t)$ of $\phi_q^{-1}(S_t)$ is a curve of degree $pqk$, and its singular locus consists of $pqk^2$ points of type $(p, q)$.

**Proof.** Note that the assumption (3) implies that $S_t \cap H_i$ is also equal with $pD_i$ for $t \neq \infty$. Let $P$ be a general point of any irreducible component of $D_i$, and let $Q$ be a point such that $\phi_q(Q) = P$, which lies on the hyperplane defined by $Y_i = 0$. By the assumption (3), $Q$ is not contained in any of the other hyperplanes defined by $Y_j = 0$ ($j \neq i$). Hence there exist analytic local coordinate systems $(w_1, \ldots, w_{n-1})$ and $(z_1, \ldots, z_{n-1})$ of $\mathbb{P}^{n-1}$ with the origins $P$ and $Q$, respectively, such that $H_i$ is given by $w_1 = 0$, $\phi_q^{-1}(H_i)$ is given by $z_1 = 0$, and $\phi_q$ is given by

$$
(z_1, \ldots, z_{n-1}) \mapsto (w_1, \ldots, w_{n-1}) = (z_1^q, z_2, \ldots, z_{n-1}).
$$

Let $t \in \mathbb{A}^1$ be general. By the assumption (3), the defining equation of $S_t$ at $P$ is of the form

$$u(w) \cdot w_1 + v(w_2, \ldots, w_{n-1})^p = 0.$$

By the assumption (4), $S_t$ is non-singular at $P$, because $P$ is a general point of an irreducible component of $D_i$. This implies that $u(P) \neq 0$. On the other hand, the divisor $D_i$, which is defined by $v(w_2, \ldots, w_{n-1}) = 0$ on the hyperplane $H_i = \{w_1 = 0\}$, is non-singular at $P$, because $D_i$ is reduced by the assumption (3) and $P$ is general. Hence we have

$$\frac{\partial v}{\partial w_j}(P) \neq 0 \quad \text{at least for one } j \geq 2.$$

The defining equation of $\phi_q^{-1}(S_t)$ is then of the form

$$\tilde{u}(z) \cdot z_1^q + v(z_2, \ldots, z_{n-1})^p = 0, \quad \text{where} \quad \tilde{u}(Q) \neq 0.$$

Then, it is easy to see that, in terms of suitable analytic coordinates $(\tilde{z}_1, \ldots, \tilde{z}_{n-1})$ with the origin $Q$, this equation can be written as follows;

$$\tilde{z}_1^q + \tilde{z}_2^p = 0.$$

Thus, when we cut $\phi_q^{-1}(S_t)$ by a general 2-dimensional plane passing through $Q$, a germ of curve singularity of type $(p, q)$ appears at $Q$.

Since the degree of $D_i$ is $k = n/p$, the inverse image $\phi_q^{-1}(D_i)$ is a reduced hypersurface of degree $qk$ in the hyperplane defined by $Y_i = 0$. Moreover $\phi_q^{-1}(D_i)$ and $\phi_q^{-1}(D_j)$ have no common irreducible components when $i \neq j$ because of the assumption (3). Hence the intersection points of $\phi_q^{-1}(\sum_{i=1}^n D_i)$ with a general plane $\mathbb{P}^2 \subset \mathbb{P}^n$ is $pqk^2$ in number. Moreover, $\mathbb{P}^2 \cap \phi_q^{-1}(S_t)$ is non-singular outside of these intersection points, because of the assumption (4). \hfill \Box

**1.3. Summary.** Suppose that we have constructed a hypersurface $S_0 \subset \mathbb{P}^{n-1}$ of degree $n \geq 3$ which satisfies the assumptions (1)-(4) in Propositions 2 and 3. Let $C_t$ be a general plane section of $\phi_q^{-1}(S_t)$, where $t$ is general. Because of Zariski’s hyperplane
section theorem [10] and Propositions 1, 2 and 3, we see that the curve $C_2$ has the same type of singularities as that of $C_{p,q,k}$, but the fundamental group $\pi_1(\mathbb{P}^2 \setminus C_2)$ is abelian. Hence $(C_1, C_2)$ is a Zariski pair, with $C_1 = C_{p,q,k}$.

§2. Construction of Series I

We carry out the construction of the previous section with $p = 3$, $k = 1$, $n = 3$ and $q$ an arbitrary integer $\geq 2$ prime to 3.

We fix a homogeneous coordinate system $(X : Y : Z)$ of $\mathbb{P}^2$, and put

$$L_1 = \{X = 0\}, \quad L_2 = \{Y = 0\}, \quad L_3 = \{Z = 0\}, \quad \text{and}$$

$$R_1 = (0 : 1 : -1) \in L_1, \quad R_2 = (1 : 0 : -1) \in L_2.$$

Let $\mathbb{P}_*(\Gamma(\mathbb{P}^2, O(3)))$ be the space of all cubic curves on $\mathbb{P}^2$, which is isomorphic to the projective space of dimension 9, and let $\mathcal{F} \subset \mathbb{P}_*(\Gamma(\mathbb{P}^2, O(3)))$ be the family of cubic curves $C$ which satisfy the following conditions;

(a) $C$ intersects $L_1$ at $R_1$ with multiplicity $\geq 3$,
(b) $C$ intersects $L_2$ at $R_2$ with multiplicity $\geq 3$, and
(c) $C$ intersects $L_3$ at a point with multiplicity $\geq 3$.

(We consider that $C$ intersects a line $L_i$ with multiplicity $\infty$, if $L_i$ is contained in $C$.)

**Proposition 4.** The family $\mathcal{F}$ consists of 3 projective lines. They meet at one point corresponding to $C_\infty := \{XYZ = 0\}$.

**Proof.** Let $F(X, Y, Z) = 0$ be the defining equation of a member $C$ of this family $\mathcal{F}$. By the condition (a), $F$ is of the form

$$F(X, Y, Z) = A(Y + Z)^3 + X \cdot G(X, Y, Z),$$

where $A$ is a constant, and $G(X, Y, Z)$ is a homogeneous polynomial of degree 2. By the condition (b), we have $F(X, 0, Z) = A(Z + X)^3$, and hence we get

$$G(X, Y, Z) = A(3Z^2 + 3ZX + X^2) + Y \cdot H(X, Y, Z),$$

where $H(X, Y, Z)$ is a homogeneous polynomial of degree 1. By the condition (c), we have $F(X, Y, 0) = A(Y + \alpha X)^3$ for some $\alpha$. Then $\alpha$ must be a cubic root of unity, and we get

$$H(X, Y, Z) = 3A\alpha^2 X + 3A\alpha Y + BZ,$$

where $B$ is a constant. Combining all of these, we get

$$F(X, Y, Z) = A(X^3 + Y^3 + Z^3) + 3A(2X^2 + \alpha XY^2 + Y^2 Z + YZ^2 + Z^2 X + ZX^2) + BXYZ$$

$$= A(X + Y + Z)^3 + 3A(\alpha^2 - 1)X^2 Y + 3A(\alpha - 1)XY^2 + (B - 6A)XYZ.$$
This curve \( C = \{ F = 0 \} \) intersects \( L_3 \) at

\[
R_3 = R_3(\alpha) := (1 : -\alpha : 0) \in L_3
\]

with multiplicity \( \geq 3 \). This means that the family \( \mathcal{F} \) consists of three lines \( \mathcal{L}(1), \mathcal{L}(\omega) \) and \( \mathcal{L}(\omega^2) \) in the projective space \( \mathbb{P}_s(\Gamma(\mathbb{P}^2, \mathcal{O}(3))) \), where \( \omega = \exp(2\pi i/3) \), such that a general cubic \( C \) in \( \mathcal{L}(\alpha) \) intersects \( L_3 \) at \( R_3(\alpha) \) with multiplicity 3. The ratio of the coefficients \( t := B/A \) gives an affine coordinate on each line \( \mathcal{L}(\alpha) \). The three lines \( \mathcal{L}(1), \mathcal{L}(\omega), \mathcal{L}(\omega^2) \) intersect at one point \( t = \infty \) corresponding to the cubic \( C_{\infty} = L_1 + L_2 + L_3 \). \( \square \)

Hence we get three pencils of cubic curves \( \{ C(1)_t ; t \in \mathcal{L}(1) \}, \{ C(\omega)_t ; t \in \mathcal{L}(\omega) \}, \) and \( \{ C(\omega^2)_t ; t \in \mathcal{L}(\omega^2) \} \). It is easy to check that these pencils satisfy the assumptions (2), (3) and (4) in the previous section. Note that the pencil \( \mathcal{L}(1) \) does not satisfy the assumption (1) because \( C(1)_6 \) is a triple line. However, the other two satisfy (1). Indeed, if a cubic curve \( C \) in the family \( \mathcal{F} \) is non-reduced, then the conditions (a)-(c) imply that it must be a triple line. Therefore the three points \( R_1, R_2 \) and \( R_3(\alpha) \) are co-linear, which is equivalent to \( \alpha = 1 \). Consequently, \( C \) must be a member of \( \mathcal{L}(1) \).

Now, by using the pencil \( \mathcal{L}(\omega) \) or \( \mathcal{L}(\omega^2) \), we complete the construction of Series I.

Note that, if \( C(1)_a \) is a non-singular member of \( \mathcal{L}(1) \), then \( \pi_1(\mathbb{P}^2 \setminus \phi_q^{-1}(C(1)_a)) \) is isomorphic to the free product \( \mathbb{Z}/(3) \ast \mathbb{Z}/(q) \). Indeed, since \( C(1)_a \) is defined by

\[
(X + Y + Z)^3 + (a - 6)XYZ = 0,
\]

the pull-back \( \phi_q^{-1}(C(1)_a) \) is defined by

\[
(U^q + V^q + W^q)^3 + (a - 6)(UVW)^q = 0,
\]

which is of the form \( \tilde{f}^3 + \tilde{g}^q = 0 \). The polynomials \( \tilde{f} \) and \( \tilde{g} \) are not general by any means. However, since the type of singularities of \( \phi_q^{-1}(C(1)_a) \) is the same as that of \( C_{3,q,1} \), we have an isomorphism \( \pi_1(\mathbb{P}^2 \setminus \phi_q^{-1}(C(1)_a)) \cong \pi_1(\mathbb{P}^2 \setminus C_{3,q,1}) \).

§3. Construction of Series II

It is enough to show the following:

**Proposition 5.** The quartic surface

\[
S_0 : F_0(x_1, x_2, x_3, x_4) := (x_1^2 + x_2^2)^2 + 2x_3x_4(x_1^2 - x_2^2) + x_3^2x_4^2 = 0
\]

in \( \mathbb{P}^3 \) satisfies the assumptions (1)-(4) with \( p = 2 \) and \( k = 2 \).

**Proof.** The assumptions (2) and (3) can be trivially checked. To check the assumptions (1) and (4), we put

\[
F_t := F_0 + t \cdot x_1x_2x_3x_4,
\]

and calculate the partial derivatives \( \partial F_t/\partial x_i \) for \( i = 1, \ldots, 4 \). Let \( Q_t \subset \mathbb{P}^3 \) be the quadric surface defined by

\[
2x_1^2 - 2x_2^2 + 2x_3x_4 + tx_1x_2 = 0.
\]
It is easy to see that $Q_t$ is irreducible for all $t \neq 1$. It is also easy to see that $Q_t$ is the unique common irreducible component of the two cubic surfaces

$$\frac{\partial F_t}{\partial x_3} = 0, \quad \text{and} \quad \frac{\partial F_t}{\partial x_4} = 0.$$ 

Suppose that a surface $S_a = \{F_a = 0\}$ in this pencil contains a non-reduced irreducible component $mT$ $(m \geq 2)$. Then, both of $\partial F_a/\partial x_3$ and $\partial F_a/\partial x_4$ must vanish on $T$. Hence $T$ must coincide with $Q_a$, and we get $S_a = 2Q_a$. Comparing the defining equations of $S_a$ and $2Q_a$, we see that there are no such $a$. Thus the assumption (1) is satisfied. To check the assumption (4), we remark that the condition dim Sing $S_t \leq 0$ is an open condition for $t$. Hence it is enough to prove, for example, dim Sing $S_2 = 0$. It is easy to show that Sing $S_2$ consists of four points $(1 : \pm \sqrt{-1} : 0 : 0)$, $(0 : 0 : 0 : 1)$ and $(0 : 0 : 1 : 0)$. □

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