ON ARITHMETIC ZARISKI PAIRS IN DEGREE 6

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Dedicated to Professor Mutsuo Oka for his sixtieth birthday.

ABSTRACT. We define a topological invariant of complex projective plane curves. As an application, we present new examples of arithmetic Zariski pairs.

1. INTRODUCTION

In this paper, we mean by a *plane curve* a complex reduced (possibly reducible) projective plane curve. The following definition is due to Artal [3]:

Definition 1.1. A pair (C, C') of plane curves of the same degree is called a *Zariski pair* if there exist tubular neighborhoods $T \subset \mathbb{P}^2$ of C and $T' \subset \mathbb{P}^2$ of C' such that (T, C) and (T', C') are diffeomorphic, while $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are not homeomorphic.

The first example of Zariski pairs was studied by Zariski in order to show that an equisingular family of plane curves need not be connected. Zariski [35] considered a six-cuspidal sextic curve C with the six cusps lying on a conic, and proved that $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to the free product of cyclic groups of order 2 and 3. Zariski [36] then showed, by means of deformation from a nine-cuspidal sextic curve (the dual curve of a smooth cubic curve), that there exists a six-cuspidal sextic curve C' with the six cusps *not* lying on a conic, and that $\pi_1(\mathbb{P}^2 \setminus C')$ is cyclic of order 6. Oka [18] constructed explicitly a non-conical six-cuspidal sextic curve C', and showed that $\pi_1(\mathbb{P}^2 \setminus C')$ is a cyclic group of order 6. Therefore the moduli space $\mathcal{M}(6A_2)$ of plane sextics possessing six cusps as their only singularities has at least two connected components that are distinguished by the fundamental groups of the complements. (See [3] and [22] for simple constructions of the pair (C, C').) Recently, Degtyarev [12] showed that $\mathcal{M}(6A_2)$ has exactly two connected components.

Many examples of Zariski pairs have been known now. The standard method to distinguish $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ topologically is to compare the fundamental groups of the complements, which are calculated by Zariski-van Kampen theorem (see [24]). Other methods are, for example, to compute Alexander polynomials, or to prove (non-)existence of finite étale Galois coverings of the complements with given branching properties.

Let F be a number field, that is, a finite extension of $\mathbb{Q}$. Let Φ be a polynomial with coefficients in F. For an embedding $\sigma : F \hookrightarrow \mathbb{C}$, we denote by Φ^σ the polynomial obtained from Φ by applying σ to the coefficients of Φ.

1991 *Mathematics Subject Classification.* 14F45 (Primary) 14J28, 14G25 (Secondary).

Definition 1.2. Plane curves C and C' are said to be *conjugate in a number field* F if there exist a homogeneous polynomial $\Phi(x_0, x_1, x_2)$ with coefficients in F and two distinct embeddings $\sigma : F \hookrightarrow \mathbb{C}$ and $\tau : F \hookrightarrow \mathbb{C}$ such that we have

$$C = \{\Phi^\sigma = 0\} \quad \text{and} \quad C' = \{\Phi^\tau = 0\}.$$

We say that C and C' are *conjugate* if they are conjugate in some number field.

Definition 1.3. A Zariski pair (C, C') is called an *arithmetic Zariski pair* if C and C' are conjugate.

A difficulty in constructing examples of arithmetic Zariski pairs comes from the fact that, if C and C' are conjugate, then $\pi_1(\mathbb{P}^2 \setminus C)$ and $\pi_1(\mathbb{P}^2 \setminus C')$ have the same pro-finite completions.

Artal, Carmona, and Cogolludo ([5], [7]) constructed an arithmetic Zariski pair in degree 12. They distinguished $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ by means of the braid monodromy. On the other hand, Artal, Carmona, Cogolludo and Marco [6] found an arithmetic Zariski pair of real line arrangements.

In this paper, we introduce an invariant N_C of the homeomorphism type of $(\mathbb{P}^2, C)$ for plane curves C of even degree. By means of this invariant combined with Degtyarev's general result [12] on the connected components of the moduli of singular sextic curves, we construct examples of arithmetic Zariski pairs in degree 6. More precisely, we show that some pairs of conjugate curves of degree 6 that were discovered by Artal, Carmona and Cogolludo [4] are in fact arithmetic Zariski pairs. We also give another example using the idea developed in [27].

In order to explain our examples, we introduce some terminologies. A *Dynkin type* is a finite formal sum

$$R = \sum_{l \geq 1} a_l A_l + \sum_{m \geq 4} d_m D_m + \sum_{n=6}^{8} e_n E_n,$$

where a_l, d_m and e_n are non-negative integers, almost all of which are zero. The *rank* of the Dynkin type R is defined by

$$\operatorname{rank}(R) = \sum a_l l + \sum d_m m + \sum e_n n.$$

An *ADE-sextic* is a plane curve of degree 6 with only simple singularities. The *type R of an ADE-sextic C* is the Dynkin type of the singularities of C. Then $\operatorname{rank}(R)$ is equal to the total Milnor number of C, and hence it is at most 19. We say that an ADE-sextic is a *maximizing sextic* if the total Milnor number is 19 (see Persson [19]). If C is an ADE-sextic, then the minimal resolution X_C of the double covering $Y_C \to \mathbb{P}^2$ that branches exactly along C is a $K3$ surface. When C is a maximizing sextic, our invariant N_C of $(\mathbb{P}^2, C)$ coincides with the transcendental lattice of X_C.

Theorem 1.4. *There exists an arithmetic Zariski pair of maximizing sextics for each of the following Dynkin types:*

$$\text{(i) } A_{16} + A_2 + A_1, \quad \text{(ii) } A_{16} + A_3, \quad \text{(iii) } A_{18} + A_1, \quad \text{(iv) } A_{10} + A_9.$$

The plan of this paper is as follows. In §2, we define an invariant N_C for curves C on a smooth projective surface S satisfying certain conditions, and show that N_C is in fact an invariant of the *Γ-equivalence class* of (S, C) (see Definition 2.5). The Γ-equivalence is an equivalence relation coarser than the homeomorphism type of (S, C), and finer than the homeomorphism type of $S \setminus C$. Applying the main result (Theorem 2.6) to the case $S = \mathbb{P}^2$, we obtain an invariant of the homeomorphism type of $(\mathbb{P}^2, C)$ for plane curves C of even degree. In §3, we review the theory of Degtyarev [12] on the connected components of the moduli space $\mathcal{M}(R)$ of *ADE*-sextics with a given Dynkin type R. This theory is crucial to our construction. In §4, we calculate the connected components of $\mathcal{M}(R)$ for some R with $\mathrm{rank}(R) = 19$. Combining this calculation with the result of [4, Theorem 5.8] and using our invariant, we show that some pairs of conjugate maximizing sextics obtained in [4] yield examples of arithmetic Zariski pairs (the examples (i)-(iii) above). In §5, we present another example of arithmetic Zariski pairs constructed by means of the theory of Hilbert class fields of imaginary quadratic fields (the example (iv) above).

The first example of non-homeomorphic conjugate complex varieties was given by Serre [21]. Since then, only few examples seem to have been treated (e.g., Abelson [1]). The argument of this paper provides us with a new method to construct examples of non-trivial effects of the automorphism of the base field on the topology of complex varieties.

The author expresses gratitude to the referee for valuable comments on the first version of this paper.

2. The invariant N_C

First we fix some notation and terminologies.

Let A be a finitely generated $\mathbb{Z}$-module. We denote by A_{tor} the torsion subgroup of A, and by $A^{\mathrm{tf}} := A/A_{\mathrm{tor}}$ the *torsion-free quotient* of A. If $b : A \times A \to \mathbb{Z}$ is a symmetric bilinear form on A, then b induces a symmetric bilinear form on A^{tf} in the natural way.

Let A be a free $\mathbb{Z}$-module of finite rank, and A' a submodule of A. The *primitive closure* of A' in A is defined to be the intersection of $A' \otimes \mathbb{Q}$ and A in $A \otimes \mathbb{Q}$. We say that A' is *primitive in* A if the primitive closure of A' is equal to A'.

A *lattice* is a free $\mathbb{Z}$-module A of finite rank equipped with a *non-degenerate* symmetric bilinear form $A \times A \to \mathbb{Z}$. Two lattices A and A' are *isomorphic* if there exists an isomorphism $A \xrightarrow{\sim} A'$ of $\mathbb{Z}$-modules that preserves the symmetric bilinear forms. The automorphism group of a lattice A is denoted by $O(A)$. If A and A' are lattices, then $A \perp A'$ denotes the orthogonal direct-sum of A and A'.

For a topological space Z, we denote by $H_2(Z)$ the singular homology group $H_2(Z, \mathbb{Z})$. When Z is an oriented topological manifold with $\dim_{\mathbb{R}}(Z) = 4$, we have the intersection pairing

$$b_Z : H_2(Z) \times H_2(Z) \to \mathbb{Z}.$$

If we further assume that Z is compact, then $H_2(Z)^{\mathrm{tf}}$ becomes a lattice by b_Z.

Let S be a smooth connected complex projective surface such that

$$\mathrm{Pic}(S) \cong \mathbb{Z} \quad \text{and} \quad \pi_1(S) = \{1\}.$$

Remark 2.1. We are mainly interested in the case where S is $\mathbb{P}^2$. There exist other examples. If S is a *general* complete intersection in $\mathbb{P}^{2+r}$ of multi-degree $(a_1, \dots, a_r)$, then S satisfies the conditions above, provided that $(a_1, \dots, a_r)$ is not (2), (3) or $(2, 2)$ (see [32]).

Let $\mathcal{H}$ be the line bundle on S such that its class is the positive generator of $\mathrm{Pic}(S)$. Let d be a positive even integer, and put

$$\mathcal{L} := \mathcal{H}^{\otimes d} \quad \text{and} \quad \mathcal{M} := \mathcal{H}^{\otimes d/2}.$$

An $\mathcal{L}$*-curve* is a reduced (possibly reducible) member of the complete linear system $|\mathcal{L}|$. Let C be an $\mathcal{L}$-curve given by $s = 0$, where s is a global section of $\mathcal{L}$, and let

$$\pi : Y \to S$$

be the finite double covering that branches exactly along C, where Y is the pull-back of the image of the global section s by the squaring morphism $\mathcal{M} \to \mathcal{M}^{\otimes 2} = \mathcal{L}$ over S. Note that Y is normal, because Y is a hypersurface in the total space of the line bundle $\mathcal{M}$ with only isolated singular points (Altman and Kleiman [2, Chapter VII, Corollary (2.14)]). Let

$$\rho : X \to Y$$

be a proper birational morphism from a smooth surface X that induces an isomorphism $\rho^{-1}(Y \setminus \pi^{-1}(C)) \cong Y \setminus \pi^{-1}(C)$. We put

$$\phi := \pi \circ \rho \ : \ X \ \to \ S.$$

Then ϕ is an étale double covering over $S \setminus C$. We denote by

$$\widetilde{M}_C \ \subset \ H_2(X)$$

the submodule generated by the homology classes of the reduced irreducible components of $\phi^{-1}(C) \subset X$. We then put

$$\begin{aligned} \widetilde{N}_C \ &:= \ \{ \, x \in H_2(X) \ | \ b_X(x, y) = 0 \ \text{ for any } \ y \in \widetilde{M}_C \, \} \quad \text{and} \\ N_C \ &:= \ (\widetilde{N}_C)^{\mathrm{tf}} \ \subset \ H_2(X)^{\mathrm{tf}}. \end{aligned}$$

Note that N_C is primitive in $H_2(X)^{\mathrm{tf}}$.

Lemma 2.2. *The restriction of* b_X *to* N_C *is non-degenerate.*

Proof. First we assume that ρ is the minimal desingularization of Y. Since $H_2(X)^{\mathrm{tf}}$ is a lattice by b_X, and N_C is the orthogonal complement of $(\widetilde{M}_C)^{\mathrm{tf}}$ in $H_2(X)^{\mathrm{tf}}$, it is enough to show that the restriction of b_X to $(\widetilde{M}_C)^{\mathrm{tf}}$ is non-degenerate. Let $h \in H_2(X)^{\mathrm{tf}}$ be the first Chern class of the line bundle $\phi^*(\mathcal{H})$. (We have a canonical isomorphism $H_2(X)^{\mathrm{tf}} \cong H^2(X)^{\mathrm{tf}}$.) Since $b_X(h, h) > 0$, the $\mathbb{Z}$-module $\langle h \rangle \subset H_2(X)^{\mathrm{tf}}$ generated by h is a positive-definite lattice of rank 1. Let $p_1, \dots, p_t$ be the singular points of Y. For each p_i, we denote by $\Sigma_i \subset H_2(X)^{\mathrm{tf}}$ the submodule generated by the homology classes of reduced irreducible curves on X that are contracted to p_i by ρ. By the theorem of Mumford [16], the $\mathbb{Z}$-module Σ_i is a negative-definite lattice. The lattice Σ_i is perpendicular to $\langle h \rangle$ and Σ_j $(j \neq i)$ with respect to b_X. Therefore the submodule

$$M_C^0 := \langle h \rangle \perp \Sigma_1 \perp \cdots \perp \Sigma_t$$

of $H_2(X)^{\mathrm{tf}}$ is a lattice by b_X. Let C_i be an irreducible component of C, and let $\widetilde{C}_i$ be the reduced irreducible curve on X such that $\phi(\widetilde{C}_i) = C_i$. Since $\mathrm{Pic}(S) \cong \mathbb{Z}$ is generated by the class of $\mathcal{H}$, there exists an integer d_i such that C_i is linearly equivalent to $\mathcal{H}^{\otimes d_i}$ on S. Then there exists $\gamma \in \Sigma_1 \perp \cdots \perp \Sigma_t$ such that $2[\widetilde{C}_i] = d_i h + \gamma$ holds in $H_2(X)^{\mathrm{tf}}$. Therefore $M_C \otimes \mathbb{Q}$ is equal to $M_C^0 \otimes \mathbb{Q}$ in $H_2(X)^{\mathrm{tf}} \otimes \mathbb{Q}$, and hence b_X is non-degenerate on M_C.

The proof for the general case is derived from the following observation. Let $X' \to X$ be the blowing up at a point P on $\phi^{-1}(C)$, and let E be the (-1)-curve on X' contracted to P. Then we have a natural isomorphism

$$H_2(X') = H_2(X) \perp \langle [E] \rangle.$$

Hence the $\mathbb{Z}$-module $N'_C \subset H_2(X')^{\mathrm{tf}}$ equipped with $b_{X'}$ is isomorphic to the $\mathbb{Z}$-module $N_C \subset H_2(X)^{\mathrm{tf}}$ equipped with b_X. □

From now on, we consider N_C as a lattice by b_X. The following has already been shown in the proof of Lemma 2.2.

Lemma 2.3. *The isomorphism class of the lattice N_C does not depend on the choice of the morphism $\rho : X \to Y$.*

Therefore we can consider the isomorphism class of the lattice N_C as an invariant of the $\mathcal{L}$-curve C.

Next we define the $\mathit{\Gamma}$-equivalence relation among $\mathcal{L}$-curves.

Definition 2.4. Let I be the closed interval $[0, 1] \subset \mathbb{R}$. We fix a base point $b \in S \setminus C$. Let C_i be an irreducible component of C. A loop $\gamma : I \to S \setminus C$ with the base point b is called a *simple loop* around C_i if there exists a continuous embedding $\delta : \Delta \hookrightarrow S$ of the closed unit disk $\Delta := \{z \in \mathbb{C} \mid |z| \le 1\}$ into S such that

(i) $\delta^{-1}(C) = \{0\}$, and $P := \delta(0)$ is a smooth point of C_i,
(ii) the local intersection number of $\delta(\Delta)$ and C_i (with the orientation coming from the complex structures) at P is 1, and
(iii) the loop γ goes from b to a point $b' \in \delta(\partial\Delta)$ along a path β, turns around C_i along $\delta(\partial\Delta)$ once in a positive direction, and goes back to b along β^{-1}.

Let $C_1, \dots, C_m$ be the irreducible components of an $\mathcal{L}$-curve C. It is easy to see that the homotopy classes of simple loops around C_i form a conjugacy class in $\pi_1(S \setminus C, b)$, which we will denote by Γ_i^+. We then put

$$\Gamma_i^- := \{\ [\gamma]^{-1}\ \mid\ [\gamma] \in \Gamma_i^+\ \} \quad \text{and} \quad \Gamma_i := \Gamma_i^+ \cup \Gamma_i^-.$$

Finally, we put

$$\mathit{\Gamma}^+(C) := \{\Gamma_1^+, \dots, \Gamma_m^+\} \quad \text{and} \quad \mathit{\Gamma}(C) := \{\Gamma_1, \dots, \Gamma_m\}.$$

Definition 2.5. Let C and C' be $\mathcal{L}$-curves. We say that (S, C) and (S, C') are *$\mathit{\Gamma}$-equivalent* (resp. *$\mathit{\Gamma}^+$-equivalent*) if there exists a homeomorphism

$$\psi\ :\ S \setminus C\ \cong\ S \setminus C'$$

such that the induced isomorphism $\pi_1(S \setminus C, b) \cong \pi_1(S \setminus C', \psi(b))$ gives rise to a bijection from $\mathit{\Gamma}(C)$ to $\mathit{\Gamma}(C')$ (resp. from $\mathit{\Gamma}^+(C)$ to $\mathit{\Gamma}^+(C')$).

Let C and C' be $\mathcal{L}$-curves. If (S, C) and (S, C') are homeomorphic, then they are $\mathit{\Gamma}$-equivalent. If there exists an orientation-preserving homeomorphism between (S, C) and (S, C'), then they are $\mathit{\Gamma}^+$-equivalent.

The following is the main result of this paper:

Theorem 2.6. *The isomorphism class of the lattice N_C is an invariant of the Γ-equivalence class of (S, C).*

Proof. Since $\pi_1(S)$ is assumed to be trivial, $\pi_1(S \setminus C)$ is generated by simple loops around irreducible components of C. Therefore a homomorphism $\pi_1(S \setminus C) \to \mathbb{Z}/2\mathbb{Z}$ that maps every element of $\Gamma_1 \cup \cdots \cup \Gamma_m$ to the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ is unique. Consequently, the homeomorphism type of

$$U := \phi^{-1}(S \setminus C) \quad \subset \quad X$$

is uniquely determined by the Γ-equivalence class of (S, C). For a compact subset K of U, we denote by B^K the image of the natural homomorphism $H_2(U \setminus K) \to H_2(U)$. We then put

$$B^\infty \ := \ \bigcap B^K,$$

where K runs through the set of all compact subsets of U, and set

$$\widetilde{B}_U := H_2(U)/B^\infty \quad \text{and} \quad B_U := (\widetilde{B}_U)^{\mathrm{tf}}.$$

Since every topological cycle is compact, the intersection pairing b_U on $H_2(U)$ sets up a symmetric bilinear form

$$\beta_U \ : \ \widetilde{B}_U \times \widetilde{B}_U \ \to \ \mathbb{Z}.$$

By construction, the isomorphism class of (B_U, β_U) is determined by the homeomorphism type of U, and hence by the Γ-equivalence class of (S, C). Therefore it is enough to show that the lattice N_C is isomorphic to (B_U, β_U).

We put $D := \phi^{-1}(C)$, and equip D with the reduced structure. Let $T \subset X$ be a tubular neighborhood of D. We put

$$T^\times := T \setminus D = T \cap U,$$

and consider the Mayer-Vietoris sequence

$$\tag{2.1} \begin{array}{ccccccc} \to & H_2(T^\times) & \xrightarrow{\ i\ } & H_2(T) \oplus H_2(U) & \xrightarrow{\ j\ } & H_2(X) & \to \\ & x & \longmapsto & (i_T(x), i_U(x)) & & & \\ & & & (y, z) & \longmapsto & j_T(y) - j_U(z) & . \end{array}$$

First note that, since T is a tubular neighborhood of D, we have

$$\tag{2.2} \mathrm{Im}(i_U) = B^\infty,$$

where $i_U : H_2(T^\times) \to H_2(U)$ is the natural homomorphism induced by the inclusion $T^\times \hookrightarrow U$.

Next we prove

$$\tag{2.3} \mathrm{Im}(j_U) = \widetilde{N}_C,$$

where $j_U : H_2(U) \to H_2(X)$ is the natural homomorphism induced by the inclusion $U \hookrightarrow X$. It is obvious that $\mathrm{Im}(j_U) \subseteq \widetilde{N}_C$. Let $[W]$ be an element of $\widetilde{N}_C$ represented by a 2-dimensional topological cycle $W \subset X$. We can assume the following:

(i) $W \cap \mathrm{Sing}(D) = \emptyset$,
(ii) $W \cap D$ consists of a finite number of points, and
(iii) locally around each point P of $W \cap D$, W is a $\mathcal{C}^\infty$-manifold intersecting D at P transversely.

Let $D^{(1)}, \dots, D^{(n)}$ be the irreducible components of D. For each $\nu = 1, \dots, n$, let $P^{(\nu)}_{+,1}, \dots, P^{(\nu)}_{+,k(\nu)}$ (resp. $P^{(\nu)}_{-,1}, \dots, P^{(\nu)}_{-,l(\nu)}$) be the intersection points of W and $D^{(\nu)}$ with local intersection number 1 (resp. -1). Since $b_X([W],[D^{(\nu)}]) = 0$, we have

$$k(\nu) = l(\nu).$$

Let $\Delta \subset \mathbb{C}$ be the closed unit disk, and let $I \subset \mathbb{R}$ be the closed interval $[0,1]$. Since $D^{(\nu)} \setminus \mathrm{Sing}(D)$ is path-connected for each ν, we have continuous maps

$$\xi^{(\nu)}_i : \Delta \times I \to X$$

for $\nu = 1, \dots, n$ and $i = 1, \dots, k(\nu)$ with the following properties:

(i) $\xi^{(\nu)}_i(\Delta \times I) \cap D \subset D^{(\nu)} \setminus \mathrm{Sing}(D)$,

(ii) $(\xi^{(\nu)}_i)^{-1}(D) = \{0\} \times I$,

(iii) $\xi^{(\nu)}_i(0,0) = P^{(\nu)}_{+,i}$, and $\xi^{(\nu)}_i$ induces a homeomorphism from $\Delta \times \{0\}$ to a closed neighborhood $\Delta^{(\nu)}_{+,i} \subset W$ of $P^{(\nu)}_{+,i}$ in W, and

(iv) $\xi^{(\nu)}_i(0,1) = P^{(\nu)}_{-,i}$, and $\xi^{(\nu)}_i$ induces a homeomorphism from $\Delta \times \{1\}$ to a closed neighborhood $\Delta^{(\nu)}_{-,i} \subset W$ of $P^{(\nu)}_{-,i}$ in W.

We then put

$$W' := \Big(W \setminus \bigcup_{\nu,i} (\Delta^{(\nu)}_{+,i} \cup \Delta^{(\nu)}_{-,i})\Big) \cup \bigcup_{\nu,i} \xi^{(\nu)}_i(\partial\Delta \times I).$$

Namely, we cut out discs $\Delta^{(\nu)}_{+,i}$ and $\Delta^{(\nu)}_{-,i}$ from W, and put tubes $\xi^{(\nu)}_i(\partial\Delta \times I)$ around the paths $\xi^{(\nu)}_i(\{0\} \times I)$ on $D^{(\nu)}$. The tube $\xi^{(\nu)}_i(\partial\Delta \times I)$ connects the pair of circles $\partial\Delta^{(\nu)}_{+,i}$ and $\partial\Delta^{(\nu)}_{-,i}$. Since the local intersection numbers of W and $D^{(\nu)}$ at $P^{(\nu)}_{+,i}$ and at $P^{(\nu)}_{-,i}$ have opposite signs, we can put an orientation on each solid tube $\xi^{(\nu)}_i(\Delta \times I)$ in such a way that $\Delta^{(\nu)}_{+,i} \subset W$ and $\xi^{(\nu)}_i(\Delta \times \{0\}) \subset \partial(\xi^{(\nu)}_i(\Delta \times I))$ (resp. $\Delta^{(\nu)}_{-,i} \subset W$ and $\xi^{(\nu)}_i(\Delta \times \{1\}) \subset \partial(\xi^{(\nu)}_i(\Delta \times I))$) have the opposite orientations. Then, with the orientation on the tubes $\xi^{(\nu)}_i(\partial\Delta \times I)$ induced from the orientation of $\xi^{(\nu)}_i(\Delta \times I)$, the space W' becomes a topological cycle. Note that W and W' are homologous in X, because $W - W'$ is the boundary of the 3-dimensional topological chain $\bigcup \xi^{(\nu)}_i(\Delta \times I)$. Moreover W' is disjoint from D. Therefore $[W] = [W']$ is contained in $\mathrm{Im}(j_U)$, and hence (2.3) is proved.

Let z be an element of $H_2(U)$. If $j_U(z) = 0$, then $(0, z) \in H_2(T) \oplus H_2(U)$ is contained in $\mathrm{Ker}(j) = \mathrm{Im}(i)$, where i and j are homomorphisms in the Mayer-Vietoris exact sequence (2.1), and hence $z \in \mathrm{Im}(i_U)$ holds. Therefore we have a natural inclusion $\mathrm{Ker}(j_U) \hookrightarrow \mathrm{Im}(i_U)$. Consider the following diagram:

$$(2.4) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(j_U) & \longrightarrow & H_2(U) & \xrightarrow{j_U} & \widetilde{N}_C & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow \lambda & & \\ 0 & \longrightarrow & \mathrm{Im}(i_U) & \longrightarrow & H_2(U) & \longrightarrow & \widetilde{B}_U & \longrightarrow & 0. \end{array}$$

The upper sequence is exact by (2.3), and the lower sequence is exact by (2.2). Therefore we obtain a surjective homomorphism $\lambda : \widetilde{N}_C \to \widetilde{B}_U$ that makes the diagram (2.4) commutative. Remark that, by the definition of the intersection pairing, we have

$$b_U(z, z') = b_X(j_U(z), j_U(z'))$$

for any $z, z' \in H_2(U)$. Hence, for any $\zeta, \zeta' \in \widetilde{N}_C \subset H_2(X)$, we have

$$b_X(\zeta, \zeta') = \beta_U(\lambda(\zeta), \lambda(\zeta')). \tag{2.5}$$

Therefore, in order to show that N_C is isomorphic to (B_U, β_U), it is enough to prove that λ induces an isomorphism $N_C \xrightarrow{\sim} B_U$ on the torsion-free quotients, or equivalently, $\lambda^{-1}((\widetilde{B}_U)_{\mathrm{tor}}) = (\widetilde{N}_C)_{\mathrm{tor}}$ holds. It is obvious that $\lambda((\widetilde{N}_C)_{\mathrm{tor}}) \subseteq (\widetilde{B}_U)_{\mathrm{tor}}$. Suppose that $\zeta \in \widetilde{N}_C$ satisfies $\lambda(\zeta) \in (\widetilde{B}_U)_{\mathrm{tor}}$. By (2.5), we have $b_X(\zeta, \zeta') = 0$ for any $\zeta' \in \widetilde{N}_C$. Since N_C is a lattice, we have $\zeta \in (\widetilde{N}_C)_{\mathrm{tor}}$. □

Corollary 2.7. *Let C and C' be plane curves of the same degree. Suppose that $\deg C = \deg C'$ is even. If $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are homeomorphic, then N_C and $N_{C'}$ are isomorphic.*

3. Sextics with only simple singularities

Let $\mathbb{P}_*(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6)))$ be the projective space of one-dimensional subspaces of the vector space $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$ of homogeneous polynomials of degree 6 on $\mathbb{P}^2$. For a Dynkin type R of rank ≤ 19, we denote by

$$\mathcal{M}(R) \quad \subset \quad \mathbb{P}_*(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6)))$$

the space of ADE-sextics of type R. Using Urabe's idea [33], Yang [34] made the complete list of Dynkin types R such that $\mathcal{M}(R) \neq \emptyset$. Degtyarev [12] refined Yang's argument, and gave a method to calculate the connected components of $\mathcal{M}(R)$ for a given R. In this section, we expound Degtyarev's theory.

We fix some notation and terminologies about lattices.

Let Λ be a lattice of rank $n = 2 + s_-$ and signature $(2, s_-)$. For a non-zero vector $\omega \in \Lambda \otimes \mathbb{C}$, we denote by $[\omega] \in \mathbb{P}_*(\Lambda \otimes \mathbb{C})$ the one-dimensional vector space spanned by ω. We put

$$\Omega_\Lambda := \{ \, [\omega] \in \mathbb{P}_*(\Lambda \otimes \mathbb{C}) \;\mid\; (\omega, \omega) = 0, \, (\omega, \bar{\omega}) > 0 \, \}.$$

It is easy to verify that Ω_Λ is a complex manifold of dimension $s_- = n-2$ consisting of two connected components.

The *dual lattice* $\Lambda^\vee$ of a lattice Λ is defined by

$$\Lambda^\vee := \{ \, v \in \Lambda \otimes \mathbb{Q} \;\mid\; (x, v) \in \mathbb{Z} \text{ for all } x \in \Lambda \, \}.$$

We have $\Lambda \subset \Lambda^\vee$. An *overlattice* of Λ is a submodule Λ' of $\Lambda^\vee$ containing Λ such that the natural $\mathbb{Q}$-valued symmetric bilinear form on $\Lambda \otimes \mathbb{Q}$ takes values in $\mathbb{Z}$ on Λ'. The *discriminant group* G_Λ of Λ is defined by

$$G_\Lambda := \Lambda^\vee / \Lambda.$$

A lattice is called *unimodular* if $\Lambda^\vee = \Lambda$. A lattice Λ is said to be *even* if $(v, v) \in 2\,\mathbb{Z}$ holds for every $v \in \Lambda$. If Λ is an even lattice, we can define a quadratic form

$$q_\Lambda \;:\; G_\Lambda \to \mathbb{Q}/2\,\mathbb{Z}$$

by $q_\Lambda(\bar{v}) := (v, v) \bmod 2\,\mathbb{Z}$, where $v \in \Lambda^\vee$ and $\bar{v} := v \bmod \Lambda$. This quadratic form is called the *discriminant form* of Λ. See Nikulin [17] for the basic properties of discriminant forms.

Let Λ be a negative-definite even lattice. A vector $d \in \Lambda$ is called a *root* if $(d,d) = -2$ holds. We say that Λ is a *root lattice* if Λ is generated by the roots in Λ. The isomorphism classes of root lattices are in one-to-one correspondence with the Dynkin types (see, for example, Ebeling [13, Section 1.4]). We denote by Σ_R^- the negative-definite root lattice of Dynkin type R. A subset F of Σ_R^- is called a *fundamental system of roots* if every element of F is a root, F is a basis of Σ_R^-, and each root $v \in \Sigma_R^-$ is written as a linear combination $v = \sum_{d \in F} k_d d$ of roots $d \in F$ with integer coefficients k_d all non-positive or all non-negative. A fundamental system F of roots exists (see [13, Section 1.4]). The intersection matrix of Σ_R^- with respect to the basis F is the Cartan matrix of type R multiplied by -1.

A lattice is called a $K3$ *lattice* if it is even, unimodular, of rank 22 and with signature $(3, 19)$. By the structure theorem of unimodular lattices, a $K3$ lattice is unique up to isomorphism (see, for example, Serre [20, Chapter V]).

We now start explaining Degtyarev's theory. Let R be a Dynkin type with

$$r := \operatorname{rank}(R) \leq 19.$$

First, we define a set $Q(R)$ and an equivalence relation $\sim$ on it. We denote by $\langle h \rangle$ the lattice of rank 1 generated by a vector h with $(h,h) = 2$. We put

$$M^0 := \Sigma_R^- \perp \langle h \rangle,$$

which is an even lattice of signature $(1, r)$. *We choose a fundamental system of roots $F \subset \Sigma_R^-$ once and for all*, and put

$$O_{F,h}(M^0) := \{ \, g \in O(M^0) \;\mid\; g(F) = F, \; g(h) = h \, \}.$$

We denote by Ms the set of even overlattices M of M^0 satisfying the following two conditions:

(m1) $\{ \, v \in M \mid (v,h) = 1, \; (v,v) = 0 \, \} = \emptyset$, and
(m2) $\{ \, v \in M \mid (v,h) = 0, \; (v,v) = -2 \, \} = \{ \, v \in \Sigma_R^- \mid (v,v) = -2 \, \}$.

(These conditions correspond to the conditions (a) and (b) in [34, Theorem 2.3].) For $M \in Ms$, we denote by $Ns(M)$ a complete set of representatives of isomorphism classes of even lattices N of rank $21 - r$ satisfying the following two conditions:

(n1) N is of signature $(2, 19 - r)$, and
(n2) the discriminant form (G_N, q_N) of N is isomorphic to $(G_M, -q_M)$.

By Nikulin [17, Proposition 1.6.1], the conditions (n1) and (n2) are equivalent to the following condition:

(n) there exists an even unimodular overlattice L of $M \perp N$ with signature $(3, 19)$ such that M and N are primitive in L.

Let N be an element of $Ns(M)$. We denote by $Ls(M, N)$ the set of even unimodular overlattices L of $M \perp N$ such that M and N are primitive in L. Note that every $L \in Ls(M, N)$ is a $K3$ lattice. We also denote by $c\Omega s(N)$ the set of connected components of the complex manifold Ω_N. Remark that we have $|c\Omega s(N)| = 2$.

We define $Q(R)$ to be the set of quartets $(M, N, L, c\Omega)$ such that $M \in Ms$, $N \in Ns(M)$, $L \in Ls(M, N)$, and $c\Omega \in c\Omega s(N)$. For quartets $(M, N, L, c\Omega)$ and $(M', N', L', c\Omega')$ in $Q(R)$, we write

$$(M, N, L, c\Omega) \;\sim\; (M', N', L', c\Omega')$$

if the following hold.

(i) There exists $g^0 \in O_{F,h}(M^0) \subset O(M^0)$ such that the induced action of g^0 on Ms maps $M \in Ms$ to $M' \in Ms$. We denote by $g_M : M \xrightarrow{\sim} M'$ the unique isomorphism satisfying $g_M | M^0 = g^0$.

(ii) Since $(G_M, -q_M)$ and $(G_{M'}, -q_{M'})$ are isomorphic, there exists a canonical bijection between $Ns(M)$ and $Ns(M')$. The elements $N \in Ns(M)$ and $N' \in Ns(M')$ are corresponding by this bijection; that is, N and N' are isomorphic.

(iii) There exists an isomorphism $g_N : N \xrightarrow{\sim} N'$ of lattices such that the bijection $Ls(M, N) \xrightarrow{\sim} Ls(M', N')$ induced by the isomorphism $g_M \oplus g_N$ from $M \perp N$ to $M' \perp N'$ maps $L \in Ls(M, N)$ to $L' \in Ls(M', N')$, and that the induced isomorphism $\Omega_N \xrightarrow{\sim} \Omega_{N'}$ maps $c\Omega$ to $c\Omega'$.

For $(M, N, L, c\Omega) \in Q(R)$, we denote by $[M, N, L, c\Omega] \in Q(R)/\sim$ the equivalence class of the relation $\sim$ containing $(M, N, L, c\Omega)$.

Remark 3.1. If $(M, N, L, c\Omega) \in Q(R)$, then M^0 is the sublattice of L generated by $F \subset L$ and $h \in L$, M is the primitive closure of M^0 in L, and N is the orthogonal complement of M in L. Hence $[M, N, L, c\Omega] = [M', N', L', c\Omega']$ holds if and only if there exists an isomorphism $L \xrightarrow{\sim} L'$ that maps F to F, h to h, and such that the induced isomorphism $\Omega_L \xrightarrow{\sim} \Omega_{L'}$ maps the connected component $c\Omega$ of $\Omega_N \subset \Omega_L$ to the connected component $c\Omega'$ of $\Omega_{N'} \subset \Omega_{L'}$.

Next we define a map ρ from the space $\mathcal{M}(R)$ to the set $Q(R)/\sim$. Let C be an ADE-sextic of type R, and let X be the minimal resolution of the double covering $Y \to \mathbb{P}^2$ that branches exactly along C. We denote by $\mathcal{L}$ the line bundle on X corresponding to the pull-back of the invertible sheaf $\mathcal{O}_{\mathbb{P}^2}(1)$. We have $([\mathcal{L}], [\mathcal{L}]) = 2$. We then put

$$L_X := H^2(X, \mathbb{Z}),$$

which is a $K3$ lattice. Let $F_{(X,\mathcal{L})} \subset L_X$ be the set of cohomology classes of (-2)-curves that are contracted by the desingularization morphism $X \to Y$, and let $\Sigma_{(X,\mathcal{L})} \subset L_X$ be the sublattice of L_X generated by $F_{(X,\mathcal{L})}$. Then $\Sigma_{(X,\mathcal{L})}$ is a negative-definite root lattice of type R. It is known that $F_{(X,\mathcal{L})}$ is a fundamental system of roots in $\Sigma_{(X,\mathcal{L})}$ (see [26, Proposition 2.4]). In particular, there exists an isomorphism of lattices from $\Sigma_{(X,\mathcal{L})}$ to Σ_R^- that maps $F_{(X,\mathcal{L})}$ to the fixed fundamental system of roots $F \subset \Sigma_R^-$ bijectively. We put

$$M^0_{(X,\mathcal{L})} := \Sigma_{(X,\mathcal{L})} \perp \langle [\mathcal{L}] \rangle,$$

and choose an isomorphism

$$\gamma^0_M \;:\; M^0_{(X,\mathcal{L})} \;\xrightarrow{\sim}\; M^0$$

satisfying $\gamma^0_M(F_{(X,\mathcal{L})}) = F$ and $\gamma^0_M([\mathcal{L}]) = h$. Let $M_{(X,\mathcal{L})}$ be the primitive closure of $M^0_{(X,\mathcal{L})}$ in L_X, and M the even overlattice of M^0 corresponding to the even overlattice $M_{(X,\mathcal{L})}$ of $M^0_{(X,\mathcal{L})}$ by γ^0_M. Then M satisfies the conditions (m1) and (m2) (see [26, Proposition 2.1]). Hence $M \in Ms$. We denote by

$$\gamma_M \;:\; M_{(X,\mathcal{L})} \;\xrightarrow{\sim}\; M$$

the isomorphism induced by γ^0_M. Let $N_{(X,\mathcal{L})}$ be the orthogonal complement of $M_{(X,\mathcal{L})}$ in L_X. Since the $K3$ lattice L_X is an even unimodular overlattice of $M_{(X,\mathcal{L})} \perp N_{(X,\mathcal{L})}$ in which $M_{(X,\mathcal{L})}$ and $N_{(X,\mathcal{L})}$ are primitive, the lattice $N_{(X,\mathcal{L})}$

satisfies the condition (n). Hence there exists a unique element N of $Ns(M)$ that is isomorphic to $N_{(X,\mathcal{L})}$. We choose an isomorphism

$$\gamma_N \;:\; N_{(X,\mathcal{L})} \;\xrightarrow{\sim}\; N.$$

By the isomorphism

$$\gamma_M \oplus \gamma_N \;:\; M_{(X,\mathcal{L})} \perp N_{(X,\mathcal{L})} \;\xrightarrow{\sim}\; M \perp N,$$

the even unimodular overlattice L_X of $M_{(X,\mathcal{L})} \perp N_{(X,\mathcal{L})}$ corresponds to an element L of $Ls(M,N)$. We denote by

$$\omega_X \in H^{2,0}(X) \subset L_X \otimes \mathbb{C}$$

the cohomology class of a non-zero holomorphic 2-form on X. Since $M_{(X,\mathcal{L})} \subset H^{1,1}(X)$, the vector ω_X defines a point $[\omega_X]$ of $\Omega_{N_{(X,\mathcal{L})}}$. Let $c\Omega$ be the connected component of Ω_N that contains the point $[\gamma_N(\omega_X)]$. Thus we obtain a quartet $(M,N,L,c\Omega) \in Q(R)$. The choices we have made during the process of finding $(M,N,L,c\Omega)$ are only on γ_M^0 and γ_N. Since γ_M^0 is unique up to $O_{F,h}(M^0)$ and γ_N is unique up to $O(N)$, the equivalence class $[M,N,L,c\Omega]$ does not depend on these choices. We thus can put

$$\rho(C) := [M,N,L,c\Omega].$$

Remark 3.2. By definition, we have $\rho(C) = [M,N,L,c\Omega]$ if and only if there exists an isomorphism $L_X \xrightarrow{\sim} L$ that maps $F_{(X,\mathcal{L})}$ to F, $[\mathcal{L}]$ to h, and such that the induced isomorphism $\Omega_{L_X} \xrightarrow{\sim} \Omega_L$ maps the point $[\omega_X] \in \Omega_{N_{(X,\mathcal{L})}} \subset \Omega_{L_X}$ to a point of the connected component $c\Omega$ of $\Omega_N \subset \Omega_L$.

We now have all the ingredients that are needed to state the main theorem of Degtyarev [12]:

Theorem 3.3. *The map ρ induces a bijection from the set of connected components of the space $\mathcal{M}(R)$ to the set $Q(R)/\!\sim$.*

The main tool of the proof is the Torelli theorem for the refined period map of marked $K3$ surfaces. See the book by Barth, Hulek, Peters and Van de Ven [8, Theorems 12.3 and 14.1 in Chapter VIII].

By definition, we have the following:

Corollary 3.4. *Let C be an ADE-sextic such that $\rho(C) = [M,N,L,c\Omega]$. Then the lattice N is isomorphic to the invariant N_C of the Γ-equivalence class of $(\mathbb{P}^2, C)$.*

We explain how to calculate the set $Q(R)/\!\sim$. By [17, Proposition 1.4.1], the even overlattices of $M^0 = \Sigma_R^- \perp \langle h \rangle$ are in one-to-one correspondence with the totally isotropic subgroups of the discriminant form (G_{M^0}, q_{M^0}). For an even overlattice M of M^0, we can determine whether M satisfies the conditions (m1) and (m2) by the method described in [25]. Since G_{M^0} is finite, we obtain the set Ms. The group $O_{F,h}(M^0)$ is isomorphic to the automorphism group of the Dynkin diagram of type R, and hence it is finite. Therefore the image of the natural homomorphism

$$O_{F,h}(M^0) \;\hookrightarrow\; O(M^0) \;\to\; O(q_{M^0})$$

is easy to calculate, where $O(q_{M^0})$ is the automorphism group of the finite quadratic form (G_{M^0}, q_{M^0}) (see [23, Section 6.2]). Consequently we obtain the set

$$\overline{Ms} := O_{F,h}(M^0)\backslash Ms$$

of the orbits of the action of $O_{F,h}(M^0)$ on Ms. For an element M of Ms, let $[M] \in \overline{Ms}$ denote the orbit containing M. We also put

$$O_{F,h,M}(M^0) := \{\ g \in O_{F,h}(M^0)\ \mid\ g \text{ fixes } M \in Ms\ \}.$$

We have a natural map

$$\mathrm{pr}\ :\ Q(R)/\!\sim\ \ \to\ \ \overline{Ms}$$

that maps $[M, N, L, c\Omega]$ to $[M]$. We denote by $\overline{Ms}^{\sharp} \subset \overline{Ms}$ the image of the map $\mathrm{pr} : Q(R)/\!\sim\ \to \overline{Ms}$; that is, we put

$$\overline{Ms}^{\sharp} := \{\ [M] \in \overline{Ms}\ \mid\ Ns(M) \neq \emptyset\ \}.$$

For $[M] \in \overline{Ms}$, we can determine whether $Ns(M)$ is empty or not by the criterion of Nikulin [17, Theorem 1.10.1]. Hence $\overline{Ms}^{\sharp}$ is calculated.

Suppose that $[M] \in \overline{Ms}^{\sharp}$. By [17, Corollary 1.9.4], the set $Ns(M)$ forms a genus. If $r := \mathrm{rank}(R) < 19$, then the isomorphism class of an indefinite lattice N of signature $(2, 19-r)$ is determined by the spinor genus by Eichler's theorem (see, for example, Cassels [9]). The method of enumeration of spinor genera in a given genus is described in Conway and Sloane [10, Chapter 15]. When $\mathrm{rank}(R) = 19$, the set $Ns(M)$ is easily calculated by Corollary 3.9 below.

For each $[M] \in \overline{Ms}^{\sharp}$, we have a natural map

$$\mathrm{pr}_{[M]}\ :\ \mathrm{pr}^{-1}([M])\ \ \to\ \ Ns(M)$$

that maps $[M', N', L', c\Omega'] \in \mathrm{pr}^{-1}([M])$ to the lattice $N \in Ns(M)$ isomorphic to $N' \in Ns(M')$. (Note that, if $[M] = [M']$, then M and M' are isomorphic, and hence $Ns(M)$ and $Ns(M')$ are canonically identified.) Let N be an element of $Ns(M)$. We put

$$F([M], N) := \mathrm{pr}_{[M]}^{-1}(N).$$

We can regard $O_{F,h,M}(M^0)$ as a subgroup of $O(M)$:

$$O_{F,h,M}(M^0) = \{\ g \in O(M)\ \mid\ g(F) = F,\ g(h) = h\ \}.$$

Then the group $O_{F,h,M}(M^0) \times O(N)$ acts on the set $Ls(M, N) \times c\Omega s(N)$ in the natural way. The fiber $F([M], N)$ of $\mathrm{pr}_{[M]}$ over N is, by definition, equal to the set of orbits of this action:

$$F([M], N) = (O_{F,h,M}(M^0) \times O(N))\backslash(Ls(M, N) \times c\Omega s(N)).$$

By [17, Proposition 1.6.1], there exists a natural bijection between the set $Ls(M, N)$ and the set of isomorphisms of finite quadratic forms from $(G_M, -q_M)$ to (G_N, q_N). Since $G_M \cong G_N$ is a finite abelian group, we obtain the set $Ls(M, N)$. Hence the set $F([M], N)$ can be calculated, provided that the group $O(N)$ and its actions on (G_N, q_N) and on $c\Omega s(N)$ are calculated.

Remark 3.5. When $\mathrm{rank}(R) = 19$, the lattice N is positive-definite. Hence $O(N)$ is finite, and we can easily make the list of elements of $O(N)$. The actions of $O(N)$ on (G_N, q_N) and on $c\Omega s(N)$ are then readily calculated.

We use the following terminology in §4 and §5.

Definition 3.6. Let $\tau : c\Omega s(N) \xrightarrow{\sim} c\Omega s(N)$ be the transposition of the two connected components of Ω_N. An orbit $U \subset Ls(M, N) \times c\Omega s(N)$ of the action of $O_{F,h,M}(M^0) \times O(N)$ is called *real* if U is stable under τ.

We review the classical theory of binary forms due to Gauss (see Edwards [14] or Conway and Sloane [10, Chapter 15], for example). For integers a, b, c, we denote by $Q[a, b, c]$ the matrix

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

For a positive integer d, we put

$$\mathcal{Q}_d := \{\ Q[a,b,c] \ \mid\ a \equiv c \equiv 0 \,(\mathrm{mod}\, 2),\ \ a > 0,\ \ c > 0,\ \ ac - b^2 = d\ \},$$

on which $GL_2(\mathbb{Z})$ acts from right by $(Q, g) \mapsto {}^T g\, Q\, g$. The set of isomorphism classes of even positive-definite lattices of rank 2 with discriminant d is canonically identified with the set $\mathcal{Q}_d / GL_2(\mathbb{Z})$ of $GL_2(\mathbb{Z})$-orbits in $\mathcal{Q}_d$.

Definition 3.7. We call an $SL_2(\mathbb{Z})$-orbit in $\mathcal{Q}_d$ an isomorphism class of even positive-definite *oriented* lattices of rank 2 with discriminant d.

For $Q[a, b, c] \in \mathcal{Q}_d$, we denote by $\Lambda[a, b, c]$ (resp. $\widetilde{\Lambda}[a, b, c]$) the lattice (resp. the oriented lattice) expressed by $Q[a, b, c]$.

Proposition 3.8. *Let d be a positive integer. Then the set*

$$\{\ \widetilde{\Lambda}[a,b,c] \ \mid\ Q[a,b,c] \in \mathcal{Q}_d,\ \ -a < 2b \le a \le c \ \text{ with } \ b \ge 0 \ \text{ if } \ a = c\ \}$$

is a complete set of representatives of isomorphism classes of even positive-definite oriented lattices of rank 2 with discriminant d.

Corollary 3.9. *Let d be a positive integer. Then the set*

$$\{\ \Lambda[a,b,c] \ \mid\ Q[a,b,c] \in \mathcal{Q}_d,\ 0 \le 2b \le a \le c\ \} \tag{3.1}$$

is a complete set of representatives of isomorphism classes of even positive-definite lattices of rank 2 with discriminant d.

Remark 3.10. Let $\Lambda[a, b, c]$ be an element of the set (3.1), and let $[\Lambda[a, b, c]] \in \mathcal{Q}_d / GL_2(\mathbb{Z})$ be the $GL_2(\mathbb{Z})$-orbit containing $\Lambda[a, b, c]$. Then the fiber of the natural map $\mathcal{Q}_d / SL_2(\mathbb{Z}) \to \mathcal{Q}_d / GL_2(\mathbb{Z})$ over $[\Lambda[a, b, c]]$ consists of two elements if

$$0 < 2b < a < c,$$

while it consists of a single element otherwise.

4. Examples of arithmetic Zariski pairs

Let $f \in \mathbb{Q}[t]$ be an irreducible polynomial. We denote by F_f the field $\mathbb{Q}[t]/(f)$. Then there exists a natural bijection $\alpha \mapsto \sigma_\alpha$ from the set of complex roots of f to the set of embeddings $F_f \hookrightarrow \mathbb{C}$ given by $\sigma_\alpha(t) := \alpha$. For a homogeneous polynomial $\Phi(x_0, x_1, x_2)$ with coefficients in F_f, we write Φ^α instead of Φ^{σ_α}.

Suppose that C and C' are conjugate ADE-sextics. Then the configurations of C and C' are the same. (See Yang [34, §3] for the precise definition of the configuration of an ADE-sextic.) In particular, there exist tubular neighborhoods $T \subset \mathbb{P}^2$ of C and $T' \subset \mathbb{P}^2$ of C' such that (T, C) and (T', C') are diffeomorphic.

Combining this fact with Corollaries 2.7 and 3.4, we see that the following pairs of conjugate maximizing sextics discovered by Artal, Carmona and Cogolludo [4, Theorem 5.8] are in fact arithmetic Zariski pairs.

Example 4.1. Consider the Dynkin type $R = A_{16} + A_2 + A_1$. We put

$$f \;:=\; 17\,t^3 - 18\,t^2 - 228\,t + 536,$$

which has two non-real roots $\alpha, \bar{\alpha}$ and a real root β. In [4], it is shown that $\mathcal{M}(R)$ consists of three connected components $\mathcal{M}(R)_\alpha$, $\mathcal{M}(R)_{\bar{\alpha}}$ and $\mathcal{M}(R)_\beta$, and that there exists a homogeneous polynomial $\Phi(x_0, x_1, x_2)$ of degree 6 with coefficients in F_f such that the conjugate sextics

$$C_\alpha = \{\Phi^\alpha = 0\}, \quad C_{\bar{\alpha}} = \{\Phi^{\bar{\alpha}} = 0\}, \quad C_\beta = \{\Phi^\beta = 0\}$$

are members of $\mathcal{M}(R)_\alpha$, $\mathcal{M}(R)_{\bar{\alpha}}$ and $\mathcal{M}(R)_\beta$, respectively. On the other hand, by the method described in the previous section, we calculate that $\overline{Ms}^\sharp = \{[M^0]\}$ and $Ns(M^0) = \{N^1, N^2\}$, where

$$N^1 = \Lambda[10, 4, 22] \quad \text{and} \quad N^2 = \Lambda[6, 0, 34].$$

The set $F([M^0], N^1)$ consists of two non-real orbits, while the set $F([M^0], N^2)$ consists of a single real orbit. Since the complex conjugation induces a homeomorphism $(\mathbb{P}^2, C_\alpha) \cong (\mathbb{P}^2, C_{\bar{\alpha}})$, the invariants N_{C_α} and $N_{C_{\bar{\alpha}}}$ must be equal. Hence N_{C_α} and $N_{C_{\bar{\alpha}}}$ are isomorphic to N^1, while N_{C_β} is isomorphic to N^2. Since N^1 and N^2 are not isomorphic, we conclude that (C_α, C_β) is an arithmetic Zariski pair.

Example 4.2. Consider the Dynkin type $R = A_{16} + A_3$. In [4], it is shown that $\mathcal{M}(R)$ consists of two connected components $\mathcal{M}_+$ and $\mathcal{M}_-$, and that there exist members C_+ of $\mathcal{M}_+$ and C_- of $\mathcal{M}_-$ that are conjugate in $\mathbb{Q}(\sqrt{17})$. On the other hand, we calculate that $\overline{Ms}^\sharp = \{[M^0]\}$ and $Ns(M^0) = \{N^1, N^2\}$, where

$$N^1 = \Lambda[4, 0, 34] \quad \text{and} \quad N^2 = \Lambda[2, 0, 68].$$

Each of $F([M^0], N^1)$ and $F([M^0], N^2)$ consists of a single real orbit. Therefore (C_+, C_-) is an arithmetic Zariski pair.

Example 4.3. Suppose that $R = A_{18} + A_1$. We put

$$f \;:=\; 19\,t^3 + 50\,t^2 + 36\,t + 8,$$

which has two non-real roots $\alpha, \bar{\alpha}$ and a real root β. Again by [4], the moduli space $\mathcal{M}(R)$ consists of three connected components $\mathcal{M}(R)_\alpha$, $\mathcal{M}(R)_{\bar{\alpha}}$, $\mathcal{M}(R)_\beta$ that have members $C_\alpha = \{\Phi^\alpha = 0\}$, $C_{\bar{\alpha}} = \{\Phi^{\bar{\alpha}} = 0\}$, $C_\beta = \{\Phi^\beta = 0\}$, respectively, for some homogeneous polynomial Φ with coefficients in F_f. On the other hand, we have $\overline{Ms}^\sharp = \{[M^0]\}$ and $Ns(M^0) = \{N^1, N^2\}$, where

$$N^1 = \Lambda[8, 2, 10] \quad \text{and} \quad N^2 = \Lambda[2, 0, 38].$$

The set $F([M^0], N^1)$ consists of two non-real orbits, while the set $F([M^0], N^2)$ consists of a single real orbit. Hence (C_α, C_β) is an arithmetic Zariski pair.

For the cases $R = A_{15} + A_4$ and $R = A_{19}$ that are also treated in [4, Theorem 5.8], the situation is as follows.

Example 4.4. Suppose that $R = A_{15} + A_4$. We have $\overline{Ms}^\sharp = \{[M^0], [M^1]\}$, where M^1 is an overlattice of M^0 with index 2. We have $Ns([M^0]) = \{N^0\}$ with $N^0 = \Lambda[8, 4, 22]$, and $F([M^0], N^0)$ consists of two non-real orbits, while we have $Ns([M^1]) = \{N^1\}$ with $N^1 = \Lambda[2, 0, 20]$, and $F([M^1], N^1)$ consists of a single real orbit. According to Yang's list [34], there exist two configurations of maximizing sextics of type $A_{15} + A_4$. By [4], there exist members C and $\overline{C}$ of distinct connected components of $\mathcal{M}(R)$ that are conjugate in $\mathbb{Q}(\sqrt{-1})$. The complex conjugation yields a homeomorphism $(\mathbb{P}^2, C) \cong (\mathbb{P}^2, \overline{C})$. Hence we must have $N_C \cong N_{\overline{C}} \cong N^0$.

Example 4.5. Suppose that $R = A_{19}$. We have $\overline{Ms^\sharp} = \{[M^0]\}$ and $Ns([M^0]) = \{N^0\}$ with $N^0 = \Lambda[2, 0, 20]$. The set $F([M^0], N^0)$ consists of two real orbits. According to [4], there exist members C_+ and C_- of $\mathcal{M}(R)$ belonging to the distinct connected components that are conjugate in $\mathbb{Q}(\sqrt{5})$. Our invariant fails to distinguish $(\mathbb{P}^2, C_+)$ and $(\mathbb{P}^2, C_-)$ topologically, because we have $N_{C_+} \cong N_{C_-} \cong N^0$. It would be an interesting problem to determine whether $(\mathbb{P}^2, C_+)$ and $(\mathbb{P}^2, C_-)$ are homeomorphic or not.

5. A singular $K3$ surface defined over a number field

Let Y be a complex $K3$ surface or a complex abelian surface such that the transcendental lattice $T(Y)$ is of rank 2. Then $T(Y)$ is an even positive-definite lattice. Moreover the Hodge structure

$$T(Y) \otimes \mathbb{C} = H^{2,0}(Y) \oplus \overline{H^{2,0}(Y)}$$

of $T(Y)$ defines a canonical orientation on $T(Y)$; namely, an ordered basis e_1, e_2 of $T(Y)$ is positive if the imaginary part of the complex number $(e_1, \omega_Y)/(e_2, \omega_Y)$ is positive, where $\omega_Y \in H^{2,0}(Y)$ is the cohomology class of a non-zero holomorphic 2-form of Y. We denote by $\widetilde{T}(Y)$ the *oriented transcendental lattice* of Y.

Definition 5.1. A (smooth) $K3$ surface X defined over a field k of characteristic 0 is called *singular* if the Picard number of $X \otimes \bar{k}$ is 20.

If X is a *complex* singular $K3$ surface, then we have the oriented transcendental lattice $\widetilde{T}(X)$. We have the following important theorem due to Shioda and Inose [28]:

Theorem 5.2. *The correspondence $X \mapsto \widetilde{T}(X)$ yields a bijection from the set of isomorphism classes of complex singular $K3$ surfaces to the set of isomorphism classes of even positive-definite oriented lattices of rank 2.*

Notice that, if C is a complex maximizing sextic, then the minimal resolution X_C of the double covering $Y_C \to \mathbb{P}^2$ that branches exactly along C is a complex singular $K3$ surface, and $T(X_C)$ is isomorphic to N_C.

Let X be a singular $K3$ surface defined over a number field F. For an embedding σ of F into $\mathbb{C}$, we denote by X^σ the complex $K3$ surface obtained from X by σ.

The following is a special case of [27, Theorem 3].

Proposition 5.3. *There exist a singular $K3$ surface X defined over a number field F and two embeddings τ and τ' of F into $\mathbb{C}$ such that*

$$\widetilde{T}(X^\tau) \cong \widetilde{\Lambda}[2, 1, 28] \quad \textit{and} \quad \widetilde{T}(X^{\tau'}) \cong \widetilde{\Lambda}[8, 3, 8].$$

Proof. We put

$$K := \mathbb{Q}(\sqrt{-55}) \subset \mathbb{C},$$

and denote by $\mathbb{Z}_K$ the ring of integers of K. For a number field L containing K, we denote by $\mathrm{Emb}(L/K)$ the set of embeddings of L into $\mathbb{C}$ whose restrictions to K

are the identity of K. We define fractional ideals $I_0, \ldots, I_3$ of $\mathbb{Z}_K$ by the following:

$$\begin{array}{lllll} I_0 & := & \mathbb{Z}_K = \mathbb{Z}+\mathbb{Z}\tau_0 & \text{where} & \tau_0 := (1+\sqrt{-55})/2, \\ I_1 & := & \mathbb{Z}+\mathbb{Z}\tau_1 & \text{where} & \tau_1 := (3+\sqrt{-55})/4, \\ I_2 & := & \mathbb{Z}+\mathbb{Z}\tau_2 & \text{where} & \tau_2 := (5+\sqrt{-55})/8, \\ I_3 & := & \mathbb{Z}+\mathbb{Z}\tau_3 & \text{where} & \tau_3 := (1+\sqrt{-55})/4. \end{array}$$

The ideal class group Cl_K of $\mathbb{Z}_K$ is a cyclic group of order 4 generated by the class $[I_1]$, and we have $[I_2] = [I_1]^2$ and $[I_3] = [I_1]^3$. We consider the Hilbert class polynomial

$$\begin{aligned} \mathcal{H}(t) &:= (t-j(\tau_0))(t-j(\tau_1))(t-j(\tau_2))(t-j(\tau_3)) \\ &= t^4 + 13136684625\,t^3 - 20948398473375\,t^2 + \\ &\qquad +172576736359017890625\,t - 18577989025032784359375 \end{aligned}$$

of $\mathbb{Z}_K$, and the Hilbert class field

$$H := K[t]/(\mathcal{H}(t))$$

of K (see Cox [11]). We put

$$\gamma := t \bmod (\mathcal{H}) \in H.$$

Then we have $\mathrm{Emb}(H/K) = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$, where σ_i is the embedding defined by $\sigma_i(\gamma) = j(\tau_i) \in \mathbb{C}$. Consider the elliptic curve

$$E \;:\; y^2 + xy = x^3 - \frac{36}{\gamma - 1728}x - \frac{1}{\gamma-1728} \tag{5.1}$$

defined over H (see Silverman [31, page 52]). Then we have $j(E) = \gamma \in H$, and hence $j(E^{\sigma_i}) = j(\tau_i)$ holds for $i = 0, \ldots, 3$, where E^{σ_i} is the complex elliptic curve defined by (5.1) with γ replaced by $\sigma_i(\gamma) = j(\tau_i)$. Therefore we have an isomorphism of Riemann surfaces

$$E^{\sigma_i} \;\cong\; \mathbb{C}/I_i \tag{5.2}$$

for $i = 0, \ldots, 3$. We then put

$$A := E \times E.$$

Note that $T(A)$ is of rank 2. By means of a double covering of the Kummer surface $\mathrm{Km}(A)$ of A, Shioda and Inose [28] constructed a singular $K3$ surface X defined over a finite extension F of H with the following properties (see also [27, Propositions 6.1 and 6.4]).

> For any $\sigma \in \mathrm{Emb}(F/K)$, the oriented trancendental lattice $\widetilde{T}(X^\sigma)$ is isomorphic to the oriented transcendental lattice $\widetilde{T}(A^\sigma)$ of the complex abelian surface $A^\sigma = E^\sigma \times E^\sigma$.

See Inose [15] and Shioda [29] for an explicit defining equation of X.

The oriented lattice $\widetilde{T}(A^\sigma)$ is calculated by Shioda and Mitani [30]. Suppose that the restriction of $\sigma \in \mathrm{Emb}(F/K)$ to H is σ_i. Then we have

$$A^\sigma \;\cong\; \mathbb{C}/I_i \times \mathbb{C}/I_i \;\cong\; \mathbb{C}/I_i^2 \times \mathbb{C}/I_0 \;\cong \begin{cases} \mathbb{C}/I_0 \times \mathbb{C}/I_0 & \text{if } i = 0 \text{ or } i = 2, \\ \mathbb{C}/I_2 \times \mathbb{C}/I_0 & \text{if } i = 1 \text{ or } i = 3, \end{cases}$$

by (5.2) and [30, (4.14)]. Hence, by [30, Section 3], we have

$$\widetilde{T}(A^\sigma) \cong \begin{cases} \widetilde{\Lambda}[2,1,28] & \text{if } \sigma|H \text{ is } \sigma_0 \text{ or } \sigma_2, \\ \widetilde{\Lambda}[8,3,8] & \text{if } \sigma|H \text{ is } \sigma_1 \text{ or } \sigma_3, \end{cases}$$

(see also [27, §6.3].) Thus we obtain the hoped-for X and τ, τ'. □

Remark 5.4. Note that the orientation reversing does not change the isomorphism classes of the oriented lattices $\widetilde{\Lambda}[2,1,28]$ and $\widetilde{\Lambda}[8,3,8]$ (see Remark 3.10). Hence, by Theorem 5.2, if a complex singular $K3$ surface Y satisfies $T(Y) \cong \Lambda[2,1,28]$ (resp. $T(Y) \cong \Lambda[8,3,8]$), then Y is isomorphic to the complex $K3$ surface X^τ (resp. to the complex $K3$ surface $X^{\tau'}$) in Proposition 5.3

Using Proposition 5.3 and Remark 5.4, we obtain the following example of arithmetic Zariski pairs.

Example 5.5. Consider the Dynkin type $R = A_{10} + A_9$. We have

$$\overline{Ms^\sharp} = \{[M^0], [M^1]\},$$

where M^1 is an overlattice of M^0 with index 2. We then have

$$\begin{array}{rcl} Ns([M^0]) & = & \{\ \Lambda[10,0,22],\ \ \Lambda[2,0,110]\ \} \quad \text{and} \\ Ns([M^1]) & = & \{\ \Lambda[2,1,28],\ \ \Lambda[8,3,8]\ \}, \end{array}$$

and each of the sets

$$\begin{array}{ll} F([M^0], \Lambda[10,0,22]), & F([M^0], \Lambda[2,0,110]), \\ F([M^1], \Lambda[2,1,28]), & F([M^1], \Lambda[8,3,8]) \end{array}$$

consists of a single real orbit. In particular, the number of the connected components of $\mathcal{M}(R)$ is four. Let C and C' be members of the connected components of $\mathcal{M}(R)$ corresponding to $F([M^1], \Lambda[2,1,28])$ and $F([M^1], \Lambda[8,3,8])$, respectively. Note that we have

$$N_C \cong T(X_C) \cong \Lambda[2,1,28] \quad \text{and} \quad N_{C'} \cong T(X_{C'}) \cong \Lambda[8,3,8].$$

By Remark 5.4, we see that X_C is isomorphic to X^τ and $X_{C'}$ is isomorphic to $X^{\tau'}$. Consider the composites

$$\phi_C \,:\, X_C \longrightarrow Y_C \xrightarrow{\pi_C} \mathbb{P}^2 \quad \text{and} \quad \phi_{C'} \,:\, X_{C'} \longrightarrow Y_{C'} \xrightarrow{\pi_{C'}} \mathbb{P}^2$$

of the finite double coverings branching along C and C' and the minimal desingularizations. Since $X_C \cong X^\tau$, there exists a morphism $\phi_L : X \otimes L \to \mathbb{P}^2$ with the Stein factorization

$$\phi_L \,:\, X \otimes L \longrightarrow Y_L \xrightarrow{\pi_L} \mathbb{P}^2$$

defined over a finite extension L of F such that, for some embedding θ of L into $\mathbb{C}$ satisfying $\theta|F = \tau$, the morphism

$$\phi_L^\theta \,:\, (X \otimes L)^\theta = X^\tau \longrightarrow Y_L^\theta \xrightarrow{\pi_L^\theta} \mathbb{P}^2$$

is isomorphic to ϕ_C. In particular, the branch curve B of the finite double covering π_L^θ is isomorphic to C as a complex plane curve. Let θ' be an embedding of L into $\mathbb{C}$ such that $\theta'|F = \tau'$, and consider the morphism

$$\phi_L^{\theta'} \,:\, (X \otimes L)^{\theta'} = X^{\tau'} \longrightarrow Y_L^{\theta'} \xrightarrow{\pi_L^{\theta'}} \mathbb{P}^2.$$

Since the branch curve B' of $\pi_L^{\theta'}$ is conjugate to the branch curve B of π_L^θ, it is a maximizing sextic of type $A_{10} + A_9$. Since $(X \otimes L)^{\theta'} = X^{\tau'}$ is isomorphic to

$X_{C'}$, the morphism $\phi^{\theta'}$ must be isomorphic to $\phi_{C'}$, and hence B' is isomorphic to C' as a complex plane curve. Therefore the conjugate pair (B, B') of plane curves is isomorphic to the pair (C, C') with $N_C \not\cong N_{C'}$, and thus yields an example of arithmetic Zariski pairs.

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