SINGULARITIES OF DUAL VARIETIES IN CHARACTERISTIC 2

ICHIRO SHIMADA

ABSTRACT. We investigate unibranched singularities of dual varieties of even-dimensional smooth projective varieties in characteristic 2.

1. INTRODUCTION

We work over an algebraically closed field k.

Let $X \subset \mathbb{P}^m$ be a smooth projective variety of dimension $n > 0$. We assume that X is not contained in any hyperplane of $\mathbb{P}^m$. Then we can consider the dual projective space
$$\mathbf{P} := (\mathbb{P}^m)^\vee$$
of $\mathbb{P}^m$ as the parameter space of the linear system $|M|$ of hyperplane sections of X, where M is a linear subspace of $H^0(X, \mathcal{O}_X(1))$. We use the same letter to denote a point $H \in \mathbf{P}$ and the corresponding hyperplane $H \subset \mathbb{P}^m$. Let $\mathcal{D} \subset X \times \mathbf{P}$ be the universal family of the hyperplane sections. Then $\mathcal{D}$ is a smooth scheme of dimension $n + m - 1$. The support of $\mathcal{D}$ is equal to the set
$$\{ (p, H) \in X \times \mathbf{P} \mid p \in H \}.$$
Let $\mathcal{C} \subset \mathcal{D}$ be the critical subscheme of the second projection $\mathcal{D} \to \mathbf{P}$. (See Notation and Terminology below for the definition of critical subschemes.) Then $\mathcal{C}$ is smooth, irreducible and of dimension $m - 1$. If $\mathcal{N}$ is the conormal sheaf of $X \subset \mathbb{P}^m$, then $\mathcal{C}$ is isomorphic to $\mathbb{P}^*(\mathcal{N})$ ([12, Remarque 3.1.5]). The support of $\mathcal{C}$ is equal to the set
$$\{ (p, H) \in \mathcal{D} \mid \text{the divisor } H \cap X \text{ of } X \text{ is singular at } p \}.$$
We denote the projections by
$$\pi_1 : \mathcal{C} \to X \quad \text{and} \quad \pi_2 : \mathcal{C} \to \mathbf{P}.$$
The image of π_2 is called the *dual variety* of $X \subset \mathbb{P}^m$. Let $\mathcal{E} \subset \mathcal{C}$ be the critical subscheme of π_2. By [12, Proposition 3.3], $\mathcal{E}$ is set-theoretically equal to
$$\{ (p, H) \in \mathcal{C} \mid \text{the Hessian of the singularity of } H \cap X \text{ at } p \text{ is degenerate} \}.$$
We will study the singularities of the dual variety by investigating the morphism π_2 at points of $\mathcal{E}$.

Since the paper of Wallace [22], properties of dual varieties peculiar to positive characteristics have been studied mainly from the point of view of the (failure of the) reflexivity. See Kleiman's paper [14] for the definition and a detailed account of the reflexivity. Many studies have been done for the analysis of the situation in which the reflexivity does not hold. See [5, 6, 7, 8, 9, 10, 11, 16], for example.

A well-known example of the situations in which the reflexivity fails is as follows. Suppose that

(1.1) $\operatorname{char} k = 2$ and $\dim X \equiv 1 \bmod 2$.

Then the critical subscheme $\mathcal{E}$ of $\pi_2 : \mathcal{C} \to \mathbf{P}$ coincides with $\mathcal{C}$ ([12, Section 1.2], [14, Corollary (18)]), and hence either the dual variety is not a hypersurface in $\mathbf{P}$, or X is not reflexive by the generalized Monge-Segre-Wallace criterion ([13, Theorem (4.4)], [14, Theorem (4)]).

Except for the case (1.1), however, the reflexivity is recovered when the linear system $|M|$ of the hyperplane sections are sufficiently ample. We have the following theorem ([12, Théorème 2.5], [7, Theorem (5.4)]):

Theorem 1.1. *Suppose that* $\operatorname{char} k \neq 2$ *or* $\dim X$ *is even. If X is embedded in* $\mathbb{P}^m$ *by a complete linear system of the form* $|\mathcal{A}^{\otimes d}|$ *with $\mathcal{A}$ being a very ample line bundle and $d \geq 2$, then the dual variety of $X \subset \mathbb{P}^m$ is a hypersurface of $\mathbf{P}$, and $X \subset \mathbb{P}^m$ is reflexive.*

See also [19, Proposition 4.9] or Proposition 6.12 of this paper for other sufficient conditions for the dual variety to be a hypersurface, and for the reflexivity to hold.

In our previous paper [19], we have discovered that, even when the reflexivity holds, the singularities of dual varieties in characteristic 3 still possess a peculiar feature. We assume that the linear system $|M|$ is sufficiently ample. In particular, the dual variety is a hypersurface in $\mathbf{P}$ and X is reflexive. We consider the projective plane curve obtained by cutting the dual variety by a general plane in $\mathbf{P}$. If k is of characteristic > 3 or 0, then the plane curve has only ordinary cusps as its unibranched singularities. In characteristic 3, the plane curve has singular points of E_6-type instead of ordinary cusps.

In this paper, we investigate the singularities of the dual variety in the case where $\operatorname{char} k = 2$ and $\dim X$ is even.

The dual variety is the *discriminant variety* associated with the linear system of hyperplane sections. See [2] for the definition of discriminant varieties. In fact, our results are proved not only for dual varieties but for discriminant varieties associated with (not necessarily very ample) linear systems. Here in Introduction, however, we present our results for dual varieties.

Let k be of characteristic 2, and let $\dim X$ be even. For simplicity, we assume that $|M|$ is sufficiently ample so that the evaluation homomorphism

$$v_p^{[4]} \,:\, M \,\to\, \mathcal{L}_p/\mathfrak{m}_p^5 \mathcal{L}_p$$

is surjective at every point p of X, where $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{O}_{X,p}$, and $\mathcal{L}_p$ is the $\mathcal{O}_{X,p}$-module $\mathcal{L} \otimes \mathcal{O}_{X,p}$. Under this assumption, the critical subscheme $\mathcal{E}$ is an irreducible divisor of $\mathcal{C}$ and hence the dual variety is a hypersurface of $\mathbf{P}$.

We can define the *universal Hessian*

$$\mathcal{H} \,:\, \pi_1^* T(X) \otimes \pi_1^* T(X) \,\to\, \pi_1^* \mathcal{L} \otimes \pi_2^* \mathcal{O}_{\mathbf{P}}(1)$$

on $\mathcal{C}$, where $T(X)$ is the tangent bundle of X. As was proved in [19, Proposition 3.14], the subscheme $\mathcal{E}$ coincides with the degeneracy subscheme of the homomorphism

$$\pi_1^* T(X) \,\to\, \pi_1^* \mathcal{L} \otimes \pi_2^* \mathcal{O}_{\mathbf{P}}(1) \otimes (\pi_1^* T(X))^{\vee}$$

induced from $\mathcal{H}$, where $(\pi_1^* T(X))^\vee$ is the dual vector bundle of $\pi_1^* T(X)$. (See Notation and Terminology below for the definition of degeneracy subschemes.) The peculiarity of the geometry of the dual variety in characteristic 2 stems from the fact that the universal Hessian is not only symmetric but also *anti*-symmetric; that is, $\mathcal{H}(x \otimes x) = 0$ and $\mathcal{H}(x \otimes y) + \mathcal{H}(y \otimes x) = 0$ hold for any local sections x and y of $\pi_1^* T(X)$. From this fact, it follows that the irreducible divisor $\mathcal{E}$ is written as $2\mathcal{R}$, where $\mathcal{R}$ is a reduced divisor of $\mathcal{C}$. We denote by $\varpi_2 : \mathcal{R} \to \mathbf{P}$ the projection.

Definition 1.2. We put $R := k[[t_1, \ldots, t_{m-4}]]$. It turns out that the equations

$$(1.2) \quad u_1 y^2 + v_1 \;=\; u_2 y^2 + v_2 \;=\; w v_1 + u_1 v_2 y \;=\; w v_2 + u_2 v_1 y \;=$$
$$=\; v_1 v_2 + w y \;=\; w^2 + u_1 u_2 y^2 \;=\; 0$$

define an $(m-1)$-dimensional singular scheme in the $(m+2)$-dimensional smooth scheme $\operatorname{Spec} R[[u_1, v_1, u_2, v_2, w, y]]$, which we will denote by Γ_{m-1}.

The scheme Γ_{m-1} is singular along the $(m-2)$-dimensional locus defined by

$$v_1 = v_2 = w = y = 0.$$

See §4 for the geometric meaning of Γ_{m-1}.

Let $P = (p, H) \in X \times \mathbf{P}$ be a general point of the reduced irreducible divisor $\mathcal{R}$ of $\mathcal{C}$. Then we have the following:

(I) There exist isomorphisms of local rings

$$\mathcal{O}_{\pi_2^{-1}(H), P} \cong k[[s, t]]/(s^2, t^2) \qquad \text{and} \qquad \mathcal{O}_{\varpi_2^{-1}(H), P} \cong k[[s, t]]/(s^2, t^2)$$

over k. In particular, the projection $\varpi_2 : \mathcal{R} \to \mathbf{P}$ is inseparable of degree 4 over its image.

(II) The formal completion

$$(\pi_2)_P^\wedge : \operatorname{Spec}(\mathcal{O}_{\mathcal{C}, P})^\wedge \to \operatorname{Spec}(\mathcal{O}_{\mathbf{P}, H})^\wedge$$

of the projection $\pi_2 : \mathcal{C} \to \mathbf{P}$ at P factors through a singular scheme isomorphic to Γ_{m-1}.

(III) Let Λ and L be general linear subspaces of $\mathbf{P}$ of dimension 2 and 3, respectively, such that $H \in \Lambda \subset L$. We put $C_\Lambda := \pi_2^{-1}(\Lambda)$ and $S_L := \pi_2^{-1}(L)$. Then S_L is smooth of dimension 2 at P, and C_Λ is a curve on S_L that has an ordinary cusp at P.

(IV) Let $\nu : \widetilde{C}_\Lambda \to C_\Lambda$ be the normalization of C_Λ at P, and let z be a formal parameter of $\widetilde{C}_\Lambda$ at the inverse image $P' \in \widetilde{C}_\Lambda$ of P. Then the formal completion at P' of the composite of $\nu : \widetilde{C}_\Lambda \to C_\Lambda$ and $\pi_2 \,|\, C_\Lambda : C_\Lambda \to \Lambda$ is written as

$$((\pi_2 \,|\, C_\Lambda) \circ \nu)^* x \;=\; a\, z^4 \;+\; (\text{terms of degree} \geq 6) \quad \text{and}$$
$$((\pi_2 \,|\, C_\Lambda) \circ \nu)^* y \;=\; b\, z^4 \;+\; (\text{terms of degree} \geq 6)$$

for some $a, b \in k$, where (x, y) is a formal parameter system of Λ at H.

This paper is organized as follows. First we define several notions in §2. The conditions on $P = (p, H) \in \mathcal{R}$ for the facts (I)-(IV) above to hold will be stated more precisely in terms of the singularity of the divisor $H \cap X$ of X at p. For this purpose, we define some classes of hypersurface singularities in §3. In §4, the $(m-1)$-dimensional singular scheme Γ_{m-1} is introduced. After recalling the

definitions and results given in [19, Section 3] in §5, we prove the main results above in more refined forms in §6. In §7, we give a remark about the degree of $\mathcal{R}$ with respect to $\mathcal{O}_{\mathbf{P}}(1)$, and derive a nontrivial divisibility relation among Chern numbers of X from the fact (I) above.

The author would like to thank Professor Hajime Kaji for many valuable comments and suggestions.

Notation and Terminology.

(1) We work over an algebraically closed field k. By a *variety*, we mean a reduced irreducible quasi-projective scheme over k. A *point* of a variety means a closed point. Let X be a variety, and $P \in X$ a point. We denote by $(X, P)^{\wedge}$ the scheme $\mathrm{Spec}(\mathcal{O}_{X,P})^{\wedge}$, where $(\mathcal{O}_{X,P})^{\wedge}$ is the formal completion of the local ring $\mathcal{O}_{X,P}$ of X at P.

(2) Let X be a smooth variety. We denote by $T_P(X)$ the Zariski tangent space to X at a point $P \in X$, and by $T(X)$ the tangent bundle of X.

(3) Let E and F be vector bundles on a variety X with rank e and f, respectively, and let $\sigma : E \to F$ be a bundle homomorphism. We put $r := \min(e, f)$. The *degeneracy subscheme* of σ is the closed subscheme of X defined locally on X by all the r-minors of the $f \times e$-matrix expressing σ.

(4) Let $f : X \to Y$ be a morphism from a smooth variety X to a smooth variety Y. The *critical subscheme* of f is the degeneracy subscheme of the homomorphism $df : T(X) \to f^* T(Y)$.

(5) For a formal power series F with coefficients in k, we denote by $F^{[d]}$ the homogeneous part of degree d in F.

2. Preliminaries

2.1. **The quotient morphism by an integrable tangent subbundle.** Let X be a smooth variety defined over an algebraically closed field of characteristic $p > 0$.

Definition 2.1. A subbundle $\mathcal{N}$ of $T(X)$ is called *integrable* if $\mathcal{N}$ is closed under the p-th power operation and the bracket product of derivations.

Proposition 2.2 ([18] Théorème 2). *Let $\mathcal{N}$ be an integrable subbundle of $T(X)$. Then there exists a unique morphism $q : X \to X^{\mathcal{N}}$ with the following properties;*

(i) *q induces a homeomorphism on the underlying topological spaces,*
(ii) *q is a radical covering of height 1, and*
(iii) *the kernel of $dq : T(X) \to q^* T(X^{\mathcal{N}})$ coincides with $\mathcal{N}$.*

Moreover, the variety $X^{\mathcal{N}}$ is smooth, and the morphism q is finite of degree p^r, where r is the rank of $\mathcal{N}$.

Definition 2.3. For an integrable subbundle $\mathcal{N}$ of $T(X)$, the morphism $q : X \to X^{\mathcal{N}}$ is called the *quotient morphism* by $\mathcal{N}$.

From the construction of the quotient morphism given in the proof of [18, Théorème 2], we obtain the following:

Proposition 2.4. *Let $f : X \to Y$ be a morphism from a smooth variety X to a smooth variety Y such that the kernel $\mathcal{K}$ of $df : T(X) \to f^* T(Y)$ is a subbundle of $T(X)$. Then $\mathcal{K}$ is integrable, and the morphism f factors through the quotient morphism $q : X \to X^{\mathcal{K}}$ by $\mathcal{K}$.*

2.2. **Anti-symmetric forms.**

Definition 2.5. Let V be a finite dimensional vector space over a field K. A bilinear form $\varphi : V \times V \to K$ is called *anti-symmetric* if the following conditions are satisfied:

(i) $\varphi(v, v) = 0$ for any $v \in V$, and

(ii) $\varphi(v, w) + \varphi(w, v) = 0$ for any $v, w \in V$.

Note that, in characteristic 2, the condition (ii) does not imply the condition (i).

Definition 2.6. An $n \times n$-matrix $A = (a_{ij})$ with components in a commutative ring is called *anti-symmetric* if the following conditions are satisfied:

(i) $a_{ii} = 0$ for $i = 1, \ldots, n$, and

(ii) $a_{ij} + a_{ji} = 0$ for $i, j = 1, \ldots, n$.

An anti-symmetric matrix in characteristic 2 is just a symmetric matrix with zero diagonal components.

The following results are well-known. We put

$$J_2 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and let $J_{2r,n}$ be the $n \times n$ matrix

$$J_{2r,n} := \begin{bmatrix} J_2 & & & 0 \\ & \ddots & & \\ & & J_2 & \\ 0 & & & 0 \end{bmatrix}$$

with r copies of J_2 along the diagonal.

Lemma 2.7. *Let A be an anti-symmetric matrix of type $n \times n$ with components in a field K. Then the rank of A is even, and there exists $T \in GL(n, K)$ such that ${}^t T A T$ is equal to $J_{2r,n}$, where $2r$ is the rank of A.*

For a positive integer m, we define a homogeneous polynomial f_{2m} of degree m in variables x_{ij} $(1 \le i < j \le 2m)$ by

$$f_{2m} := \sum \operatorname{sign} \begin{pmatrix} 1 & 2 & \ldots & & 2m \\ i_1 & j_1 & \ldots & i_m & j_m \end{pmatrix} x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_m j_m},$$

where the summation ranges over all the lists $[[i_1, j_1], \ldots, [i_m, j_m]]$ satisfying the conditions $i_\nu < j_\nu$ $(\nu = 1, \ldots, m)$, $i_1 < i_2 < \cdots < i_m$, and

$$\{i_1, j_1, \ldots, i_m, j_m\} = \{1, 2, \ldots, 2m\}.$$

Lemma 2.8. *Let*

(2.1)
$$A = \begin{bmatrix} 0 & x_{12} & x_{13} & \ldots \\ -x_{12} & 0 & x_{23} & \ldots \\ -x_{13} & -x_{23} & 0 & \ldots \\ & \ldots & & \ddots \end{bmatrix}$$

be an anti-symmetric matrix of type $2m \times 2m$ with components being variables x_{ij} $(i < j)$. Then $\det A$ is equal to f_{2m}^2.

For a positive integer i, we put

$$\tau(i) := \begin{cases} i-1 & \text{if } i \text{ is even,} \\ i+1 & \text{if } i \text{ is odd.} \end{cases}$$

Corollary 2.9. *Let A be an anti-symmetric matrix as in Lemma 2.8. We put*

$$g_A := \sqrt{\det(J_{2r,2m} + A)}\,,$$

which is a polynomial of x_{ij} by Lemma 2.8.
  (1) *If $2r \leq 2m - 4$, then g_A has no terms of degree ≤ 1.*
  (2) *Suppose that $2r = 2m - 2$. Then g_A is equal, up to sign, to*

$$z \;+\; \sum_{j=1}^{m-1} \varepsilon_j\, x_{2j-1,2j}\, z \;+\; \sum_{i=1}^{2m-2} \varepsilon'_i\, x_{i,2m-1}\, x_{\tau(i),2m} \;+\; (\text{terms of degree} \geq 3),$$

where $z := x_{2m-1,2m}$, and ε_j and ε'_i are ± 1.

Using (2.1), we consider the affine space

$$\mathbb{A}^{m(2m-1)} = \operatorname{Spec} k[\ldots, x_{ij}, \ldots] \quad (1 \leq i < j \leq 2m)$$

as the space of anti-symmetric matrices of type $2m \times 2m$.

Lemma 2.10. *The hypersurface in $\mathbb{A}^{m(2m-1)}$ defined by $f_{2m} = 0$ is irreducible.*

Proof. By Lemma 2.7, the hypersurface defined by $f_{2m} = 0$ is the closure of the locus

$$\{\, {}^t T J_{2m-2,2m} T \ \mid\ T \in GL(2m, k) \,\},$$

and hence is irreducible. □

2.3. **The formula of Harris-Tu-Pragacz.** Let X be a smooth variety, E a vector bundle on X, and L a line bundle on X.

Definition 2.11. A bundle homomorphism

$$\sigma : E \otimes_{\mathcal{O}_X} E \to L$$

is called *anti-symmetric* if the following conditions are satisfied:
  (i) $\sigma(x \otimes x) = 0$ for any local section x of E, and
  (ii) $\sigma(x \otimes y) + \sigma(y \otimes x) = 0$ for any local sections x and y of E.

Definition 2.12. For an anti-symmetric homomorphism $\sigma : E \otimes E \to L$, we define the *degeneracy subscheme* $\mathbf{R}(\sigma)$ as follows. Let U be an arbitrary Zariski open subset of X over which E and L are trivialized. Let A_U be the anti-symmetric matrix with components in $\Gamma(U, \mathcal{O}_X)$ expressing σ over U. By Lemma 2.8, there exists $f_U \in \Gamma(U, \mathcal{O}_X)$ such that $\det A_U = f_U^2$. Then the collection of the closed subschemes $\{f_U = 0\}$ of Zariski open subsets U patches together to form a closed subscheme $\mathbf{R}(\sigma)$ of X, which is the degeneracy subscheme of σ.

Definition 2.13. For a closed subscheme W of X, we denote by $[W]$ the class of W in the Chow group of X.

The following was proved by Harris-Tu [4] in characteristic 0 and by Pragacz [17] in any characteristics.

Theorem 2.14. *Suppose that the rank e of E is even, and that $\mathbf{R}(\sigma)$ is of codimension 1 in X. Then we have*

$$[\mathbf{R}(\sigma)] \;=\; \left(\frac{e}{2}\, c_1(L) \,-\, c_1(E) \right) \,\cap\, [X]$$

in the Chow group of X.

Remark 2.15. Suppose that $\mathbf{R}(\sigma)$ is of codimension 1 in X. Let

$$\sigma^{\tilde{}} : E \,\rightarrow\, L \otimes E^{\vee}$$

be the homomorphism induced from σ, where $E^{\vee}$ is the dual vector bundle of E. Then the degeneracy subscheme of $\sigma^{\tilde{}}$ is also a divisor of X, and is equal to $2\,\mathbf{R}(\sigma)$ as a divisor of X.

2.4. **Ordinary cusps.** Let P be a point of a smooth surface S, and let (s,t) be a formal parameter system of S at P. For $\phi \in (\mathcal{O}_{S,P})^{\wedge} = k[[s,t]]$, we denote by $\phi^{[d]}$ the homogeneous part of degree d in (s,t). Let C be a divisor of S that contains P and is singular at P.

Definition 2.16. We say that $\phi \in k[[s,t]]$ is *degenerate* if $\phi^{[0]} = \phi^{[1]} = 0$ and $\phi^{[2]} = l(s,t)^2$ hold for some linear form $l(s,t)$ of (s,t). It is obvious that this definition of degeneracy does not depend on the choice of the formal parameter system (s,t). We say that C has a *degenerate singularity* at P if a (and hence any) formal power series defining C at P is degenerate.

The equivalence of the following conditions on the singularity of C at P, which is well-known in characteristic 0, holds also in characteristic 2.

   (i) For an arbitrary formal parameter system (s,t) of S at P, any formal power series ϕ defining C at P is degenerate and the linear form $\sqrt{\phi^{[2]}}$ is not zero and does not divide $\phi^{[3]}$.

  (ii) There exist a formal parameter system (s,t) of S at P and a defining formal power series ϕ of C at P such that ϕ is degenerate and that the linear form $\sqrt{\phi^{[2]}}$ is not zero and does not divide $\phi^{[3]}$.

 (iii) There exists a formal parameter system (s,t) of S at P such that C is defined by $s^2 + t^3 = 0$ locally at P.

Definition 2.17. We say that C has an *ordinary cusp* at P if the conditions (i)-(iii) above are satisfied.

3. Hypersurface singularities in characteristic 2

From now on until the end of the paper, we assume that the base field k is of characteristic 2.

Let X be a smooth variety of dimension n. Let p be a point of X, and let D be a hypersurface of X that passes through p and is singular at p. Let $(x_1, \ldots, x_n)$ be a formal parameter system of X at p. Suppose that D is defined by $\phi = 0$ locally at p, where ϕ is a formal power series of $(x_1, \ldots, x_n)$. The $n \times n$ matrix

$$H_{\phi,p} := \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}(p) \right)$$

defining the Hessian

$$T_p(X) \times T_p(X) \;\rightarrow\; k$$

of D at p is *anti*-symmetric because we are in characteristic 2.

Definition 3.1. A formal parameter system $(x_1, \ldots, x_n)$ of X at p is said to be *admissible with respect to ϕ* if the matrix $H_{\phi,p}$ is of the form $J_{2r,n}$, where $2r$ is the rank of the Hessian of D at p.

Remark 3.2. By Lemma 2.7, a formal parameter system admissible with respect to ϕ is obtained from an arbitrarily given formal parameter system by a *linear* transformation of parameters.

From now on to the end of this section, we assume that n is even.

We put

$$\rho := \ f_n\left(\ldots, \ \frac{\partial^2\phi}{\partial x_i \partial x_j}, \ \ldots\right) \ = \ \sqrt{\det\left(\frac{\partial^2\phi}{\partial x_i \partial x_j}\right)} \ ,$$

where f_n is the polynomial defined in §2.2.

Definition 3.3. Let $C(D,p)$ be the subscheme of $(X,p)^\wedge = \mathrm{Spec}(\mathcal{O}_{X,p})^\wedge$ defined by

$$\phi = \frac{\partial\phi}{\partial x_1} = \cdots = \frac{\partial\phi}{\partial x_n} = 0,$$

and let $R(D,p)$ be the subscheme of $(X,p)^\wedge$ defined by

$$\phi = \frac{\partial\phi}{\partial x_1} = \cdots = \frac{\partial\phi}{\partial x_n} = \rho = 0.$$

It is obvious that each of $C(D,p)$ and $R(D,p)$ depends only on D and p, and not on the choice of ϕ and $(x_1, \ldots, x_n)$.

Using the admissible formal parameter system with respect to ϕ, or the subschemes $C(D,p)$ and $R(D,p)$ of $(X,p)^\wedge$, we can define various classes of hypersurface singularities in characteristic 2.

Suppose that the rank of the Hessian matrix $H_{\phi,p}$ is $n-2$. Let

$$(x, \xi) = (x_1, \ldots, x_{n-2}, \xi_1, \xi_2)$$

be an admissible formal parameter system with respect to ϕ. We put

$$G := \{ \ M \in GL(n-2, k) \ | \ {}^t M J_{n-2,n-2} M = J_{n-2,n-2} \ \}.$$

The following lemma is trivial:

Lemma 3.4. *Let*

$$(x', \xi') = (x'_1, \ldots, x'_{n-2}, \xi'_1, \xi'_2)$$

be another formal parameter system of X at p. Then (x', ξ') is admissible with respect to ϕ if and only if (x, ξ) and (x', ξ') are related by the transformation of the form

$$\left[\begin{array}{c} x \\ \hline \xi \end{array}\right] = \left[\begin{array}{c|c} A & 0 \\ \hline * & B \end{array}\right] \left[\begin{array}{c} x' \\ \hline \xi' \end{array}\right] + (\text{terms of degree} \geq 2)$$

such that $A \in G$ and $B \in GL(2, k)$.

We denote the coefficients in the formal power series expansion of ϕ in the parameters (x, ξ) as indicated in Table 3.1. The following Proposition follows immediately from Lemma 3.4.

$$x_1\, x_2\ +\ \ldots\ +\ x_{n-3}\, x_{n-2}\ +$$

+	$a_1\, x_1^2$	+	$\cdots$	+	$a_{n-2}\, x_{n-2}{}^2$	+	$\alpha_1 \xi_1^2$	+	$\alpha_2 \xi_2^2$	+	
+	$b_1\, x_1\, \xi_1^2$	+	$\cdots$	+	$b_{n-2}\, x_{n-2}\, \xi_1^2$	+	$\beta_1\, \xi_1^3$	+	$\beta_2\, \xi_1^2\, \xi_2$	+	
+	$c_1\, x_1\, \xi_2^2$	+	$\cdots$	+	$c_{n-2}\, x_{n-2}\, \xi_2^2$	+	$\gamma_1\, \xi_1\, \xi_2^2$	+	$\gamma_2\, \xi_2^3$	+	
+	$d_1\, x_1\, \xi_1\, \xi_2$	+	$\cdots$	+	$d_{n-2}\, x_{n-2}\, \xi_1\, \xi_2$	+					

+ (terms of degree 3 in (x,ξ) and degree ≥ 2 in x)

+ $f_{40}\, \xi_1^4$ + $f_{31}\, \xi_1^3\, \xi_2$ + $f_{22}\, \xi_1^2\, \xi_2^2$ + $f_{13}\, \xi_1\, \xi_2^3$ + $f_{04}\, \xi_2^4$ +

+ (terms of degree 4 in (x,ξ) and degree≥ 1 in x)

+ (terms of degree ≥ 5 in (x,ξ))

TABLE 3.1. The coefficients of a formal power series in (x,ξ)

Proposition 3.5. *Suppose that the rank of $H_{\phi,p}$ is $n-2$. Then the following condition is independent of the choice of ϕ and (x,ξ): at least one of the coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ is not zero.*

Definition 3.6. We say that the singularity of D at p is *of type $(\widetilde{\mathrm{A}})$* if $H_{\phi,p}$ is of rank $n-2$ and at least one of the coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ is not zero.

Proposition-Definition 3.7. (1) *The following three conditions are equivalent:*

(i) $\dim_k \mathcal{O}_{C(D,p),p} \leq 4$,
(ii) $\mathcal{O}_{C(D,p),p} \cong k[[s,t]]/(s^2, t^2)$, *and*
(iii) *the rank of the Hessian $H_{\phi,p}$ is $n-2$, and*

$$(3.1) \qquad \operatorname{rank} \begin{bmatrix} \alpha_1 & \beta_1 & \beta_2 \\ \alpha_2 & \gamma_1 & \gamma_2 \end{bmatrix} = 2.$$

We say that the singularity of D at p is of type (C) *if these conditions are satisfied.*
(2) *We put*

$$T_1 := \sum_{i=1}^{n-2} d_i b_{\tau(i)} + f_{31} \quad and \quad T_2 := \sum_{i=1}^{n-2} d_i c_{\tau(i)} + f_{13}.$$

The following three conditions are equivalent:

(i) $\dim_k \mathcal{O}_{R(D,p),p} \leq 4$,
(ii) $\mathcal{O}_{R(D,p),p} \cong k[[s,t]]/(s^2, t^2)$, *and*
(iii) *the rank of the Hessian $H_{\phi,p}$ is $n-2$, and*

$$(3.2) \qquad \operatorname{rank} \begin{bmatrix} \alpha_1 & \beta_1 & \beta_2 & T_1 \\ \alpha_2 & \gamma_1 & \gamma_2 & T_2 \end{bmatrix} = 2.$$

We say that the singularity of D at p is of type (R) *if these conditions are satisfied.*

Remark 3.8. As a corollary of Proposition-Definition 3.7, we see that the conditions (3.1) and (3.2) do not depend on the choice of the admissible formal parameter system (x,ξ). This fact can also be proved directly by means of Lemma 3.4.

For the proof of Proposition-Definition 3.7, we use the following easy lemma:

Lemma 3.9. *Let $g_1, \ldots, g_l$ be degenerate formal power series in $k[[s,t]]$, and let J be the ideal of $k[[s,t]]$ generated by $g_1, \ldots, g_l$. We denote by a_i and b_i the coefficients of s^2 and t^2 in g_i, respectively;*

$$g_i = a_i s^2 + b_i t^2 + (\text{terms of degree} \geq 3).$$

Then the following equivalence holds:

$$\dim_k k[[s,t]]/J \leq 4 \iff J = (s^2, t^2) \iff \operatorname{rank} \begin{bmatrix} a_1 & \ldots & a_l \\ b_1 & \ldots & b_l \end{bmatrix} = 2.$$

Proof of Proposition-Definition 3.7. Since the proof of the assertion (1) is similar to and simpler than that of (2), we prove only the assertion (2). For simplicity, we put

$$D_i \phi := \frac{\partial \phi}{\partial x_i},$$

and let I denote the ideal of $(\mathcal{O}_{X,P})^\wedge$ generated by $\phi, D_1\phi, \ldots, D_n\phi$ and ρ. Suppose that rank $H_{\phi,p} = n - 2l$, and let $(x_1, \ldots, x_n)$ be an admissible formal parameter system with respect to ϕ. Then the linear parts of ϕ and $D_i\phi$ are given as follows;

$$\phi^{[1]} = 0, \quad (D_i\phi)^{[1]} = \begin{cases} x_{\tau(i)} & \text{if } i \leq n - 2l \\ 0 & \text{if } i > n - 2l. \end{cases}$$

By Corollary 2.9, we also have

$$\rho^{[1]} = \begin{cases} 0 & \text{if } n - 2l \leq n - 4 \\ d_1 x_1 + \cdots + d_{n-2} x_{n-2} & \text{if } n - 2l = n - 2. \end{cases}$$

Therefore the classes of $1 \in k$ and $x_{n-2l+1}, \ldots, x_n$ span a $(2l+1)$-dimensional linear subspace in $(\mathcal{O}_{R(D,p),p})^\wedge = k[[x_1, \ldots, x_n]]/I$. Consequently the condition that $\dim_k \mathcal{O}_{R(D,p),p} \leq 4$ implies that rank $H_{\phi,p}$ is $n - 2$. Hence it suffices to show the equivalence of the conditions (i), (ii) and (iii) under the assumption that rank $H_{\phi,p} = n - 2$.

We use the admissible formal parameter system (x, ξ) and Table 3.1. Let I' be the ideal of $k[[x,\xi]]$ generated by $D_1\phi, \ldots, D_{n-2}\phi$, and put

$$s := \xi_1 \bmod I', \quad t := \xi_2 \bmod I'.$$

For $i = 1, \ldots, n-2$, we have

$$D_i\phi = x_{\tau(i)} + b_i \xi_1^2 + c_i \xi_2^2 + d_i \xi_1 \xi_2 + \begin{pmatrix} \text{terms of degree} \geq 2 \text{ in } (x, \xi) \text{ and degree} \\ \geq 1 \text{ in } x, \text{ or degree} \geq 3 \text{ in } (x, \xi) \end{pmatrix}.$$

Hence we obtain

$$(3.3) \qquad x_i \bmod I' = b_{\tau(i)} s^2 + c_{\tau(i)} t^2 + d_{\tau(i)} st + (\text{terms of degree} \geq 3 \text{ in } (s,t))$$

for $i = 1, \ldots, n-2$. In particular, we have $k[[x,\xi]]/I' = k[[s,t]]$. Using (3.3), we have

$$\begin{aligned}
g_1 &:= \phi \bmod I' &&= \alpha_1 s^2 + \alpha_2 t^2 + (\text{terms of degree} \geq 3 \text{ in } (s,t)), \\
g_2 &:= (D_{n-1}\phi) \bmod I' &&= \beta_1 s^2 + \gamma_1 t^2 + (\text{terms of degree} \geq 3 \text{ in } (s,t)), \\
g_3 &:= (D_n\phi) \bmod I' &&= \beta_2 s^2 + \gamma_2 t^2 + (\text{terms of degree} \geq 3 \text{ in } (s,t)).
\end{aligned}$$

On the other hand, using (3.3) again, we see that

$$\frac{\partial^2 \phi}{\partial \xi_1 \partial \xi_2} \bmod I', \quad \frac{\partial^2 \phi}{\partial x_i \partial \xi_1} \bmod I', \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x_i \partial \xi_2} \bmod I'$$

are equal to

$$f_{31} s^2 + f_{13} t^2 + \sum_{i=1}^{n-2} d_i \left(b_{\tau(i)} s^2 + c_{\tau(i)} t^2 + d_{\tau(i)} st \right) + (\text{terms of degree} \geq 3 \text{ in } (s,t)),$$

$d_i t + (\text{terms of degree} \geq 2 \text{ in } (s,t)) \quad \text{and}$

$d_i s + (\text{terms of degree} \geq 2 \text{ in } (s,t)),$

respectively. Moreover, we have

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} \bmod I' = \begin{cases} 1 + (\text{terms of degree} \geq 1 \text{ in } (s,t)) & \text{if } i = \tau(j), \\ (\text{terms of degree} \geq 1 \text{ in } (s,t)) & \text{if } i \neq \tau(j), \end{cases}$$

for $i, j = 1, \ldots, n-2$. By Corollary 2.9 and using $\sum_{i=1}^{n-2} d_i d_{\tau(i)} = 0$, we obtain

$$g_4 := \rho \bmod I' = T_1 s^2 + T_2 t^2 + (\text{terms of degree} \geq 3 \text{ in } (s,t)).$$

Since

$$(\mathcal{O}_{R(D,p),p})^\wedge = k[[x,\xi]]/I = k[[s,t]]/(g_1, g_2, g_3, g_4),$$

we obtain the equivalence of the conditions (i)-(iii) from Lemma 3.9. □

The following remark is crucial in the proof of Proposition 6.12.

Remark 3.10. Each of the conditions that the singularity of D at p is of type $(\widetilde{A})$, (C) and (R) is an *open* condition on the coefficients of the formal power series ϕ with rank $H_{\phi,p} \leq n-2$ in the following sense. We denote by $\mathfrak{m}$ the maximal ideal of $k[[x_1, \ldots, x_n]]$. For a positive integer $d \geq 2$, we put

$$V_d := \mathfrak{m}^2/\mathfrak{m}^{d+1}.$$

For $\phi \in \mathfrak{m}^2$, let $\bar{\phi} \in V_d$ denote the class of ϕ modulo $\mathfrak{m}^{d+1}$. By Lemma 2.10, the locus

$$\Delta := \{ \bar{\phi} \in V_2 \mid \text{rank } H_{\phi,p} \leq n-2 \}$$

is a closed irreducible hypersurface of V_2. For $d \geq 3$, let Δ_d denote the pull-back of Δ by the natural linear homomorphism $V_d \to V_2$. Note that the types $(\widetilde{A})$, (C) and (R) of the hypersurface singularities are defined by using the coefficients of terms of degree ≤ 3, ≤ 3 and ≤ 4 of the formal power series defining the hypersurface, respectively. Hence the loci

$$\begin{aligned} \Delta(\widetilde{A}) &:= \{ \bar{\phi} \in V_3 \mid \text{the singularity of } \phi = 0 \text{ at } p \text{ is of type } (\widetilde{A}) \}, \\ \Delta(C) &:= \{ \bar{\phi} \in V_3 \mid \text{the singularity of } \phi = 0 \text{ at } p \text{ is of type } (C) \}, \\ \Delta(R) &:= \{ \bar{\phi} \in V_4 \mid \text{the singularity of } \phi = 0 \text{ at } p \text{ is of type } (R) \}, \end{aligned}$$

are well defined. The loci $\Delta(\widetilde{A})$ and $\Delta(C)$ are Zariski open dense subsets of Δ_3, and the locus $\Delta(R)$ is a Zariski open dense subset of Δ_4.

4. THE SINGULAR SCHEME Γ_{m-1}

We define a homomorphism
$$\gamma^* \,:\, k[[u_1, v_1, u_2, v_2, w, y]] \,\to\, k[[x_1, x_2, z]]$$
over k by
$$\gamma^* u_1 = x_1^2, \quad \gamma^* v_1 = x_1 z, \quad \gamma^* u_2 = x_2^2, \quad \gamma^* v_2 = x_2 z, \quad \gamma^* w = x_1 x_2 z, \quad \gamma^* y = z,$$
and denote by
$$\Gamma := \mathrm{Spec}(k[[u_1, v_1, u_2, v_2, w, y]] \,/\, \mathrm{Ker}\,\gamma^*)$$
the scheme-theoretic image of the morphism
$$\gamma \,:\, \mathrm{Spec}\,k[[x_1, x_2, z]] \,\to\, \mathrm{Spec}\,k[[u_1, v_1, u_2, v_2, w, y]]$$
induced from γ^*. The critical subscheme of γ is defined by $z^2 = 0$. Calculating the Gröbner basis (see [1]) of the ideal
$$\langle\, u_1 - x_1^2,\ v_1 - x_1 z,\ u_2 - x_2^2,\ v_2 - x_2 z,\ w - x_1 x_2 z,\ y - z \,\rangle$$
in the polynomial ring $k[x_1, x_2, z, u_1, v_1, u_2, v_2, w, y]$ under the pure lexicographic order
$$x_1 \,>\, x_2 \,>\, z \,>\, w \,>\, v_2 \,>\, v_1 \,>\, y \,>\, u_2 \,>\, u_1,$$
we see that Γ is a 3-dimensional singular scheme defined by the equations (1.2) given in Introduction. The singular locus of Γ is equal to the 2-dimensional plane
$$v_1 = v_2 = w = y = 0.$$
For a k-algebra R, we denote by
$$\gamma^R \,:\, \mathrm{Spec}\,R[[x_1, x_2, z]] \,\to\, \mathrm{Spec}\,R[[u_1, v_1, u_2, v_2, w, y]]$$
the morphism obtained from γ by $k \hookrightarrow R$, and by
$$\Gamma^R := \mathrm{Spec}(R[[u_1, v_1, u_2, v_2, w, y]] \,/\, \mathrm{Ker}\,(\gamma^R)^*),$$
the scheme-theoretic image of γ^R.

Theorem 4.1. *Let V be a smooth variety of dimension $l \geq 3$, and let $\phi : V \to W$ be a morphism to a smooth variety W. Suppose that there exists a smooth hypersurface H of V such that, at every point P of H, the kernel of*
$$d_P\phi \,:\, T_P(V) \,\to\, T_{\phi(P)}(W)$$
is of dimension 2 and is contained in $T_P(H)$.

Let P be a point of H. We denote by R the ring of formal power series $k[[x_3, \dots, x_{l-1}]]$ of $l - 3$ variables. Then there exists an isomorphism
$$\iota \,:\, (V, P)^{\wedge} \,\xrightarrow{\sim}\, \mathrm{Spec}\,R[[x_1, x_2, z]]$$
over k with the following properties:

(i) *the divisor $(H, P)^{\wedge}$ of $(V, P)^{\wedge}$ is mapped by the isomorphism ι to the divisor given by $z = 0$, and*

(ii) *there exists a unique morphism $\psi : \Gamma^R \to (W, \phi(P))^{\wedge}$ such that the formal completion $\phi_P^{\wedge}$ of ϕ at P factors as the following diagram:*

$$
\begin{array}{ccc}
(V, P)^{\wedge} & \xrightarrow{\ \phi_P^{\wedge}\ } & (W, \phi(P))^{\wedge} \\[4pt]
{\scriptstyle \iota}\downarrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \psi} \\[4pt]
\mathrm{Spec}\,R[[x_1, x_2, z]] & \xrightarrow[\ \gamma^R\ ]{} & \Gamma^R.
\end{array}
$$

Proof. We denote by
$$\phi_H \;:\; H \;\to\; W$$
the restriction of ϕ to H. The kernel $\mathcal{K}$ of the homomorphism
$$d\phi_H \;:\; T(H) \;\to\; \phi_H^* T(W)$$
is an integrable subbundle of $T(H)$ with rank 2 by the assumption. Let P be an arbitrary point of H. By [18, Proposition 6], there exist a basis (D_1, D_2) of the $\mathcal{O}_{H,P}$-module $\mathcal{K} \otimes \mathcal{O}_{H,P}$ and a system of uniformizing variables $(y_1, \ldots y_{l-1})$ of H at P such that
$$D_i(y_j) = \delta_{ij} \qquad (i = 1, 2, \;\; j = 1, \ldots, l-1).$$
In other words, the submodule $\mathcal{K} \otimes \mathcal{O}_{H,P}$ of $T(H) \otimes \mathcal{O}_{H,P}$ is equal to
$$\mathcal{O}_{H,P} \frac{\partial}{\partial y_1} \;\oplus\; \mathcal{O}_{H,P} \frac{\partial}{\partial y_2}.$$
We can choose a formal parameter system $(x_1, \ldots, x_{l-1}, z)$ of V at P such that H is defined by $z = 0$, and that the restriction $x_i \,|\, H$ of x_i to H is equal to y_i. Then, for every element $g \in (\mathcal{O}_{W, \phi(P)})^\wedge$, we have
$$\frac{\partial\,(\phi_H^* g)}{\partial y_1} \;=\; \frac{\partial\,(\phi^* g)}{\partial x_1} \bigg|_H \;\equiv\; 0 \quad \text{and} \quad \frac{\partial\,(\phi_H^* g)}{\partial y_2} \;=\; \frac{\partial\,(\phi^* g)}{\partial x_2} \bigg|_H \;\equiv\; 0;$$
that is, $\phi^* g$ is contained in the subring A of $(\mathcal{O}_{V,P})^\wedge = k[[x_1, \ldots, x_{l-1}, z]]$ defined by
$$A := \left\{ f \in (\mathcal{O}_{V,P})^\wedge \;\;\bigg|\;\; \frac{\partial f}{\partial x_1} \text{ and } \frac{\partial f}{\partial x_2} \text{ are contained in the ideal } (z) \right\}.$$
Putting $R := k[[x_3, \ldots, x_{l-1}]]$, we have
$$A \;=\; R[[x_1^2, x_2^2, z]] \;+\; z\, R[[x_1, x_2, z]] \;\;\subset\;\; R[[x_1, x_2, z]] \;=\; (\mathcal{O}_{V,P})^\wedge.$$
Therefore, the subring A is the image of the homomorphism
$$(\gamma^R)^* \;:\; R[[u_1, v_1, u_2, v_2, w, y]] \;\to\; R[[x_1, x_2, z]].$$
We denote by
$$j \;:\; R[[u_1, v_1, u_2, v_2, w, y]] \,/\, (\mathrm{Ker}\,(\gamma^R)^*) \;\xrightarrow{\sim}\; A$$
the induced isomorphism. Let $(t_1, \ldots, t_q)$ be a formal parameter system of W at $\phi(P)$, where $q = \dim W$. Since $\phi^* t_i \in A$, there exists a unique element
$$\psi_i \;\in\; R[[u_1, v_1, u_2, v_2, w, y]] \,/\, (\mathrm{Ker}\,(\gamma^R)^*)$$
such that $j(\psi_i) = \phi^* t_i$. Define a morphism
$$\psi \;:\; \Gamma^R \;\to\; (W, \phi(P))^\wedge$$
by $\psi^* t_i = \psi_i$. Then $\phi_P^\wedge$ factors as $\psi \circ \gamma^R \circ \iota$, where ι is the isomorphism given by the canonical isomorphism $R[[x_1, x_2, z]] \xrightarrow{\sim} (\mathcal{O}_{V,P})^\wedge$. $\qquad\square$

Definition 4.2. When R is the ring of formal power series in $(m-4)$ variables with coefficients in k, the scheme-theoretic image Γ^R is an $(m-1)$-dimensional singular scheme, which we will denote by Γ_{m-1}.

Remark 4.3. A normal form theorem similar to Theorem 4.1 for a morphism from a smooth surface to a smooth curve was proved in [20].

5. The discriminant variety

In this section, we recall the definitions and results in [19, Section 3], which are valid in any characteristics.

Let $\overline{X}$ be a projective variety of dimension $n > 0$, $\mathcal{L}$ a line bundle on $\overline{X}$, and M a linear subspace of $H^0(\overline{X}, \mathcal{L})$ with dimension $m + 1 \geq 2$. We denote by

$$\mathbf{P} := \mathbb{P}_*(M)$$

the parameter space of the m-dimensional linear system $|M|$ of divisors on $\overline{X}$ corresponding to M, and put

$$X := \overline{X} \setminus (\mathrm{Sing}(\overline{X}) \cup \mathrm{Bs}(|M|)),$$

where $\mathrm{Sing}(\overline{X})$ is the singular locus of $\overline{X}$ and $\mathrm{Bs}(|M|)$ is the base locus of $|M|$. We denote by $\Psi : X \to \mathbf{P}^\vee$ the morphism induced by $\mathbf{P}$, and set

$$X^\circ := \{\, p \in X \mid \text{the homomorphism } d_p \Psi : T_p(X) \to T_{\Psi(p)}(\mathbf{P}^\vee) \text{ is injective} \,\}.$$

Note that $X^\circ = \overline{X}$ if $\overline{X}$ is smooth and $|M|$ is very ample.

Assumption 5.1. Throughout the paper, we assume that $m > n$, and that X° is dense in $\overline{X}$.

For a non-zero vector $f \in M$, we denote by $[f] \in \mathbf{P}$ the corresponding point of $\mathbf{P}$, by $\overline{D}_{[f]}$ the divisor of $\overline{X}$ defined by $f = 0$, and by $D_{[f]}$ the intersection $\overline{D}_{[f]} \cap X$. Let

$$\mathrm{pr}_1 : X \times \mathbf{P} \to X \quad \text{and} \quad \mathrm{pr}_2 : X \times \mathbf{P} \to \mathbf{P}$$

be the projections. We put

$$\widetilde{\mathcal{L}} := \mathrm{pr}_1^* \mathcal{L} \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbf{P}}(1).$$

There exists a canonical isomorphism $M^\vee \xrightarrow{\sim} H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1))$ unique up to multiplicative constants. Combining this isomorphism with the inclusion $M \hookrightarrow H^0(X, \mathcal{L})$, we obtain a natural homomorphism

$$\mathrm{Hom}(M, M) = M \otimes M^\vee \;\to\; H^0(X \times \mathbf{P}, \widetilde{\mathcal{L}}).$$

We denote by

$$\sigma \in H^0(X \times \mathbf{P}, \widetilde{\mathcal{L}})$$

the global section of $\widetilde{\mathcal{L}}$ corresponding to the identity homomorphism of M. Note that σ is determined uniquely up to multiplicative constants. The zero locus of σ is equal to

$$\{\, (p, [f]) \in X \times \mathbf{P} \mid p \in D_{[f]} \,\}.$$

Let $\mathcal{D}$ be the subscheme of $X \times \mathbf{P}$ defined by $\sigma = 0$, and let

$$p_1 : \mathcal{D} \to X \quad \text{and} \quad p_2 : \mathcal{D} \to \mathbf{P}$$

be the projections. Then $\mathcal{D}$ is smooth of dimension $m + n - 1$, and $p_2 : \mathcal{D} \to \mathbf{P}$ is the universal family of the divisors $D_{[f]}$ ($[f] \in \mathbf{P}$). We have a natural homomorphism

$$d\sigma \,:\, T(X \times \mathbf{P}) \otimes \mathcal{O}_{\mathcal{D}} \;\to\; \widetilde{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{D}}$$

defined by $\Theta \mapsto (\Theta \sigma)\,|\,\mathcal{D}^\circ$, where Θ is a local section of $T(X \times \mathbf{P})$ considered as a derivation of $\mathcal{O}_{X \times \mathbf{P}}$. We denote by

$$d\sigma_X \,:\, p_1^* T(X) \to \widetilde{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{D}}$$

the restriction of $d\sigma$ to the direct factor $p_1^*T(X)$ of

$$T(X \times \mathbf{P}) \otimes \mathcal{O}_{\mathcal{D}} = p_1^*T(X) \oplus p_2^*T(\mathbf{P}).$$

Let $\mathcal{C}$ be the critical subscheme of $p_2 : \mathcal{D} \to \mathbf{P}$.

Proposition 5.2. (1) *The scheme $\mathcal{C}$ is equal to the degeneracy subscheme of $d\sigma_X$.*
   (2) *The intersection $\mathcal{C}^\circ$ of $\mathcal{C}$ and $X^\circ \times \mathbf{P}$ is smooth, irreducible and of dimension $m - 1$.*

Corollary 5.3. (1) *The support of $\mathcal{C}$ is equal to the set*

$$\{ (p, [f]) \in \mathcal{D} \mid D_{[f]} \text{ is singular at } p \}.$$

   (2) *Let $P = (p, [f])$ be a point of $\mathcal{C}$, and let $\pi_2' : \mathcal{C} \to \mathbf{P}$ be the projection. Then, as a subscheme of $(\mathrm{pr}_2^{-1}([f]), P)^\wedge \cong (X, p)^\wedge$, the completion $(\pi_2'^{-1}([f]), P)^\wedge$ of the fiber of π_2' at P coincides with $C(D_{[f]}, p)$ defined in Definition 3.3.*

We denote by

$$\pi_1 : \mathcal{C}^\circ \to X^\circ \quad \text{and} \quad \pi_2 : \mathcal{C}^\circ \to \mathbf{P}$$

the projections from $\mathcal{C}^\circ = \mathcal{C} \cap (X^\circ \times \mathbf{P})$. We can define the *universal Hessian*

$$\mathcal{H} : \pi_1^*T(X^\circ) \otimes \pi_1^*T(X^\circ) \to \widetilde{\mathcal{L}} \otimes \mathcal{O}_{\mathcal{C}^\circ}$$

on $\mathcal{C}^\circ$. Because we are in characteristic 2, the universal Hessian $\mathcal{H}$ on $\mathcal{C}^\circ$ is *anti-symmetric*. Let

$$\mathcal{H}\tilde{} : \pi_1^*T(X^\circ) \to \widetilde{\mathcal{L}} \otimes (\pi_1^*T(X^\circ))^\vee$$

be the homomorphism obtained from $\mathcal{H}$. Let $\mathcal{E}$ be the critical subscheme of the projection $\pi_2 : \mathcal{C}^\circ \to \mathbf{P}$.

Proposition 5.4. *The scheme $\mathcal{E}$ is equal to the degeneracy subscheme of $\mathcal{H}\tilde{}$.*

Corollary 5.5. *The support of $\mathcal{E}$ is equal to the set of all point $(p, [f]) \in \mathcal{C}^\circ$ such that the Hessian of the hypersurface singularity $p \in D_{[f]}$ is degenerate.*

Since the rank of an anti-symmetric form is always even, we obtain the following:

Corollary 5.6. *Suppose that n is odd. Then $\mathcal{E}$ coincides with $\mathcal{C}^\circ$.*

6. Main results

From now on until the end of the paper, we assume that n is even.

Definition 6.1. Let $\mathcal{R} \subset \mathcal{C}^\circ$ be the degeneracy subscheme $\mathbf{R}(\mathcal{H})$ of the anti-symmetric form $\mathcal{H}$, and let

$$\varpi_1 : \mathcal{R} \to X^\circ \quad \text{and} \quad \varpi_2 : \mathcal{R} \to \mathbf{P}$$

be the projections.

Proposition 6.2. (1) *The subscheme $\mathcal{R}$ is of codimension ≤ 1 in $\mathcal{C}^\circ$. If $\mathcal{R}$ is of codimension 1 in $\mathcal{C}^\circ$, then $\mathcal{E}$ coincides with $2\,\mathcal{R}$ as a divisor of $\mathcal{C}^\circ$.*
   (2) *Let $P = (p, [f])$ be a point of $\mathcal{R}$. Then, as a subscheme of $(\mathrm{pr}_2^{-1}([f]), P)^\wedge \cong (X, p)^\wedge$, the completion $(\varpi_2^{-1}([f]), P)^\wedge$ of the fiber of ϖ_2 at P coincides with $R(D_{[f]}, p)$ defined in Definition 3.3.*

Proof. The assertion (1) follows from Remark 2.15 and Proposition 5.4. The assertion (2) follows from the definition. $\square$

It is possible that $\mathcal{R}$ is of codimension 0 in $\mathcal{C}^\circ$. The following is a classical example due to Wallace [22]. See also [21].

Example 6.3. Let X be the Fermat hypersurface of degree $2^\nu + 1$ in $\mathbb{P}^{n+1}$ with $\nu \geq 1$, and let $|M|$ be the complete linear system $|\mathcal{O}_X(1)|$. Then the rank of the Hessian of $D_{[f]}$ at p is zero at every point $(p, [f])$ of $\mathcal{C}^\circ = \mathcal{C}$. In particular, $\mathcal{R}$ coincides with $\mathcal{C}^\circ$.

Let $P = (p, [f])$ be a point of $\mathcal{R}$, so that $D_{[f]}$ is singular at p and the Hessian of $D_{[f]}$ at p is of rank $\leq n - 2$. We investigate the formal completion of the morphism $\pi_2 : \mathcal{C}^\circ \to \mathbf{P}$ at P. For this purpose, we introduce a good formal parameter system of $X^\circ \times \mathbf{P}$ at P.

Because $p \in X^\circ$, we can choose a basis $\boldsymbol{b}_0, \dots, \boldsymbol{b}_m$ of the vector space M in such a way that the following conditions (i)-(iv) hold. Let $\tilde{\boldsymbol{b}}_i$ be the global section of $\mathcal{L}$ corresponding to $\boldsymbol{b}_i$, and let D_i be the divisor defined by $\tilde{\boldsymbol{b}}_i = 0$. Then

 (i) $\boldsymbol{b}_0 = f$, so that $D_0 = D_{[f]}$ passes through p and is singular at p,
 (ii) $p \in D_i$ for $i = 0, 1, \dots, m - 1$, and $p \notin D_m$,
 (iii) $D_1, \dots, D_n$ are smooth at p and intersect transversely at p, and
 (iv) $D_{n+1}, \dots, D_{m-1}$ are singular at p.

For $i = 0, \dots, m - 1$, we put

$$\phi_i := \tilde{\boldsymbol{b}}_i / \tilde{\boldsymbol{b}}_m,$$

which is a rational function on X regular at p. Then $(\phi_1, \dots, \phi_n)$ forms a local parameter system of X at p by (iii). We put

$$x_i := \phi_i \quad (i = 1, \dots, n).$$

We will regard $(x_1, \dots, x_n)$ as a formal parameter system of X at p, and consider ϕ_0 and $\phi_{n+1}, \dots, \phi_{m-1}$ as formal power series of $(x_1, \dots, x_n)$, which have no terms of degree ≤ 1 by (i) and (iv). Note that the rank of the anti-symmetric matrix $H_{\phi_0, p}$ is $\leq n - 2$, because $P \in \mathcal{R}$. Using Remark 3.2, we can further assume the following by a linear transformation of $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$:

 (v) the formal parameter system $(x_1, \dots, x_n)$ is admissible with respect to ϕ_0.

Let $(Y_0, \dots, Y_m)$ be the linear coordinates of M with respect to the basis $\boldsymbol{b}_0, \dots, \boldsymbol{b}_m$, and let $(y_1, \dots, y_m)$ be the affine coordinate system of $\mathbf{P}$ given by

$$y_i := Y_i / Y_0.$$

Then the point $[f] \in \mathbf{P}$ is the origin. We regard

$$(x_1, \dots, x_n, y_1, \dots, y_m)$$

as a formal parameter system of $X^\circ \times \mathbf{P}$ at $P = (p, [f])$.

We can regard the coordinates Y_i as global sections of $\mathcal{O}_\mathbf{P}(1)$. Recall that σ is the canonical section of $\widetilde{\mathcal{L}} := \mathrm{pr}_1^* \mathcal{L} \otimes \mathrm{pr}_2^* \mathcal{O}_\mathbf{P}(1)$. Multiplying σ by a suitable non-zero constant, we have

$$\sigma = \sum_{i=0}^{m} \tilde{\boldsymbol{b}}_i \otimes Y_i.$$

We put

$$\Phi := \left(\sum_{i=0}^{m} \tilde{\boldsymbol{b}}_i \otimes Y_i \right) / (\tilde{\boldsymbol{b}}_m \otimes Y_0),$$

which is a rational function of $X^\circ \times \mathbf{P}$ regular at P. By definition, we have

$$\Phi = \phi_0 + y_1 x_1 + \cdots + y_n x_n + y_{n+1}\phi_{n+1} + \cdots + y_{m-1}\phi_{m-1} + y_m.$$

The scheme $\mathcal{D}$ is defined by $\Phi = 0$ locally at P. For simplicity, we put

$$D_i\Phi := \frac{\partial\Phi}{\partial x_i}.$$

By Proposition 5.2, the subscheme $\mathcal{C}^\circ$ of $X^\circ \times \mathbf{P}$ is defined by

$$\Phi = D_1\Phi = \cdots = D_n\Phi = 0$$

locally at P. We put

$$R := f_n\left(\ldots, \frac{\partial^2\Phi}{\partial x_i \partial x_j}, \ldots\right) = \sqrt{\det\left(\frac{\partial^2\Phi}{\partial x_i \partial x_j}\right)},$$

where f_n is the polynomial defined in §2.2. The subscheme $\mathcal{R}$ of $X^\circ \times \mathbf{P}$ is defined by

$$\Phi = D_1\Phi = \cdots = D_n\Phi = R = 0$$

locally at P.

Note that the divisor $D_{[f]}$ of X is defined by $\phi_0 = 0$ locally at p. Let $2r$ be the rank of the Hessian $H_{\phi_0,p}$ of $D_{[f]}$ at p. Since $(x_1, \ldots, x_n)$ is admissible with respect to ϕ_0, we see that the linear parts of the formal power series Φ and $D_i\Phi$ are as follows:

$$(6.1) \qquad \begin{aligned} \Phi^{[1]} &= y_m, \\ (D_i\Phi)^{[1]} &= x_{\tau(i)} + y_i \quad \text{for } i = 1, \ldots, 2r, \\ (D_i\Phi)^{[1]} &= y_i \qquad\qquad \text{for } i = 2r+1, \ldots, n. \end{aligned}$$

When $2r = n - 2$, we use the formal parameter system $(x_1, \ldots, x_{n-2}, \xi_1, \xi_2)$ admissible with respect to ϕ_0 as in §3, and adopt Table 3.1 in order to denote the coefficients of ϕ_0. We also denote by $e_j \in k$ the coefficients of the term $\xi_1\xi_2$ in ϕ_j for $j = n+1, \ldots, m-1$. By Corollary 2.9, we see that the linear part of R is equal to

$$(6.2) \qquad \begin{aligned} R^{[1]} &= \begin{cases} 0 & \text{if } 2r < n-2, \\ \left(\dfrac{\partial^2\Phi}{\partial\xi_1\partial\xi_2}\right)^{[1]} & \text{if } 2r = n-2, \end{cases} \\[2em] &= \begin{cases} 0 & \text{if } 2r < n-2, \\ \begin{aligned} &d_1 x_1 + \cdots + d_{n-2}x_{n-2} + \\ &\quad + e_{n+1}y_{n+1} + \cdots + e_{m-1}y_{m-1} \end{aligned} & \text{if } 2r = n-2. \end{cases} \end{aligned}$$

6.1. **A normal form theorem.** We put

$$\mathcal{R}^{\mathrm{sm}} := \{ P \in \mathcal{R} \mid \mathcal{R} \text{ is smooth of dimension } m-2 \text{ at } P \},$$

and denote by $\varpi_2^{\mathrm{sm}} : \mathcal{R}^{\mathrm{sm}} \to \mathbf{P}$ the projection.

Theorem 6.4. *Let $P = (p, [f])$ be a point of $\mathcal{R}$.*

(1) If $P \in \mathcal{R}^{\mathrm{sm}}$, then the rank of the Hessian $H_{\phi_0,p}$ of $D_{[f]}$ at p is $n-2$. Conversely, suppose that $H_{\phi_0,p}$ is of rank $n-2$. Then P is a point of $\mathcal{R}^{\mathrm{sm}}$ if and only if at least one of

$$d_1, \ldots, d_{n-2}, e_{n+1}, \ldots, e_{m-1}$$

is not zero.

(2) *The kernel $\mathcal{K}$ of the homomorphism*

$$d\varpi_2^{\mathrm{sm}} \,:\, T(\mathcal{R}^{\mathrm{sm}}) \,\to\, (\varpi_2^{\mathrm{sm}})^* T(\mathbf{P})$$

is a subbundle of $T(\mathcal{R}^{\mathrm{sm}})$ with rank 2.

Proof. The assertion (1) follows immediately from (6.1) and (6.2). Suppose that $P \in \mathcal{R}^{\mathrm{sm}}$. Then the kernel of $d_P \varpi_2^{\mathrm{sm}} : T_P(\mathcal{R}^{\mathrm{sm}}) \to T_{[f]}(\mathbf{P})$ is of dimension 2 and generated by the vectors

$$\left(\frac{\partial}{\partial x_{n-1}} \right)_P \quad \text{and} \quad \left(\frac{\partial}{\partial x_n} \right)_P$$

of $T_P(X^\circ \times \mathbf{P})$. Since this fact holds at every point P of $\mathcal{R}^{\mathrm{sm}}$, we see that $\mathcal{K}$ is a subbundle of $T(\mathcal{R}^{\mathrm{sm}})$ with rank 2. $\qquad\square$

Corollary 6.5. *Suppose that $(n, m) = (2, 3)$. Then $\mathcal{R}^{\mathrm{sm}}$ is empty.*

By definition, the kernel $\mathcal{K}$ of $d\varpi_2^{\mathrm{sm}}$ is an integrable subbundle of $T(\mathcal{R}^{\mathrm{sm}})$. From Proposition 2.4, we obtain the following:

Corollary 6.6. *The morphism $\varpi_2^{\mathrm{sm}} : \mathcal{R}^{\mathrm{sm}} \to \mathbf{P}$ factors through the quotient morphism $q : \mathcal{R}^{\mathrm{sm}} \to (\mathcal{R}^{\mathrm{sm}})^{\mathcal{K}}$ by $\mathcal{K}$, which is finite of degree 4.*

Combining Theorems 4.1 and 6.4, we obtain the following normal form theorem for the morphism $\pi_2 : \mathcal{C}^\circ \to \mathbf{P}$ at a point $P = (p, [f])$ of $\mathcal{R}^{\mathrm{sm}}$.

Corollary 6.7. *Let $P = (p, [f])$ be a point of $\mathcal{R}^{\mathrm{sm}}$. Then there exist an isomorphism*

$$\iota \,:\, (\mathcal{C}^\circ, P)^\wedge \,\xrightarrow{\sim}\, \operatorname{Spec} k[[w_1, \ldots, w_{m-1}]]$$

and a morphism

$$\psi \,:\, \Gamma_{m-1} \,\to\, (\mathbf{P}, [f])^\wedge$$

such that the formal completion

$$(\pi_2)_P^\wedge \,:\, (\mathcal{C}^\circ, P)^\wedge \,\to\, (\mathbf{P}, [f])^\wedge$$

of π_2 at P factors as $\psi \circ \gamma^R \circ \iota$, where

$$\gamma^R \,:\, \operatorname{Spec} k[[w_1, \ldots, w_{m-1}]] \,\to\, \Gamma_{m-1}$$

is the morphism defined in §4.

6.2. **General plane sections.** Let $P = (p, [f])$ be a point of $\mathcal{R}$. We choose linear subspaces L and Λ of $\mathbf{P}$ with dimension 3 and 2, respectively, such that

$$[f] \,\in\, \Lambda \,\subset\, L.$$

We set

$$S_L := \pi_2^{-1}(L) \,\subset\, \mathcal{C}^\circ \quad \text{and} \quad C_\Lambda := \pi_2^{-1}(\Lambda) \,\subset\, \mathcal{C}^\circ,$$

and denote by

$$\pi_L \,:\, S_L \,\to\, L \quad \text{and} \quad \pi_\Lambda \,:\, C_\Lambda \,\to\, \Lambda$$

the restrictions of $\pi_2 : \mathcal{C}^\circ \to \mathbf{P}$ to S_L and to C_Λ, respectively.

Proposition 6.8. *Suppose that L is chosen generically over k. The scheme S_L is smooth of dimension 2 at P if and only if the rank of the Hessian $H_{\phi_0, p}$ of $D_{[f]}$ at p is $n - 2$.*

Proof. We can assume that L is defined by

$$(6.3) \qquad y_i = A_i\, y_{n-1} + B_i\, y_n + C_i\, y_m \qquad (i \neq n-1, n, m),$$

where A_i, B_i, C_i are generic over k. Then the assertion follows from the linear parts (6.1) of the defining equations of $\mathcal{C}^\circ$. $\qquad\square$

Proposition 6.9. *Suppose that L and Λ are chosen generically over k, and that S_L is smooth of dimension 2 at P.*

   (1) *If C_Λ is of codimension 1 in S_L, then C_Λ has a degenerate singularity at P.*

   (2) *The following two conditions are equivalent:*

    (i) *C_Λ is of codimension 1 in S_L, and the multiplicity of C_Λ at P is 2,*

    (ii) *the singularity of $D_{[f]}$ at p is of type $(\widetilde{\mathrm{A}})$.*

   (3) *Suppose that C_Λ is of codimension 1 in S_L, that the multiplicity of C_Λ at P is 2, and that P is a point of $\mathcal{R}^{\mathrm{sm}}$. Then P is an ordinary cusp of C_Λ if and only if the singularity of $D_{[f]}$ at p is of type (R).*

Proof. By Proposition 6.8 and the assumption, the rank of $H_{\phi_0, p}$ is $n-2$. We use the formal parameter system $(x_1, \ldots, x_{n-2}, \xi_1, \xi_2)$ admissible with respect to ϕ_0 as in §3, and refer to Table 3.1 for the name of coefficients of ϕ_0. We also use the defining equations (6.3) of the linear subspace L. By (6.1) and (6.3), we see that

$$s := \xi_1 \,|\, S_L \quad \text{and} \quad t := \xi_2 \,|\, S_L$$

form a formal parameter system of S_L at P. We put

$$u := y_{n-1} \,|\, L, \quad v := y_n \,|\, L \quad \text{and} \quad w := y_m \,|\, L,$$

which form an affine coordinate system of L with the origin $[f]$. For a formal power series F of $(x_1, \ldots, x_{n-2}, \xi_1, \xi_2, y_1, \ldots, y_m)$, we denote by F_L the formal power series of

$$(x_1, \ldots, x_{n-2}, s, t, u, v, w)$$

obtained from F by making the substitutions

$$y_{n-1} = u, \quad y_n = v, \quad y_m = w,$$
$$y_i = A_i u + B_i v + C_i w \quad (i \neq n-1, n, m), \quad \text{and}$$
$$\xi_1 = s, \quad \xi_2 = t.$$

Regarding

$$\Phi_L = (D_1 \Phi)_L = \cdots = (D_n \Phi)_L = 0$$

as equations with indeterminates $x_1, \ldots, x_{n-2}, u, v, w$ and with coefficients in $(\mathcal{O}_{S_L, P})^\wedge = k[[s, t]]$, and solving them, we obtain the formal power series expansion

$$\begin{aligned}
x_i \,|\, S_L &= X_i(s, t) &= \textstyle\sum X_{i, \mu\nu}\, s^\mu t^\nu \quad (i = 1, \ldots, n-2), \\
u \,|\, S_L &= U(s, t) &= \textstyle\sum U_{\mu\nu}\, s^\mu t^\nu, \\
v \,|\, S_L &= V(s, t) &= \textstyle\sum V_{\mu\nu}\, s^\mu t^\nu, \\
w \,|\, S_L &= W(s, t) &= \textstyle\sum W_{\mu\nu}\, s^\mu t^\nu,
\end{aligned}$$

of the functions $x_1|S_L, \ldots, x_{n-2}|S_L, u|S_L, v|S_L, w|S_L$ on S_L in (s, t). The formal power series U, V and W give the formal completion of $\pi_L : S_L \to L$ at P. We will calculate the homogeneous parts $U^{[d]}$, $V^{[d]}$, $W^{[d]}$ of degree d of them up to $d = 3$. From $\Phi_L = 0$ and $(D_{n-1}\Phi)_L = (D_n \Phi)_L = 0$, we obtain

$$U^{[1]} = V^{[1]} = W^{[1]} = 0.$$

From $(D_i\Phi)_L = 0$ for $i = 1, \ldots, n-2$, we obtain

$$X_i^{[1]} = 0 \qquad (i = 1, \ldots, n-2).$$

Looking at the homogeneous parts of degree 2 in (s,t) of Φ_L and $(D_i\Phi)_L$, we obtain the following equations:

$$
\begin{aligned}
\alpha_1\, s^2 + \alpha_2\, t^2 + W^{[2]} &= 0, \\
b_i\, s^2 + c_i\, t^2 + d_i\, st + X_{\tau(i)}^{[2]} + A_i\, U^{[2]} + B_i\, V^{[2]} + C_i\, W^{[2]} &= 0 \quad (i \le n-2), \\
\beta_1\, s^2 + \gamma_1\, t^2 + U^{[2]} &= 0, \\
\beta_2\, s^2 + \gamma_2\, t^2 + V^{[2]} &= 0.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
U^{[2]} &= \beta_1\, s^2 + \gamma_1\, t^2, \\
V^{[2]} &= \beta_2\, s^2 + \gamma_2\, t^2, \\
W^{[2]} &= \alpha_1\, s^2 + \alpha_2\, t^2,
\end{aligned}
$$

and

$$
\begin{aligned}
X_i^{[2]} &= (b_{\tau(i)}\, s^2 + c_{\tau(i)} t^2 + d_{\tau(i)}\, st) + A_{\tau(i)} U^{[2]} + B_{\tau(i)} V^{[2]} + C_{\tau(i)} W^{[2]} \\
&= \ (b_{\tau(i)} + A_{\tau(i)}\, \beta_1 + B_{\tau(i)}\, \beta_2 + C_{\tau(i)}\, \alpha_1)\, s^2 \\
&\quad + (c_{\tau(i)} + A_{\tau(i)}\, \gamma_1 + B_{\tau(i)}\, \gamma_2 + C_{\tau(i)}\, \alpha_2)\, t^2 \\
&\quad + d_{\tau(i)}\, st.
\end{aligned}
$$

Looking at the homogeneous part of degree 3 in (s,t) of Φ_L, we get the equation

$$\beta_1\, s^3 + \beta_2\, s^2 t + \gamma_1\, st^2 + \gamma_2\, t^3 + U^{[2]} s + V^{[2]} t + W^{[3]} = 0.$$

Thus we obtain

$$W^{[3]} = 0.$$

Looking at the homogeneous part of degree 3 in (s,t) of $(D_{n-1}\Phi)_L$ and $(D_n\Phi)_L$, we obtain the equations

$$
\sum_{i=1}^{n-2} d_i X_i^{[2]} t + f_{31} s^2 t + f_{13} t^3 + U^{[3]} + \sum_{j=n+1}^{m-1} (A_j U^{[2]} + B_j V^{[2]} + C_j W^{[2]})\, e_j t = 0,
$$

$$
\sum_{i=1}^{n-2} d_i X_i^{[2]} s + f_{31} s^3 + f_{13} st^2 + V^{[3]} + \sum_{j=n+1}^{m-1} (A_j U^{[2]} + B_j V^{[2]} + C_j W^{[2]})\, e_j s = 0,
$$

where e_j is the coefficient of $\xi_1 \xi_2$ in ϕ_j. We put

$$
\begin{aligned}
Q(s,t) &:= \sum_{i=1}^{n-2} d_i X_i^{[2]} + f_{31} s^2 + f_{13} t^2 + \sum_{j=n+1}^{m-1} (A_j U^{[2]} + B_j V^{[2]} + C_j W^{[2]})\, e_j \\
&= \ Q_s\, s^2 + Q_t\, t^2,
\end{aligned}
$$

where Q_s and Q_t are given in Table 6.1. (Note that the coefficient of st in $Q(s,t)$ is $\sum_{i=1}^{n-2} d_i d_{\tau(i)} = 0$.) Then we obtain

$$U^{[3]} = t\, Q(s,t) \quad \text{and} \quad V^{[3]} = s\, Q(s,t).$$

Suppose that Λ is defined in L by

$$Du + Ev + Fw = 0,$$

$$Q_s \; := \; \sum_{i=1}^{n-2}(b_{\tau(i)} + A_{\tau(i)}\beta_1 + B_{\tau(i)}\beta_2 + C_{\tau(i)}\alpha_1)d_i +$$
$$+ \; f_{31} \; + \; \sum_{j=n+1}^{m-1}(A_j\beta_1 + B_j\beta_2 + C_j\alpha_1)e_j,$$
$$Q_t \; := \; \sum_{i=1}^{n-2}(c_{\tau(i)} + A_{\tau(i)}\gamma_1 + B_{\tau(i)}\gamma_2 + C_{\tau(i)}\alpha_2)d_i +$$
$$+ \; f_{13} \; + \; \sum_{j=n+1}^{m-1}(A_j\gamma_1 + B_j\gamma_2 + C_j\alpha_2)e_j.$$

TABLE 6.1. The coefficients Q_s and Q_t

where D, E, F are generic over k. From the assumption, A_i, B_i, C_i and D, E, F are algebraically independent over k. The scheme C_Λ is defined in S_L by $\Gamma = 0$ locally at P, where Γ is the formal power series of (s, t) defined by

$$\Gamma := DU + EV + FW.$$

Obviously, we have $\Gamma^{[0]} = \Gamma^{[1]} = 0$. We also have

$$\Gamma^{[2]} \; = \; (D\beta_1 + E\beta_2 + F\alpha_1)\, s^2 \; + \; (D\gamma_1 + E\gamma_2 + F\alpha_2)\, t^2 \; = \; \ell^2,$$

where

$$\ell \; := \; \sqrt{D\beta_1 + E\beta_2 + F\alpha_1}\; s \; + \; \sqrt{D\gamma_1 + E\gamma_2 + F\alpha_2}\; t.$$

Hence, if $\Gamma \neq 0$, then the curve C_Λ on S_L has a degenerate singularity at P. Moreover, $\Gamma^{[2]}$ is not zero if and only if at least one of $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ is not zero. Thus the assertions (1) and (2) are proved.

Suppose that $P \in \mathcal{R}^{\mathrm{sm}}$ and that $\Gamma^{[2]} = \ell^2 \neq 0$. The degenerate singular point P of C_Λ is an ordinary cusp if and only if the linear form ℓ does not divide

$$\Gamma^{[3]} = (D\, t + E\, s)\, (\sqrt{Q_s}\, s + \sqrt{Q_t}\, t)^2.$$

The two linear forms ℓ and $Dt + Es$ are proportional if and only if

$$(6.4) \qquad D^2\, (D\beta_1 + E\beta_2 + F\alpha_1) \; + \; E^2\, (D\gamma_1 + E\gamma_2 + F\alpha_2) \; = \; 0$$

holds. Since D, E, F are generic over k, and at least one of $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ is not zero by $\Gamma^{[2]} \neq 0$, the equality (6.4) does not hold. The two linear forms ℓ and $\sqrt{Q_s}s + \sqrt{Q_t}t$ are proportional if and only if

$$(6.5) \qquad Q_t\, (D\beta_1 + E\beta_2 + F\alpha_1) \; + \; Q_s\, (D\gamma_1 + E\gamma_2 + F\alpha_2) \; = \; 0$$

holds. Since $P \in \mathcal{R}^{\mathrm{sm}}$, we see from Theorem 6.4 (1) that at least one of $d_1, \ldots, d_{n-2}$, $e_{n+1}, \ldots, e_{m-1}$ is not zero. Expanding the left hand side of (6.5) as a polynomial of A_i, B_i, C_i and D, E, F, we see that (6.5) holds for a generic choice of A_i, B_i, C_i, D, E, F if and only if the condition (3.2) does *not* hold. Therefore the assertion (3) is proved. $\qquad\square$

Proposition 6.10. *Suppose that C_Λ is of codimension 1 in S_L at P, and is reduced and irreducible locally at P. Let $\nu : \widetilde{C}_\Lambda \to C_\Lambda$ be the normalization of C_Λ at P,*

and let z be a formal parameter of $\widetilde{C}_\Lambda$ at the point $P' \in \widetilde{C}_\Lambda$ such that $\nu(P') = P$.
Then the image of

$$(\pi_\Lambda \circ \nu)_{P'}^{*,\wedge} \; : \; (\mathcal{O}_{\Lambda,[f]})^\wedge \to (\mathcal{O}_{\widetilde{C}_\Lambda,P'})^\wedge$$

is contained in the subring

$$T := \{\, f = \sum f_\mu z^\mu \;\mid\; f_1 = f_2 = f_3 = f_5 = 0 \,\}$$

of $(\mathcal{O}_{\widetilde{C}_\Lambda,P'})^\wedge = k[[z]]$.

Proof. We use the same notation as in the proof of Proposition 6.9. The formal
completion at P' of the composite of $\nu : \widetilde{C}_\Lambda \to C_\Lambda$ and the inclusion $\iota : C_\Lambda \hookrightarrow S_L$
is given by

$$\begin{aligned}
(\iota \circ \nu)^*s &= a\,z^2 + b\,z^3 + \text{(terms of degree} \geq 4) \quad \text{and} \\
(\iota \circ \nu)^*t &= c\,z^2 + d\,z^3 + \text{(terms of degree} \geq 4),
\end{aligned}$$

where $a, b, c, d \in k$. Since Λ is general in L, the variables

$$u' := u \,|\, \Lambda \quad \text{and} \quad v' := v \,|\, \Lambda$$

form a formal parameter system of Λ at $[f]$. The formal completion at P' of the
morphism $\pi_\Lambda \circ \nu : \widetilde{C}_\Lambda \to \Lambda$ is given by

$$\begin{aligned}
(\pi_\Lambda \circ \nu)^*u' &= U(\, az^2 + bz^3 + \cdots, \; cz^2 + dz^3 + \cdots) \quad \text{and} \\
(\pi_\Lambda \circ \nu)^*v' &= V(\, az^2 + bz^3 + \cdots, \; cz^2 + dz^3 + \cdots).
\end{aligned}$$

Since $U^{[1]}$ and $V^{[1]}$ are both zero, and the coefficients of st in $U^{[2]}$ and $V^{[2]}$ are
also zero, we see that $(\pi_\Lambda \circ \nu)^*u'$ and $(\pi_\Lambda \circ \nu)^*v'$ are contained in the subring T of
$k[[z]]$. □

6.3. The case where the linear system is sufficiently ample.

Definition 6.11. For a point p of X and a positive integer d, we denote by

$$v_p^{[d]} \; : \; M \;\to\; \mathcal{L}_p/\mathfrak{m}_p^{d+1}\mathcal{L}_p$$

the evaluation homomorphism at p, where $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{O}_{X,p}$, and $\mathcal{L}_p$
is the $\mathcal{O}_{X,p}$-module $\mathcal{L} \otimes \mathcal{O}_{X,p}$.

Proposition 6.12. (1) *Suppose that* $v_p^{[2]}$ *is surjective at every point* p *of* X°. *Then*
$\mathcal{R}$ *is irreducible.*

(2) *Suppose that* $v_p^{[3]}$ *is surjective at every point* p *of* X°. *Then* $\mathcal{R}^{\mathrm{sm}}$ *is dense in*
$\mathcal{R}$. *Moreover, if* $P = (p, [f])$ *is a general point of* $\mathcal{R}$, *then the singularity of* $D_{[f]}$ *at*
p *is of type* $(\widetilde{\mathrm{A}})$ *and of type* (C).

(3) *Suppose that* $v_p^{[4]}$ *is surjective at every point* p *of* X°. *If* $P = (p, [f])$ *is a*
general point of $\mathcal{R}$, *then the singularity of* $D_{[f]}$ *at* p *is of type* (R).

Proof. Considering the first projection $\mathcal{R} \to X^\circ$, we deduce the statements of
Proposition from Remark 3.10 and Theorem 6.4 (1). □

Combining Proposition 6.12 with the results proved so far, we obtain the facts
(I)-(IV) stated in Introduction.

7. Divisibility of a Chern number

We assume that $\overline{X}$ is smooth and $\mathcal{L}$ is very ample, and put $|M| = |\mathcal{L}|$. Then we have $X^\circ = X = \overline{X}$ and $\mathcal{C} = \mathcal{C}^\circ$. We further assume that $\mathcal{R}^{\mathrm{sm}}$ is dense in $\mathcal{R}$. Under these assumptions, we calculate

$$\deg \mathcal{R} := \int_{X \times \mathbf{P}} c_1(\mathrm{pr}_2^* \, \mathcal{O}_{\mathbf{P}}(1))^{m-2} \cap [\mathcal{R}].$$

We put

$$\lambda := c_1(\mathrm{pr}_1^* \, \mathcal{L}) \quad \text{and} \quad h := c_1(\mathrm{pr}_2^* \, \mathcal{O}_{\mathbf{P}}(1)).$$

Then we have

$$c_1(\widetilde{\mathcal{L}}) = h + \lambda.$$

For simplicity, we write $c_i(X)$ for $\mathrm{pr}_1^* \, c_i(X)$. By Proposition 5.2 and Thom-Porteous formula [3, Chapter 14], we have

$$[\mathcal{C}] = \left((\lambda + h) \sum_{i=0}^{n} (-1)^i c_i(X)(\lambda + h)^{n-i} \right) \cap [X \times \mathbf{P}]$$

in the Chow group of $X \times \mathbf{P}$. By Theorem 2.14, we have

$$[\mathcal{R}] = \left(\frac{n}{2} (h + \lambda) - c_1(X) \right) \cap [\mathcal{C}].$$

We put

$$G := \left(\frac{n}{2} (h + \lambda) - c_1(X) \right) \left((\lambda + h) \sum_{i=0}^{n} (-1)^i c_i(X)(\lambda + h)^{n-i} \right).$$

Then we obtain

$$\deg \mathcal{R} = \int_X (\text{the coefficient of } h^2 \text{ in } G) \cap [X].$$

By Corollary 6.6, the integer $\deg \mathcal{R}$ is divisible by 4. If the coefficient of h^2 in G is regarded as a polynomial of λ, then the constant term is equal to $(-1)^n(n \, c_n(X)/2 + c_1(X)c_{n-1}(X))$. Putting $\mathcal{L} = \mathcal{A}^{\otimes 4}$ with $\mathcal{A}$ a sufficiently ample line bundle, we obtain the following:

Corollary 7.1. *Let X be a smooth projective variety in characteristic 2 of dimension n being even. Then the integer*

$$\int_X \left(\frac{n}{2} \, c_n(X) + c_1(X)c_{n-1}(X) \right) \cap [X]$$

is divisible by 4.

In fact, this divisibility relation follows from the Hirzebruch-Riemann-Roch theorem by the argument of Libgober and Wood. See [15, Remark 2.4].

References

[1] David Cox, John Little, and Donal O'Shea, *Ideals, varieties, and algorithms*, second ed., Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1997, An introduction to computational algebraic geometry and commutative algebra.

[2] Igor Dolgachev and Anatoly Libgober, *On the fundamental group of the complement to a discriminant variety*, Algebraic geometry (Chicago, Ill., 1980), Lecture Notes in Math., vol. 862, Springer, Berlin, 1981, pp. 1–25.

[3] William Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Gren-
zgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and
Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-
Verlag, Berlin, 1998.

[4] Joe Harris and Loring W. Tu, *On symmetric and skew-symmetric determinantal varieties*,
Topology **23** (1984), no. 1, 71–84.

[5] A. Hefez and A. Thorup, *Reflexivity of Grassmannians and Segre varieties*, Comm. Algebra
15 (1987), no. 6, 1095–1108.

[6] Abramo Hefez, *Nonreflexive curves*, Compositio Math. **69** (1989), no. 1, 3–35.

[7] Abramo Hefez and Steven L. Kleiman, *Notes on the duality of projective varieties*, Geometry
today (Rome, 1984), Progr. Math., vol. 60, Birkhäuser Boston, Boston, MA, 1985, pp. 143–
183.

[8] Masaaki Homma, *On duals of smooth plane curves*, Proc. Amer. Math. Soc. **118** (1993),
no. 3, 785–790.

[9] Hajime Kaji, *On the Gauss maps of space curves in characteristic p*, Compositio Math. **70**
(1989), no. 2, 177–197.

[10] ———, *On the duals of Segre varieties*, Geom. Dedicata **99** (2003), 221–229.

[11] Hajime Kaji and Atsushi Noma, *On the generic injectivity of the Gauss map in positive
characteristic*, J. Reine Angew. Math. **482** (1997), 215–224.

[12] N. M. Katz, *Pinceaux de Lefschetz: théorème d'existence*, Groupes de monodromie en
géométrie algébrique. II, Springer-Verlag, Berlin, 1973, Séminaire de Géométrie Algébrique
du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz, Lecture Notes in
Mathematics, Vol. 340, pp. 212–253.

[13] Steven L. Kleiman, *About the conormal scheme*, Complete intersections (Acireale, 1983),
Lecture Notes in Math., vol. 1092, Springer, Berlin, 1984, pp. 161–197.

[14] ———, *Tangency and duality*, Proceedings of the 1984 Vancouver conference in algebraic
geometry (Providence, RI), CMS Conf. Proc., vol. 6, Amer. Math. Soc., 1986, pp. 163–225.

[15] Anatoly S. Libgober and John W. Wood, *Uniqueness of the complex structure on Kähler
manifolds of certain homotopy types*, J. Differential Geom. **32** (1990), no. 1, 139–154.

[16] Rita Pardini, *Some remarks on plane curves over fields of finite characteristic*, Compositio
Math. **60** (1986), no. 1, 3–17.

[17] Piotr Pragacz, *Cycles of isotropic subspaces and formulas for symmetric degeneracy loci*,
Topics in algebra, Part 2 (Warsaw, 1988), Banach Center Publ., vol. 26, PWN, Warsaw,
1990, pp. 189–199.

[18] C. S. Seshadri, *L'opéation de Cartier. Applications*, Séminaire C. Chevalley, 3ième année:
1958/59. Variétés de Picard, École Normale Supérieure, Paris, 1960, pp. 83–135.

[19] Ichiro Shimada, *Singularities of dual varieties in characteristic* 3, to appear in Geom. Dedi-
cata.

[20] ———, *On supercuspidal families of curves on a surface in positive characteristic*, Math.
Ann. **292** (1992), no. 4, 645–669.

[21] ———, *Lattices of algebraic cycles on Fermat varieties in positive characteristics*, Proc.
London Math. Soc. (3) **82** (2001), no. 1, 131–172.

[22] Andrew H. Wallace, *Tangency and duality over arbitrary fields*, Proc. London Math. Soc. (3)
6 (1956), 321–342.

Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-
0810, JAPAN

E-mail address: `shimada@math.sci.hokudai.ac.jp`