ON THE CONNECTED COMPONENTS OF THE MODULI OF POLARIZED $K3$ SURFACES

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1. Introduction

This is a note of my understanding of the paper

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and the table of maximizing sextics and their transcendenrtal lattices.

We work over the complex number field $\mathbb{C}$.

Let X be a $K3$ surface. A line bundle $\mathcal{L}$ on X is called a *polarization* if $\mathcal{L}$ is nef, $\mathcal{L}^2 > 0$, and the complete linear system $|\mathcal{L}|$ has no fixed components. A pair $(X, \mathcal{L})$ of a $K3$ surface X and a polarization $\mathcal{L}$ on X is called a *polarized* $K3$ *surface.* If $(X, \mathcal{L})$ is a polarized $K3$ surface, then $|\mathcal{L}|$ is base-point free by Saint-Donat [25, Corollary 3.2], and hence $|\mathcal{L}|$ defines a morphism $\Phi_{|\mathcal{L}|}$ from X to a projective space of dimension $N := \dim |\mathcal{L}| = \mathcal{L}^2/2 + 1$. (See Nikulin [15, Proposition 0.1].) Let

$$X \longrightarrow Y_{|\mathcal{L}|} \longrightarrow \mathbb{P}^N$$

be the Stein factorization of $\Phi_{|\mathcal{L}|}$. It is known that the normal $K3$ surface $Y_{|\mathcal{L}|}$ has only rational double points as its singularities. (See Artin [5, 6].) We say that $(X, \mathcal{L})$ *is of type* (R, n), where R is the Dynkin type of the rational double points on $Y_{|\mathcal{L}|}$, and $n = \mathcal{L}^2$ is the degree of $(X, \mathcal{L})$.

In this note, we describe the connected components of the moduli of polarized $K3$ surfaces of a given type (R, n) in purely lattice-theoretic terms. The main result is stated in Section 3.

As an application, we show that the moduli of polarized $K3$ surfaces of type $(6A_2, 2)$ has exactly two connected components. This fact implies the following theorem:

Theorem 1.1. *Let* $\mathcal{M}(6A_2) \subset \mathbb{P}_*(H^0(\mathbb{P}^2, \mathcal{O}(6)))$ *be the space of reduced projective plane curves of degree* 6 *with six ordinary cusps as their only singularities. Then* $\mathcal{M}(6A_2)$ *has exactly two connected components.*

The fact that $\mathcal{M}(6A_2)$ is not connected was classically observed by Zariski [34] and Oka [16]. Let us call members of $\mathcal{M}(6A_2)$ *six-cuspidal sextic curves.* Zariski considered a six-cuspidal sextic curve C with the six cusps lying on a conic, which is obtained as the branch curve of a generic projection of a general cubic surface. He showed that $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$ of cyclic groups of order 2 and 3. He also showed that, if there exists a six-cuspidal sextic curve C' with the six cusps *not* lying on a conic, then $\pi_1(\mathbb{P}^2 \setminus C')$ is not isomorphic to

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$\pi_1(\mathbb{P}^2 \setminus C)$. Oka completed Zariski's work by constructing explicitly such a six-cuspidal sextic curve C', and showed that $\pi_1(\mathbb{P}^2 \setminus C')$ is a cyclic group of order 6. Therefore $\mathcal{M}(6A_2)$ has *at least* two connected components that are distinguished by the fundamental groups of the complements. (See [27] for a simple construction of the pair (C, C').) The precise number of the connected components of $\mathcal{M}(6A_2)$, however, seems to have been unknown.

As another application, we investigate the moduli of maximizing sextics by using computer.

Definition 1.2. We say that a reduced plane curve $C \subset \mathbb{P}^2$ of degree 6 is a *maximizing sextic* if C has only simple singularities and the total Milnor number of C attains the possible maximum 19. The *Dynkin type of a maximizing sextic* C is the Dynkin type of the singularities of C.

At the end of this note, we give a table (Table MS) of maximizing sextic and their transcendental lattices. See Section 9 for the explanation of other entries of Table MS.

Many studies have been done about the (non-)connectedness of the moduli of sextic curves with a prescribed type of singularities, mainly from the view point of fundamental groups of complements, or from its variations like Alexander polynomials, or branched coverings of $\mathbb{P}^2$. See, for example, Artal Bartolo [1], Artal Bartolo, Carmona Ruber and Cogolludo Agustín [2], Artal Bartolo, Carmona Ruber, Cogolludo Agustín and Tokunaga [3], Artal Bartolo and Tokunaga [4]. Using the notion of *torus decomposition* introduced by Kulikov [13], Oka studied the fundamental groups of the complements and the moduli of sextic curves intensively (Eyral and Oka [11], Oka [17, 18, 19, 20, 21], Oka and Pho [22, 23]). On the other hand, sextic curves with at most simple singularities have been studied by the theory of $K3$ surfaces. See, for example, Persson [24] and Tokunaga [30, 31]. In particular, using Urabe's idea [32], Yang [33] made the complete list of Dynkin types of sextic curves with at most simple singularities.

According to Yang's list [33], there exist 519 Dynkin types of maximizing sextics. The space $\mathcal{M}(R)$ is not connected for 208 Dynkin types among them. For many Dynkin types (for example, $E_8+E_7+A_4$) of these 208 types, the non-connectedness of $\mathcal{M}(R)$ comes from the fact that the period domain

$$\Omega_N := \{\, [\omega] \in \mathbb{P}_*(N \otimes \mathbb{C}) \;|\; (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0 \,\}$$

of a lattice N with signature $(2, t_-)$ has two connected components.

The plan of this note is as follows. After fixing notions and notation about lattices in Section 2, we state Main Theorem in Section 3. First we define the set $\mathrm{Conn}(R, n)$ of connected components of the moduli of polarized $K3$ surfaces of type (R, n). Then we define a set $Q(R, n)/\!\sim$ that can be calculated from R and n by purely lattice-theoretic means. Main Theorem states that there exists a natural bijection between $\mathrm{Conn}(R, n)$ and $Q(R, n)/\!\sim$. In Section 4, we state some well-known facts about the Kähler cone of a $K3$ surface. In Section 5, we review the theory of the refined period map of marked $K3$ surfaces, which plays a central role in the proof of Main Theorem in Section 6. In Section 7, we explain how to calculate the set $Q(R, n)/\!\sim$, and present a theorem of Nikulin [14] that simplifies the calculation of $Q(R, n)/\!\sim$. Using this result, we prove in Section 8 that the cardinality of $Q(6A_2, 2)/\sim$ is two. In Section 9, we explain how to make Table MS, and investigate several examples discovered by Artal Bartolo, Carmona Ruber and Cogolludo Agustín [2], and by Oka and Pho [22].

In order to increase the legibility of this note, we use the following rule in fixing notation. Geometric objects associated with a $K3$ surface X are denoted with *subscript* like $H_X, S_X, \Gamma_X, \dots$, and corresponding lattice-theoretic objects associated with a point $[\omega]$ of the period domain are denoted with *superscript* like $H^{[\omega]}, S^{[\omega]}, \Gamma^{[\omega]}, \dots$.

2. Notions and terminologies about lattices

In this section, we summarize notions and terminologies about lattices that are necessary to state Main Theorem.

A *lattice* is a free $\mathbb{Z}$-module Λ of finite rank equipped with a non-degenerate symmetric bilinear form

$$(\ ,\ )\ :\ \Lambda \times \Lambda \to \mathbb{Z}.$$

The group of isometries of a lattice Λ is denoted by $O(\Lambda)$. We use the same symbol $(\ ,\ )$ to denote the bilinear forms

$$(\Lambda \otimes k) \times (\Lambda \otimes k) \to k$$

induced on the k-vector spaces $\Lambda \otimes k$ for $k = \mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. Let $\varphi : \Lambda \to \Lambda'$ be a homomorphism of lattices. We use the same letter φ to denote the linear maps $\Lambda \otimes k \to \Lambda' \otimes k$ induced from φ for $k = \mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

The *primitive closure* of a submodule S of a lattice Λ is the submodule $(S \otimes \mathbb{Q}) \cap \Lambda$ of Λ, where the intersection is taken in $\Lambda \otimes \mathbb{Q}$. A submodule $S \subset \Lambda$ is called *primitive* if $(S \otimes \mathbb{Q}) \cap \Lambda = S$ holds. The orthogonal complement $S^\perp$ of S is defined by

$$S^\perp := \{\ x \in \Lambda\ \mid\ (x, y) = 0\ \text{ for all }\ y \in S\ \}.$$

Note that $S^\perp$ is always primitive.

The *signature* of a lattice Λ is (t_+, t_-), where t_+ and t_- are the numbers of positive and negative eigenvalues of the symmetric matrix expressing the symmetric bilinear form $(\ ,\ )$. A lattice Λ is called *positive-definite* (resp. *negative-definite*) if $t_- = 0$ (resp. $t_+ = 0$).

Let Λ be a lattice of rank $n = t_+ + t_-$ and signature (t_+, t_-) with $t_+ \geq 2$. For a non-zero vector $\omega \in \Lambda \otimes \mathbb{C}$, we denote by $[\omega] \in \mathbb{P}_*(\Lambda \otimes \mathbb{C})$ the one-dimensional vector space spanned by ω. We put

$$\Omega_\Lambda := \{\ [\omega] \in \mathbb{P}_*(\Lambda \otimes \mathbb{C})\ \mid\ (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\ \},$$

and call it the *period domain of* Λ. Then Ω_Λ is an open subset of a smooth quadratic hypersurface of $\mathbb{P}_*(\Lambda \otimes \mathbb{C})$, and hence is a complex manifold of dimension $n - 2$. The following proposition is easy to prove:

Proposition 2.1. *The complex manifold Ω_Λ is connected if $t_+ > 2$, while it has exactly two connected components if $t_+ = 2$.*

The *dual lattice* $\Lambda^\vee$ of a lattice Λ is defined by

$$\Lambda^\vee := \{\ v \in \Lambda \otimes \mathbb{Q}\ \mid\ (x, v) \in \mathbb{Z}\ \text{ for all }\ x \in \Lambda\ \}.$$

We have $\Lambda \subset \Lambda^\vee$. An *overlattice* of Λ is a submodule Λ' of $\Lambda^\vee$ containing Λ such that the natural $\mathbb{Q}$-valued symmetric bilinear form on $\Lambda \otimes \mathbb{Q}$ takes values in $\mathbb{Z}$ on Λ'. The *discriminant group* G_Λ of Λ is defined by

$$G_\Lambda := \Lambda^\vee / \Lambda.$$

A lattice is called *unimodular* if $\Lambda^\vee = \Lambda$.

A lattice Λ is said to be *even* if $(v, v) \in 2\,\mathbb{Z}$ holds for every $v \in \Lambda$. If Λ is an even lattice, we can define a quadratic form

$$q_\Lambda \;:\; G_\Lambda \to \mathbb{Q}/2\,\mathbb{Z}$$

by $q_\Lambda(\bar{v}) := (v, v) \bmod 2\,\mathbb{Z}$, where $v \in \Lambda^\vee$ and $\bar{v} := v \bmod \Lambda$. This quadratic form is called the *discriminant form* of Λ. (See Nikulin [14].) The automorphism group of a quadratic form $q : G \to \mathbb{Q}/2\mathbb{Z}$ on a finite abelian group G is denoted by $O(q)$. For an even lattice Λ, we have a natural homomorphism $O(\Lambda) \to O(q_\Lambda)$, which is denoted by $g \mapsto \bar{g}$.

Let Λ be a (positive- or negative-)definite even lattice. A vector $d \in \Lambda$ is called a *root* if $|(d, d)| = 2$ holds. We say that Λ is a *root lattice* if Λ is generated by the roots in Λ. The isomorphism classes of root lattices (of a fixed sign) are in one-to-one correspondence with the *Dynkin types*

$$R = \sum_{l \ge 1} a_l A_l + \sum_{m \ge 4} d_m D_m + \sum_{n=6}^{8} e_n E_n,$$

where a_l, d_m and e_n are non-negative integers, almost all of which are zero. (See, for example, Ebeling [10, Chapter 1.4].) We denote by Σ_R^+ and Σ_R^- the positive- and the negative-definite root lattices of Dynkin type R. The rank of R is defined to be the rank of Σ_R^+ or Σ_R^-:

$$\operatorname{rank}(R) = \sum a_l l + \sum d_m m + \sum e_n n.$$

Let Λ be an even definite lattice. We denote by D_Λ the set of roots in Λ, and by $\Sigma \subset \Lambda$ the root sublattice of Λ generated by D_Λ. A subset F of D_Λ is called a *fundamental system of roots in* Λ if F is a basis of Σ and each $v \in D_\Lambda$ is written as a linear combination $v = \sum_{d \in F} k_d d$ of elements d of F with integer coefficients k_d all non-positive or all non-negative. A fundamental system F of roots exists whenever $D_\Lambda \neq \emptyset$. (See Ebeling [10, Chapter 1.4] or Section 4 of this note.) If the root lattice Σ is of type R, then the intersection matrix of Σ with respect to the basis F is the Cartan matrix of type R (multiplied by -1 if Σ is negative-definite).

A lattice is called a $K3$ *lattice* if it is even, unimodular, of rank 22 and with signature $(3, 19)$. By the structure theorem of unimodular lattices, a $K3$ lattice is unique up to isometries. (See, for example, Serre [26, Chapter V].) Let X be a $K3$ surface. We put

$$L_X := H^2(X, \mathbb{Z}).$$

By the cup product, we regard L_X as a lattice. Then L_X is a $K3$ lattice.

3. Main Theorem

Let R be a Dynkin type of rank ≤ 19, and n an even positive integer.

First we define the set $\mathrm{Conn}(R,n)$ of connected components of the moduli of polarized $K3$ surfaces of type (R,n).

Definition 3.1. Let $(X,\mathcal{L})$ and $(X',\mathcal{L}')$ be polarized $K3$ surfaces. An isomorphism $f:(X,\mathcal{L})\xrightarrow{\sim}(X',\mathcal{L}')$ of polarized $K3$ surfaces is an isomorphism $f:X\xrightarrow{\sim}X'$ of complex surfaces such that $f^*\mathcal{L}'\cong\mathcal{L}$ holds.

It is obvious that isomorphic polarized $K3$ surfaces have the same type.

Definition 3.2. A *family of polarized* $K3$ *surfaces of type* (R,n) is a pair $(\pi:\mathcal{X}\to B,\mathcal{L}_{\mathcal{X}})$, where $\pi:\mathcal{X}\to B$ is a smooth analytic family of $K3$ surfaces over a complex manifold B, and $\mathcal{L}_{\mathcal{X}}$ is a line bundle on $\mathcal{X}$ such that, for each point $t\in B$, the restriction $\mathcal{L}_t$ of $\mathcal{L}_{\mathcal{X}}$ to the fiber $X_t:=\pi^{-1}(t)$ gives rise to a polarized $K3$ surface $(X_t,\mathcal{L}_t)$ of type (R,n).

Definition 3.3. Let $(X,\mathcal{L})$ and $(X',\mathcal{L}')$ be polarized $K3$ surfaces of type (R,n). We write $(X,\mathcal{L})\approx_{\mathrm{conn}}(X',\mathcal{L}')$ if there exists a family of polarized $K3$ surfaces of type (R,n) with a connected base space that contains two fibers isomorphic to $(X,\mathcal{L})$ and $(X',\mathcal{L}')$. We then write $(X,\mathcal{L})\sim_{\mathrm{conn}}(X',\mathcal{L}')$ if there exists a finite sequence

$$(X,\mathcal{L})=(X_0,\mathcal{L}_0),\ \ (X_1,\mathcal{L}_1),\ \ \dots,\ \ (X_N,\mathcal{L}_N)=(X',\mathcal{L}')$$

of polarized $K3$ surfaces of type (R,n) such that $(X_{i-1},\mathcal{L}_{i-1})\approx_{\mathrm{conn}}(X_i,\mathcal{L}_i)$ holds for $i=1,\dots,N$. It is obvious that $\sim_{\mathrm{conn}}$ is an equivalence relation.

Definition 3.4. We define *the connected components of the moduli of polarized* $K3$ *surfaces of type* (R,n) to be the equivalence classes of the relation $\sim_{\mathrm{conn}}$, and denote the set of these connected components by $\mathrm{Conn}(R,n)$.

Next we define a set $Q(R,n)$ and an equivalence relation $\sim$ on it. They are described in purely lattice-theoretic terms. Let r be the rank of R. We denote by $\langle h\rangle$ the lattice of rank 1 generated by a vector h with $(h,h)=n$. We put

$$M^0:=\Sigma_R^-\oplus\langle h\rangle,$$

which is an even lattice of signature $(1,r)$. *We choose a fundamental system of roots* $F\subset\Sigma_R^-$ *once and for all*, and put

$$\mathrm{Aut}(F,h):=\{\ g\in O(M^0)\ \mid\ g(F)=F, g(h)=h\ \}.$$

Note that $\mathrm{Aut}(F,h)$ is isomorphic to the automorphism group of the Dynkin diagram of type R. We denote by Ms the set of even overlattices M of M^0 satisfying the following two conditions:

(m1) $\{\ v\in M\ \mid\ (v,h)=1,(v,v)=0\ \}=\emptyset$, and
(m2) $\{\ v\in M\ \mid\ (v,h)=0,(v,v)=-2\ \}=\{\ v\in\Sigma_R^-\ \mid\ (v,v)=-2\ \}$.

The condition (m1) is called the *polarization condition*, and the condition (m2) is called the *no-new-roots condition.* See Section 4 for the reason of this naming. For $M\in Ms$, we denote by $Ns(M)$ a complete set of representatives of isomorphism classes of even lattices N of rank $21-r$ satisfying the following two conditions:

(n1) N is of signature $(2,19-r)$, and
(n2) the discriminant form (G_N,q_N) of N is isomorphic to $(G_M,-q_M)$.

By Nikulin [14, Proposition 1.6.1], the conditions (n1) and (n2) are equivalent to the following condition:

(n) there exists an even unimodular overlattice L of $M \oplus N$ with signature $(3, 19)$ such that M and N are primitive in L.

Let N be an element of $Ns(M)$. We denote by $Ls(M, N)$ the set of even unimodular overlattices L of $M \oplus N$ such that M and N are primitive in L. Note that every $L \in Ls(M, N)$ is a $K3$ lattice. If $N \in Ns(M)$, the complex manifold Ω_N has exactly two connected components. We denote by $c\Omega s(N)$ the set of connected components of Ω_N.

We define $Q(R, n)$ to be the set of quartets $(M, N, L, c\Omega)$ such that $M \in Ms$, $N \in Ns(M)$, $L \in Ls(M, N)$, and $c\Omega \in c\Omega s(N)$. For quartets $(M, N, L, c\Omega)$ and $(M', N', L', c\Omega')$ in $Q(R, n)$, we write

$$(M, N, L, c\Omega) \;\sim\; (M', N', L', c\Omega')$$

if the following hold.

(i) There exists $g^0 \in \mathrm{Aut}(F, h) \subset O(M^0)$ such that the induced action of g^0 on Ms maps $M \in Ms$ to $M' \in Ms$. We denote by $g_M : M \xrightarrow{\sim} M'$ the unique isometry satisfying $g_M | M^0 = g^0$.

(ii) Since $(G_M, -q_M)$ and $(G_{M'}, -q_{M'})$ are isomorphic, there exists a canonical bijection between $Ns(M)$ and $Ns(M')$. The elements $N \in Ns(M)$ and $N' \in Ns(M')$ are corresponding by this bijection; that is, N and N' are isometric.

(iii) There exists an isometry $g_N : N \xrightarrow{\sim} N'$ such that the bijection $Ls(M, N) \xrightarrow{\sim} Ls(M', N')$ induced by the isometry

$$g_M \oplus g_N : M \oplus N \xrightarrow{\sim} M' \oplus N'$$

maps $L \in Ls(M, N)$ to $L' \in Ls(M', N')$, and that the induced isomorphism $\Omega_N \xrightarrow{\sim} \Omega_{N'}$ maps $c\Omega$ to $c\Omega'$.

For $(M, N, L, c\Omega) \in Q(R, n)$, we denote by $[M, N, L, c\Omega] \in Q(R, n)/\!\sim$ the equivalence class of the relation $\sim$ containing $(M, N, L, c\Omega)$.

Remark 3.5. If $(M, N, L, c\Omega) \in Q(R, n)$, then M^0 is a sublattice of L generated by $F \subset L$ and $h \in L$, M is the primitive closure of M^0 in L, and N is the orthogonal complement of M in L. Hence $[M, N, L, c\Omega] = [M', N', L', c\Omega']$ holds if and only if there exists an isometry $L \xrightarrow{\sim} L'$ that maps F to F, h to h, and such that the induced isomorphism $\Omega_L \xrightarrow{\sim} \Omega_{L'}$ maps the connected component $c\Omega$ of $\Omega_N \subset \Omega_L$ to the connected component $c\Omega'$ of $\Omega_{N'} \subset \Omega_{L'}$.

Thirdly, we define a map ρ from the set of polarized $K3$ surfaces of type (R, n) to the set $Q(R, n)/\!\sim$. As before, we put $L_X := H^2(X, \mathbb{Z})$. Let $F_{(X,\mathcal{L})} \subset L_X$ be the set of cohomology classes of (-2)-curves contracted by the birational morphism $X \to Y_{|\mathcal{L}|}$, and let $\Sigma_{(X,\mathcal{L})} \subset L_X$ be the sublattice of L_X generated by $F_{(X,\mathcal{L})}$. Then $\Sigma_{(X,\mathcal{L})}$ is a negative-definite root lattice of type R. It is known that $F_{(X,\mathcal{L})}$ is a fundamental system of roots in $\Sigma_{(X,\mathcal{L})}$. (See Proposition 4.8 in Section 4.) In particular, there exists an isometry from $\Sigma_{(X,\mathcal{L})}$ to Σ_R^- that maps $F_{(X,\mathcal{L})}$ to F bijectively. We put

$$M^0_{(X,\mathcal{L})} := \Sigma_{(X,\mathcal{L})} \oplus \langle [\mathcal{L}] \rangle,$$

and choose an isometry

$$\gamma^0_M : M^0_{(X,\mathcal{L})} \xrightarrow{\sim} M^0$$

satisfying $\gamma^0_M(F_{(X,\mathcal{L})})=F$ and $\gamma^0_M([\mathcal{L}])=h$. Let $M_{(X,\mathcal{L})}$ be the primitive closure of $M^0_{(X,\mathcal{L})}$ in L_X, and M the even overlattice of M^0 corresponding to the even overlattice $M_{(X,\mathcal{L})}$ of $M^0_{(X,\mathcal{L})}$ by γ^0_M. Then M satisfies the conditions (m1) and (m2). (See Proposition 4.7 and Corollary 4.9 in Section 4.) Hence $M\in Ms$. Let $N_{(X,\mathcal{L})}$ the orthogonal complement of $M_{(X,\mathcal{L})}$ in L_X. Since the $K3$ lattice L_X is an even unimodular overlattice of $M_{(X,\mathcal{L})}\oplus N_{(X,\mathcal{L})}$ in which $M_{(X,\mathcal{L})}$ and $N_{(X,\mathcal{L})}$ are primitive, the lattice $N_{(X,\mathcal{L})}$ satisfies the condition (n). Hence there exists a unique element N of $Ns(M)$ that is isometric to $N_{(X,\mathcal{L})}$. Let

$$\gamma_M : M_{(X,\mathcal{L})} \xrightarrow{\sim} M$$

be the isometry induced by γ^0_M. We choose an isometry

$$\gamma_N : N_{(X,\mathcal{L})} \xrightarrow{\sim} N.$$

By the isometry

$$\gamma_M\oplus\gamma_N : M_{(X,\mathcal{L})}\oplus N_{(X,\mathcal{L})} \xrightarrow{\sim} M\oplus N,$$

the even unimodular overlattice L_X of $M_{(X,\mathcal{L})}\oplus N_{(X,\mathcal{L})}$ corresponds to an element L of $Ls(M,N)$. We denote by

$$\omega_X\in H^{2,0}(X)\subset L_X\otimes\mathbb{C}$$

the cohomology class of a non-zero holomorphic 2-form on X. Since $M_{(X,\mathcal{L})}\subset H^{1,1}(X)$, the vector ω_X defines a point $[\omega_X]$ of $\Omega_{N_{(X,\mathcal{L})}}$. Let $c\Omega$ be the connected component of Ω_N that contains the point $[\gamma_N(\omega_X)]$. Thus we obtain a quartet $(M,N,L,c\Omega)$. The choices we have made during the process of finding $(M,N,L,c\Omega)$ are only on γ^0_M and γ_N. Since γ^0_M is unique up to $\mathrm{Aut}(F,h)$ and γ_N is unique up to $O(N)$, the equivalence class $[M,N,L,c\Omega]$ does not depend on these choices. We thus can put

$$\rho(X,\mathcal{L}):=[M,N,L,c\Omega].$$

Remark 3.6. By definition, we have $\rho(X,\mathcal{L})=[M,N,L,c\Omega]$ if and only if there exists an isometry $L_X\xrightarrow{\sim}L$ that maps $F_{(X,\mathcal{L})}$ to F, $[\mathcal{L}]$ to h, and such that the induced isomorphism $\Omega_{L_X}\xrightarrow{\sim}\Omega_L$ maps the point $[\omega_X]\in\Omega_{N_{(X,\mathcal{L})}}\subset\Omega_{L_X}$ to a point of the connected component $c\Omega$ of $\Omega_N\subset\Omega_L$.

We now have all the ingredients that are needed to state Main Theorem.

Main Theorem. *The map ρ induces a bijection from* $\mathrm{Conn}(R,n)$ *to* $Q(R,n)/{\sim}$.

4. The Kähler cone of a $K3$ surface

In this section, we review some well-known results about the Kähler cone of a $K3$ surface.

First we introduce a notion from the lattice theory. Let Λ be an even definite lattice. As in Section 2, we denote by D_Λ the set of roots in Λ, and by $\Sigma\subset\Lambda$ the root sublattice of Λ generated by D_Λ. We assume that $D_\Lambda\neq\emptyset$. Let $t:\Lambda\to\mathbb{R}$ be a linear form such that $t(d)\neq 0$ for any $d\in D_\Lambda$. We put

$$(D_\Lambda)^+_t:=\{\ d\in D_\Lambda\ \mid\ t(d)>0\ \}.$$

An element $d\in(D_\Lambda)^+_t$ is called *decomposable* if there exist vectors $d_1,d_2\in(D_\Lambda)^+_t$ such that $d=d_1+d_2$; otherwise, we call d *indecomposable*. The following proposition is proved, for example, in Ebeling [10, Proposition 1.4].

Proposition 4.1. *The set F_t of indecomposable elements in $(D_\Lambda)^+_t$ is a fundamental system of roots in Λ. Conversely, if F is a fundamental system of roots in Λ, then there exists a linear form $t' : \Lambda \to \mathbb{R}$ such that $t'(d) \neq 0$ for any $d \in D_\Lambda$, and that F is equal to the set $F_{t'}$ of indecomposable elements in $(D_\Lambda)^+_{t'}$.*

We call F_t the *fundamental system of roots associated with* $t : \Lambda \to \mathbb{R}$. We put

$$^0(\Lambda \otimes \mathbb{R}) := \{\, x \in \Lambda \otimes \mathbb{R} \;\mid\; (x, d) \neq 0 \text{ for all } d \in D_\Lambda \,\}.$$

For each $x \in {}^0(\Lambda \otimes \mathbb{R})$, we obtain a fundamental system of roots associated with a linear form $v \mapsto (v, x)$.

Corollary 4.2. *There exists a one-to-one correspondence between the set of fundamental systems of roots in Λ and the set of connected components of $^0(\Lambda \otimes \mathbb{R})$.*

Let X be a $K3$ surface. Recall that we put $L_X := H^2(X, \mathbb{Z})$, and let ω_X be a basis of the 1-dimensional $\mathbb{C}$-vector space $H^{2,0}(X)$. Then we obtain a point $[\omega_X]$ of Ω_{L_X}. We put

$$\begin{array}{rcl} H_X & := & \{\, x \in L_X \otimes \mathbb{R} \;\mid\; (x, \omega_X) = 0 \,\}, \\ \Gamma_X & := & \{\, x \in H_X \;\mid\; (x, x) > 0 \,\}, \\ S_X & := & H_X \cap L_X, \\ D_X & := & \{\, d \in S_X \;\mid\; (d, d) = -2 \,\}, \\ {}^0\Gamma_X & := & \{\, x \in \Gamma_X \;\mid\; (x, d) \neq 0 \text{ for all } d \in D_X \,\}. \end{array}$$

Then we have $H_X = H^{1,1}(X) \cap H^2(X, \mathbb{R})$. The space Γ_X is a disjoint union of two cones Γ_X^+ and $-\Gamma_X^+$, where Γ_X^+ be the connected component of Γ_X that contains a Kähler class of X. The lattice S_X is the Picard lattice of X.

Definition 4.3. An element $v \in S_X$ is called *effective* if v is the cohomology class of an effective divisor on X. An element $v \in S_X$ is called *nef* if $(v', v) \geq 0$ holds for any effective element $v' \in S_X$. The *Kähler cone* K_X of X is the set of vectors $\kappa \in H_X$ satisfying $(v', \kappa) > 0$ for any effective element $v' \in S_X$.

Remark 4.4. Every Kähler class on X is contained in K_X. The converse is in fact also true: every vector in K_X is a Kähler class on X. See Corollary 5.2.

The following proposition is an immediate consequence of the definition.

Proposition 4.5. *A vector $v \in S_X$ is nef if and only if v is contained in the closure of the Kähler cone K_X in H_X.*

We set

$$\Delta_X := \{\, d \in D_X \;\mid\; d \text{ is effective} \,\}.$$

By the theorem of Riemann-Roch, we have

$$D_X = \Delta_X \sqcup (-\Delta_X) \quad \text{(disjoint)}.$$

For $d \in D_X$, we put

$$d^\perp := \{\, x \in H_X \;\mid\; (x, d) = 0 \,\},$$

and call $d^\perp$ the *wall* associated with $d \in D_X$. The family of the walls $d^\perp$ associated with vectors $d \in D_X$ is locally finite in the cone Γ_X, and partitions Γ_X into the connected components of $^0\Gamma_X$. The following proposition is well-known. (See, for example, [7, Corollary 3.9, Chapter VIII]).

Proposition 4.6. *The Kähler cone K_X of X is a connected component of ${}^0\Gamma_X$. More precisely, the Kähler cone K_X is the unique connected component of $\Gamma_X^+ \cap {}^0\Gamma_X$ such that $(x, d) > 0$ holds for every $d \in \Delta_X$ and every $x \in K_X$. In particular, K_X is bounded by the walls $d^\perp$ associated with $d \in \Delta_X$.*

Next we characterize the cohomology classes of polarizations.

Proposition 4.7. *An element v of S_X is the cohomology class of a polarization if and only if $(v, v) > 0$, v is nef, and the set*

$$\{\, e \in S_X \mid (v, e) = 1, (e, e) = 0 \,\}$$

is empty.

Proof. See Nikulin [15, Proposition 0.1], and the argument in the proof of (4)⇒(1) in Urabe [32, Proposition 1.7]. □

Suppose that there exists a polarization $\mathcal{L}$ on X. Then the orthogonal complement $\langle[\mathcal{L}]\rangle^\perp$ of $\langle[\mathcal{L}]\rangle$ in S_X is negative-definite. Let $D_{(X,\mathcal{L})}$ be the set of roots in $\langle[\mathcal{L}]\rangle^\perp$:

$$D_{(X,\mathcal{L})} := \langle[\mathcal{L}]\rangle^\perp \cap D_X.$$

Recall from Section 3 that $F_{(X,\mathcal{L})}$ is the set of cohomology classes of (-2)-curves that are contracted by the birational morphism $X \to Y_{|\mathcal{L}|}$. It is obvious that $F_{(X,\mathcal{L})} \subset D_{(X,\mathcal{L})}$.

Proposition 4.8. *The set $F_{(X,\mathcal{L})}$ is equal to the fundamental system of roots in $\langle[\mathcal{L}]\rangle^\perp$ associated with the linear form $t_\kappa : \langle[\mathcal{L}]\rangle^\perp \to \mathbb{R}$ given by $t_\kappa(v) := (v, \kappa)$, where κ is a vector in the Kähler cone K_X of X.*

Proof. We denote by $(D_{(X,\mathcal{L})})_\kappa^+$ the set of $d \in D_{(X,\mathcal{L})}$ such that $t_\kappa(d) > 0$, and by $F_\kappa \subset (D_{(X,\mathcal{L})})_\kappa^+$ the set of indecomposable elements in $(D_{(X,\mathcal{L})})_\kappa^+$. We will show that $F_{(X,\mathcal{L})} = F_\kappa$.

First note that, by the theorem of Riemann-Roch, an element d of $D_{(X,\mathcal{L})}$ is contained in $(D_{(X,\mathcal{L})})_\kappa^+$ if and only if d is effective; that is,

$$(D_{(X,\mathcal{L})})_\kappa^+ = \langle[\mathcal{L}]\rangle^\perp \cap \Delta_X.$$

Since every element of $F_{(X,\mathcal{L})}$ is effective, we have $F_{(X,\mathcal{L})} \subset (D_{(X,\mathcal{L})})_\kappa^+$. Suppose that $[E] \in F_{(X,\mathcal{L})}$ were decomposable in $(D_{(X,\mathcal{L})})_\kappa^+$, where E is a (-2)-curve contracted by $X \to Y_{|\mathcal{L}|}$. Then there would exist vectors $[D_1], [D_2] \in (D_{(X,\mathcal{L})})_\kappa^+$ with D_1 and D_2 being effective such that $[E] = [D_1] + [D_2]$. Then we would have $D_1 + D_2 \in |E|$, which is absurd, because $|E|$ consists of a single reduced irreducible member E. Therefore we have $F_{(X,\mathcal{L})} \subset F_\kappa$.

Conversely, let $[D_1], \dots, [D_m]$ be the elements of F_κ. Since $F_\kappa \subset (D_{(X,\mathcal{L})})_\kappa^+$, we can assume that $D_1, \dots, D_m$ are effective. We will show that each D_i is a (-2)-curve contracted by $X \to Y_{|\mathcal{L}|}$. Let $D_i = F_i + M_i$ be the decomposition of D_i into the sum of the fixed part F_i and the movable part M_i. Since $\mathcal{L}$ is nef, we have $F_i\mathcal{L} \geq 0$ and $M_i\mathcal{L} \geq 0$. Since $D_i\mathcal{L} = 0$, we have $F_i\mathcal{L} = 0$ and $M_i\mathcal{L} = 0$. In particular, $[M_i]$ is contained in the negative-definite lattice $\langle[\mathcal{L}]\rangle^\perp$. Therefore $M_i \neq 0$ would imply $M_i^2 < 0$, which contradicts the movability of M_i. Hence we have $M_i = 0$ and $D_i = F_i$. Consequently, each irreducible component of the reduced part of D_i is a (-2)-curve. Let $E_1, \dots, E_l$ be the irreducible components of the reduced part of D_i. Then we have $D_i = a_1E_1 + \cdots + a_lE_l$, where $a_1, \dots, a_l$ are positive integers. Since $\mathcal{L}$ is nef and $D_i\mathcal{L} = 0$, we have $E_1\mathcal{L} = \cdots = E_l\mathcal{L} = 0$, and hence the (-2)-curves $E_1, \ldots, E_l$ are contracted by $\Phi_{|\mathcal{L}|}$. Therefore $[E_1], \ldots, [E_l]$ are elements of $F_{(X,\mathcal{L})} \subset F_\kappa$. Thus, for each $k = 1, \ldots, l$, there exists j_k such that $[E_k] = [D_{j_k}]$. Then we have $[D_i] = a_1[D_{j_1}] + \cdots + a_l[D_{j_l}]$. Since the elements $[D_1], \ldots, [D_m]$ of F_κ form a basis of the root sublattice $\langle D_{(X,\mathcal{L})}\rangle$ of $\langle[\mathcal{L}]\rangle^\perp$ by the definition of the fundamental system of roots and $a_1, \ldots, a_l$ are positive integers, we must have $l = 1$ and $a_1 = 1$; that is, $D_i = E_1$. Hence $[D_i] \in F_{(X,\mathcal{L})}$ holds, and $F_\kappa \subset F_{(X,\mathcal{L})}$ is proved. □

Corollary 4.9. *The Dynkin type of the rational double points on* $Y_{|\mathcal{L}|}$ *is equal to the Dynkin type of the set of roots in the negative-definite sublattice* $\langle[\mathcal{L}]\rangle^\perp$ *of* S_X.

By Proposition 4.8, we can recover the geometric data $F_{(X,\mathcal{L})}$ by lattice-theoretic means if we know $[\mathcal{L}] \in L_X$, $[\omega_X] \in \Omega_{L_X}$ and the Kähler cone $K_X \subset {}^0\Gamma_X$. Conversely, combining all the results in this section, we have the following:

Corollary 4.10. *Let* $(X, \mathcal{L})$ *be a polarized* $K3$ *surface. Then the Kähler cone* K_X *of* X *is the unique connected component of* ${}^0\Gamma_X$ *that contains* $[\mathcal{L}]$ *in the closure and satisfies* $(d, \kappa) > 0$ *for any* $d \in F_{(X,\mathcal{L})}$ *and any* $\kappa \in K_X$.

Remark 4.11. Recall that $\Sigma_{(X,\mathcal{L})} \subset \langle[\mathcal{L}]\rangle^\perp$ is the root sublattice generated by $F_{(X,\mathcal{L})}$; that is, $\Sigma_{(X,\mathcal{L})} = \langle D_{(X,\mathcal{L})}\rangle$. As before, we put

$$^0(\Sigma_{(X,\mathcal{L})} \otimes \mathbb{R}) := \{\ x \in \Sigma_{(X,\mathcal{L})} \otimes \mathbb{R}\ \mid\ (x, d) \neq 0\ \text{ for all }\ d \in D_{(X,\mathcal{L})}\ \}.$$

By Corollary 4.2, there exists a unique connected component A of ${}^0(\Sigma_{(X,\mathcal{L})} \otimes \mathbb{R})$ such that $(d, \alpha) > 0$ holds for every $d \in F_{(X,\mathcal{L})}$ and every $\alpha \in A$. A wall $d^\perp \subset H_X$ associated with $d \in D_X$ contains the vector $[\mathcal{L}]$ only when $d \in D_{(X,\mathcal{L})}$. Therefore there exists a unique connected component of ${}^0\Gamma_X$ that contains $[\mathcal{L}] + \epsilon\alpha$ for sufficiently small positive real numbers ϵ and every vector $\alpha \in A$. This connected component is just the Kähler cone K_X of X.

5. The refined period map

In this section, we review the theory of the refined period map of marked $K3$ surfaces following the book by Barth, Hulek, Peters and Van de Ven [7, Chapter VIII].

Let L be a $K3$ lattice. An L-marked $K3$ surface is a pair (X, ϕ) of a $K3$ surface X and an isometry $\phi : L_X \xrightarrow{\sim} L$. A family of L-marked $K3$ surfaces is a pair $(\pi : \mathcal{X} \to B, \Phi)$, where $\pi : \mathcal{X} \to B$ is a smooth analytic family of $K3$ surfaces over a complex manifold B, and Φ is an isomorphism $R^2\pi_*\mathbb{Z} \xrightarrow{\sim} L \times B$ of locally constant systems over B such that, at every point t of B, the restriction $L_{X_t} \xrightarrow{\sim} L$ of Φ to the fibers is an isometry of lattices, where $X_t := \pi^{-1}(t)$. There exists a universal family

$$(\pi_1 : \mathcal{X}_1 \to \mathcal{Y}_1, \Phi_1)$$

of L-marked $K3$ surfaces over a *non-Hausdorff* smooth complex manifold $\mathcal{Y}_1$ of dimension 20. See [7, Chapter VIII, Section 12] for the construction of this universal family. For a point t of $\mathcal{Y}_1$, we denote by (X_t, ϕ_t) the corresponding L-marked $K3$ surface. Let $\omega_{X_t} \in L_{X_t} \otimes \mathbb{C}$ be the cohomology class of a non-zero holomorphic

2-form on X_t. Then we obtain a point $[\phi_t(\omega_{X_t})]$ of Ω_L, which is called the *period point of* (X_t, ϕ_t) and is denoted by $\tau_1(t)$. It is proved that the map

$$\tau_1 : \mathcal{Y}_1 \to \Omega_L$$

is a holomorphic map. We call τ_1 the *period map.*

We refine the period map τ_1 to τ_2. Consider the real vector bundle $R^2\pi_{1*}\mathbb{R}$ of rank 22 over $\mathcal{Y}_1$. A point of this vector bundle is given by (t, x), where $t \in \mathcal{Y}_1$ and $x \in L_{X_t} \otimes \mathbb{R}$. We then put

$$\mathcal{Y}_2 := \{ \ (t, \kappa) \in R^2\pi_{1*}\mathbb{R} \ \mid \ \kappa \text{ is a Kähler class of } X_t \ \}.$$

Then $\mathcal{Y}_2$ is the base space of the universal family of the triples (X, ϕ, κ_X), where (X, ϕ) is an L-marked $K3$ surface and $\kappa_X \in K_X$ is a Kähler class of X. This space $\mathcal{Y}_2$ is the source space of τ_2. For $[\omega] \in \Omega_L$, we put

$$\begin{array}{rcl} H^{[\omega]} & := & \{ \ x \in L \otimes \mathbb{R} \ \mid \ (x, \omega) = 0 \ \}, \\ \Gamma^{[\omega]} & := & \{ \ x \in H^{[\omega]} \ \mid \ (x, x) > 0 \ \}, \\ S^{[\omega]} & := & H^{[\omega]} \cap L, \\ D^{[\omega]} & := & \{ \ d \in S^{[\omega]} \ \mid \ (d, d) = -2 \ \}, \\ {}^0\Gamma^{[\omega]} & := & \{ \ x \in \Gamma^{[\omega]} \ \mid \ (x, d) \neq 0 \ \text{ for all } \ d \in D^{[\omega]} \ \}. \end{array}$$

We then put

$$(K\Omega_L)^0 := \{ \ ([\omega], x) \in \Omega_L \times (L \otimes \mathbb{R}) \ \mid \ x \in {}^0\Gamma^{[\omega]} \ \}.$$

Let t is a point of $\mathcal{Y}_1$, and let

$$[\omega_t] := \phi_t([\omega_{X_t}]) = \tau_1(t) \in \Omega_L$$

be the period point of the L-marked $K3$ surface (X_t, ϕ_t). Then the L-marking $\phi_t : L_{X_t} \xrightarrow{\sim} L$ maps H_{X_t} to $H^{[\omega_t]}$, Γ_{X_t} to $\Gamma^{[\omega_t]}$, S_{X_t} to $S^{[\omega_t]}$, D_{X_t} to $D^{[\omega_t]}$, and hence ϕ_t maps ${}^0\Gamma_{X_t}$ to ${}^0\Gamma^{[\omega_t]}$. Since every Kähler class of X_t is contained in the Kähler cone $K_{X_t} \subset {}^0\Gamma_{X_t}$, we can define a map

$$\tau_2 : \mathcal{Y}_2 \to (K\Omega_L)^0,$$

which is called the *refined period map*, by

$$\tau_2(t, \kappa) := (\tau_1(t), \phi_t(\kappa)).$$

Then we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{Y}_2 \arrow[r, "\tau_2"] \arrow[d] & (K\Omega_L)^0 \arrow[d] \\ \mathcal{Y}_1 \arrow[r, "\tau_1"] & \Omega_L, \end{tikzcd} \tag{5.1}$$

where the vertical arrows are the natural projections. The following theorem plays a crucial role in the proof of Main Theorem:

Theorem 5.1 (Theorems 12.3 and 14.1 in Chapter VIII of [7]). *The refined period map τ_2 is an isomorphism.*

In particular, $\mathcal{Y}_2$ is a Hausdorff real analytic manifold of real dimension 60.

Corollary 5.2. *Every vector in the Kähler cone of a $K3$ surface is a Kähler class.*

6. Proof of Main Theorem

First we prove that, if polarized $K3$ surfaces $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ of type (R, n) satisfy $(X, \mathcal{L}) \sim_{\mathrm{conn}} (X', \mathcal{L}')$, then $\rho(X, \mathcal{L}) = \rho(X', \mathcal{L}')$ holds. By the definition of the relation $\sim_{\mathrm{conn}}$, it is enough to prove $\rho(X, \mathcal{L}) = \rho(X', \mathcal{L}')$ in the following two cases:

(i) $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ are isomorphic;

(ii) $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ are fibers of a family of polarized $K3$ surfaces of type (R, n) over a complex ball B.

If there is an isomorphism $f : (X, \mathcal{L}) \xrightarrow{\sim} (X', \mathcal{L}')$, then the isometry $f^* : L_{X'} \xrightarrow{\sim} L_X$ maps $[\mathcal{L}']$ to $[\mathcal{L}]$, $F_{(X',\mathcal{L}')}$ to $F_{(X,\mathcal{L})}$, and $[\omega_{X'}] \in \Omega_{L_{X'}}$ to $[\omega_X] \in \Omega_{L_X}$. Therefore we have $\rho(X, \mathcal{L}) = \rho(X', \mathcal{L}')$ by Remarks 3.5 and 3.6. Thus the case (i) is done. We consider the case (ii). Let $(\pi : \mathcal{X} \to B, \mathcal{L}_{\mathcal{X}})$ be a family of polarized $K3$ surfaces of type (R, n) over a complex ball B, and let b and b' be points of B such that $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ are the fibers over b and b', respectively. For $t \in B$, we put $X_t := \pi^{-1}(t)$ and $\mathcal{L}_t := \mathcal{L}_{\mathcal{X}}|X_t$, and set

$$\begin{array}{lclcl}
L_{\mathcal{X}} & := & R^2\pi_*\mathbb{Z} \;=\; \{\, (t,x) \mid t \in B, x \in L_{X_t} \,\} & = & \cup_{t\in B} L_{X_t}, \\
H_{\mathcal{X}} & := & \{\, (t,x) \in L_{\mathcal{X}} \otimes \mathbb{R} \mid (x, \omega_{X_t}) = 0 \,\} & = & \cup_{t\in B} H_{X_t}, \\
\Gamma_{\mathcal{X}} & := & \{\, (t,x) \in H_{\mathcal{X}} \mid (x,x) > 0 \,\} & = & \cup_{t\in B} \Gamma_{X_t}, \\
S_{\mathcal{X}} & := & \{\, (t,x) \in L_{\mathcal{X}} \mid x \in H_{X_t} \,\} & = & \cup_{t\in B} S_{X_t}, \\
D_{\mathcal{X}} & := & \{\, (t,d) \in S_{\mathcal{X}} \mid (d,d) = -2 \,\} & = & \cup_{t\in B} D_{X_t}, \\
{}^0\Gamma_{\mathcal{X}} & := & \{\, (t,x) \in \Gamma_{\mathcal{X}} \mid (x,d) \neq 0 \text{ for all } d \in D_{X_t} \,\} & = & \cup_{t\in B} {}^0\Gamma_{X_t}.
\end{array}$$

Then $H_{\mathcal{X}} \to B$ is a real analytic vector bundle of rank 20. We form the union of the Kähler cones K_{X_t} in $H_{\mathcal{X}}$:

$$K_{\mathcal{X}} := \{\, (t,x) \in {}^0\Gamma_{\mathcal{X}} \mid x \in K_{X_t} \,\} = \cup_{t\in B} K_{X_t}.$$

We have the following lemma:

Lemma 6.1 (Lemma (9.3) in Chapter VIII of [7])**.** *The union $K_{\mathcal{X}}$ of the Kähler cones is open in $H_{\mathcal{X}}$.*

Let ℓ be the section of $L_{\mathcal{X}} \to B$ given by $\ell(t) := [\mathcal{L}_t]$. Then ℓ can be regarded as continuous sections of $H_{\mathcal{X}} \to B$ and $\Gamma_{\mathcal{X}} \to B$. For each $t \in B$, we put

$$\langle \ell(t) \rangle^{\perp} := \{\, v \in S_{X_t} \mid (v, [\mathcal{L}_t]) = 0 \,\},$$

which is a negative-definite sublattice of S_{X_t}. Since the fibers $(X_t, \mathcal{L}_t)$ of the family $(\pi : \mathcal{X} \to B, \mathcal{L}_{\mathcal{X}})$ are of type (R, n), the family

$$D_{(\mathcal{X},\ell)} := \{\, (t,d) \in D_{\mathcal{X}} \mid (d, [\mathcal{L}_t])) = 0 \,\} \;=\; \cup_{t\in B} D_{(X_t,\mathcal{L}_t)}$$

of the set of roots in $\langle \ell(t) \rangle^{\perp}$ is constant over B. (Note that, since B is a complex ball, the monodromy action on $D_{(X_t,\mathcal{L}_t)}$ is trivial.) Therefore the family

$$\Gamma_{\mathcal{X}} \setminus {}^0\Gamma_{\mathcal{X}} = \{\, (t,x) \in \Gamma_{\mathcal{X}} \mid (x,d) = 0 \text{ for some } d \in D_{X_t} \,\}$$

of configurations of the walls $\{d^{\perp} \mid d \in D_{X_t}\}$ in Γ_{X_t} is constant over B *locally along the section ℓ of* $\Gamma_{\mathcal{X}} \to B$. Hence there exists an open tubular neighborhood $T \subset \Gamma_{\mathcal{X}}$ of $\ell(B) \subset \Gamma_{\mathcal{X}}$ with the following properties:

- T is a locally trivial fiber space over B with the fiber $T_t := T \cap \Gamma_{X_t}$ being a small open ball in the real vector space H_{X_t} with the center $\ell(t)$ for every $t \in B$,

- $T \cap {}^0\Gamma_{\mathcal{X}}$ is also a locally trivial fiber space over B, and
- for each $t \in B$, the inclusion $T_t \cap {}^0\Gamma_{X_t} \hookrightarrow {}^0\Gamma_{X_t}$ induces a one-to-one correspondence between the set of connected components of $T_t \cap {}^0\Gamma_{X_t}$ and the set of connected components of ${}^0\Gamma_{X_t}$ containing $\ell(t)$ in the closure.

Since the cohomology class $[\mathcal{L}_t] = \ell(t) \in S_{X_t}$ is nef at each point $t \in B$, there exists a unique connected component of $T_t \cap {}^0\Gamma_{X_t}$ that is contained in the Kähler cone K_{X_t} of X_t by Propositions 4.5 and 4.6. Since the union $K_{\mathcal{X}}$ of the Kähler cones is open in $H_{\mathcal{X}}$ by Lemma 6.1, we see that $T \cap K_{\mathcal{X}}$ is a connected component of $T \cap {}^0\Gamma_{\mathcal{X}}$, and is a locally trivial fiber space over B with connected fibers.

We choose a path $\gamma : I \to B$ from the point $b \in B$ corresponding to $(X, \mathcal{L})$ to the point $b' \in B$ corresponding to $(X', \mathcal{L}')$. Then γ induces an isometry

$$\varphi : L_X \xrightarrow{\sim} L_{X'}.$$

Since ℓ is a global section of $L_{\mathcal{X}} \to B$, the isometry φ maps $\ell(b) = [\mathcal{L}]$ to $\ell(b') = [\mathcal{L}']$. Since $T \cap K_{\mathcal{X}}$ is a locally trivial fiber space over B with connected fibers, there exists a lift $\widetilde{\gamma} : I \to T \cap K_{\mathcal{X}}$ of γ. Thus we obtain a family of Kähler classes $\widetilde{\gamma}(s) \in K_{X_{\gamma(s)}}$ continuously varying with $s \in I$. Since the set

$$F_{\gamma(s)} := F_{(X_{\gamma(s)}, \mathcal{L}_{\gamma(s)})}$$

of the cohomology classes of (-2)-curves contracted by $X_{\gamma(s)} \to Y_{|\mathcal{L}_{\gamma(s)}|}$ is the set of indecomposable elements in

$$(D_{(X_{\gamma(s)}, \mathcal{L}_{\gamma(s)})})^+_{\widetilde{\gamma}(s)} := \{ \, d \in D_{(X_{\gamma(s)}, \mathcal{L}_{\gamma(s)})} \ \mid \ (d, \widetilde{\gamma}(s)) > 0 \, \},$$

and $D_{(X_t, \mathcal{L}_t)}$ is constant with respect to t, we see that $F_{\gamma(s)}$ is constant with respect to $s \in I$. Hence φ maps $F_{\gamma(0)} = F_{(X,\mathcal{L})}$ to $F_{\gamma(1)} = F_{(X',\mathcal{L}')}$ bijectively. Therefore φ induces an isometry

$$M^0_{(X,\mathcal{L})} = \langle F_{\gamma(0)} \rangle \oplus \langle \ell(\gamma(0)) \rangle \ \ \xrightarrow{\sim} \ \ M^0_{(X',\mathcal{L}')} = \langle F_{\gamma(1)} \rangle \oplus \langle \ell(\gamma(1)) \rangle,$$

and hence it induces an isometry $M_{(X,\mathcal{L})} \xrightarrow{\sim} M_{(X',\mathcal{L}')}$ on their primitive closures, and an isometry $N_{(X,\mathcal{L})} \xrightarrow{\sim} N_{(X',\mathcal{L}')}$ on their orthogonal complements. Since the period map is holomorphic and B is connected, the isometry $N_{(X,\mathcal{L})} \xrightarrow{\sim} N_{(X',\mathcal{L}')}$ maps the connected component of $\Omega_{N_{(X,\mathcal{L})}}$ containing $[\omega_X]$ to the connected component of $\Omega_{N_{(X',\mathcal{L}')}}$ containing $[\omega_{X'}]$. Therefore $\rho(X, \mathcal{L}) = \rho(X', \mathcal{L}')$ holds by Remarks 3.5 and 3.6. Thus we have proved that the map ρ induces a map

$$\widetilde{\rho} : \mathrm{Conn}(R, n) \to Q(R, n)/\!\sim.$$

Next we show that $\widetilde{\rho}$ is a bijection.

Definition 6.2. A *cohomological family of polarized $K3$ surfaces of type (R, n)* is a pair $(\pi : \mathcal{X} \to B, \eta)$, where $\pi : \mathcal{X} \to B$ is a smooth analytic family of $K3$ surfaces over a complex manifold B, and η is a section of the locally constant system $R^2\pi_*\mathbb{Z}$ over B such that, for each point $t \in B$, the element $\eta(t) \in H^2(X_t, \mathbb{Z})$ is the cohomology class of a line bundle $\mathcal{L}_t$ on $X_t := \pi^{-1}(t)$ that makes $(X_t, \mathcal{L}_t)$ a polarized $K3$ surface of type (R, n).

Definition 6.3. Let $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ be polarized $K3$ surfaces of type (R, n). We write $(X, \mathcal{L}) \approx_{\mathrm{cohom}} (X', \mathcal{L}')$ if $(X, [\mathcal{L}])$ and $(X', [\mathcal{L}'])$ are isomorphic to members of a cohomological family of polarized $K3$ surfaces of type (R, n) with a connected base space.

Lemma 6.4. *If $(X, \mathcal{L}) \approx_{cohom} (X', \mathcal{L}')$ holds, then $(X, \mathcal{L}) \sim_{conn} (X', \mathcal{L}')$ holds.*

Proof. Suppose that $(\pi : \mathcal{X} \to B, \eta)$ is a cohomological family of polarized $K3$ surfaces of type (R, n). If $U \subset B$ is an open ball, then we have natural isomorphisms $H^2(\pi^{-1}(U), \mathbb{Z}) \cong H^0(U, R^2\pi_*\mathbb{Z})$ and $H^2(\pi^{-1}(U), \mathcal{O}) \cong H^0(U, R^2\pi_*\mathcal{O})$. From the commutative diagram

$$\begin{tikzcd}
H^1(\pi^{-1}(U), \mathcal{O}^\times) \arrow[r] & H^2(\pi^{-1}(U), \mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & H^2(\pi^{-1}(U), \mathcal{O}) \arrow[d, "\wr"] \\
& H^0(U, R^2\pi_*\mathbb{Z}) \arrow[r] & H^0(U, R^2\pi_*\mathcal{O})
\end{tikzcd}$$

with the upper sequence being the exponential exact sequence, we see that there exists a line bundle $\mathcal{L}_U$ on $\pi^{-1}(U)$ such that $\eta(t)$ is the cohomology class of the line bundle $\mathcal{L}_U|X_t$ on $X_t := \pi^{-1}(t)$ for any $t \in U$. Then $(\pi^{-1}(U) \to U, \mathcal{L}_U)$ is a family of polarized $K3$ surfaces of type (R, n). □

Remark 6.5. In fact, we will show that the implication of the opposite direction also holds and hence $\approx_{\mathrm{cohom}}$ is an equivalence relation. See Corollary 6.7.

Let $[M, N, L, c\Omega]$ be an element of $Q(R, n)/\sim$. The goal is to construct a cohomological family $(\pi : \mathcal{X} \to \mathcal{Z}, \zeta)$ of polarized $K3$ surfaces of type (R, n) with the following properties:

- the base space $\mathcal{Z}$ is connected,
- if $(X, [\mathcal{L}])$ is a member of this family, then $\rho(X, \mathcal{L}) = [M, N, L, c\Omega]$ holds, and
- if $(X, \mathcal{L})$ is a polarized $K3$ surface of type (R, n) such that $\rho(X, \mathcal{L})$ is equal to $[M, N, L, c\Omega]$, then $(X, [\mathcal{L}])$ is isomorphic to a member of this family.

Note that L is a $K3$ lattice. Since $N \subset L$, we have an embedding $\Omega_N \hookrightarrow \Omega_L$ of complex manifolds. For $v \in L$, we put

$$P(v) := \{ \, [\omega] \in \Omega_N \mid (v, \omega) = 0 \, \}.$$

Then $P(v)$ is a complex hypersurface of Ω_N if $v \notin M$, while $P(v) = \Omega_N$ if $v \in M$, because M is the orthogonal complement of N in L. We put

$$\begin{array}{rcl} LE_h & := & \{ \, e \in L \mid (e, h) = 1, (e, e) = 0 \, \} \quad \text{and} \\ LD_h & := & \{ \, d \in L \mid (d, h) = 0, (d, d) = -2 \, \}. \end{array}$$

Then LD_h contains the set

$$LD_h{}^0 := \{ \, d \in \Sigma_R^- \mid (d, d) = -2 \, \}$$

of the roots in the root lattice $\Sigma_R^- \subset L$. Since M satisfies the conditions (m1) and (m2), we have

$$M \cap LE_h = \emptyset \quad \text{and} \quad M \cap LD_h = LD_h{}^0.$$

Therefore $P(e)$ for $e \in LE_h$ and $P(d)$ for $d \in LD_h \backslash LD_h{}^0$ are complex hypersurfaces of Ω_N. It is easy to see that the family of these hypersurfaces is locally finite on Ω_N. We put

$$\begin{array}{rcl} \Omega_N^\sharp & := & \Omega_N \setminus \Big(\displaystyle\bigcup_{e \in LE_h} P(e) \, \cup \bigcup_{d \in LD_h \backslash LD_h{}^0} P(d) \Big) \quad \text{and} \\ c\Omega^\sharp & := & \Omega_N^\sharp \cap c\Omega. \end{array}$$

Since a complex hypersurface is of real codimension 2, the space $c\Omega^\sharp$ is connected. We define $(Kc\Omega)^{\sharp 0}$ by the following diagram of the fiber product:

$$\begin{tikzcd}
(Kc\Omega)^{\sharp 0} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & (K\Omega_L)^0 \arrow[d] \\
c\Omega^\sharp \arrow[r, hook] & \Omega_L.
\end{tikzcd}$$

Let $[\omega]$ be a point of $c\Omega^\sharp$. The fiber of $(Kc\Omega)^{\sharp 0} \to c\Omega^\sharp$ over $[\omega]$ is ${}^0\Gamma^{[\omega]}$. We choose a connected component $K^{[\omega]}$ of ${}^0\Gamma^{[\omega]}$ by the following procedure. Recall that we have chosen a fundamental system of roots F in Σ_R^-, which is a subset of ${LD_h}^0$. For $d \in D^{[\omega]}$, let $d^\perp$ denote the real hyperplane $\{x \in H^{[\omega]} \mid (d,x)=0\}$ of $H^{[\omega]}$. Then the family of walls $\{d^\perp \mid d \in D^{[\omega]}\}$ partitions $\Gamma^{[\omega]}$ into the connected components of ${}^0\Gamma^{[\omega]}$. Since $[\omega] \notin P(d)$ for any $d \in LD_h \setminus {LD_h}^0$, the set of roots

$$\{\ d \in D^{[\omega]} \ \mid\ (d,h)=0\ \} \quad = \quad D^{[\omega]} \cap LD_h$$

coincides with ${LD_h}^0$, which means that a wall $d^\perp$ associated with $d \in D^{[\omega]}$ contains the vector h only when $d \in {LD_h}^0$. Therefore, by Corollary 4.2, there exists a unique connected component $K^{[\omega]}$ of ${}^0\Gamma^{[\omega]}$ that contains $h+\epsilon\alpha$ for sufficiently small positive real numbers ϵ and any vector $\alpha \in \Sigma_R^- \otimes \mathbb{R}$ satisfying $(d,\alpha)>0$ for every $d \in F$.

Remark 6.6. This connected component $K^{[\omega]}$ of ${}^0\Gamma^{[\omega]}$ is characterized by the following properties; the closure of $K^{[\omega]}$ in $H^{[\omega]}$ contains the vector h, and $(\kappa,d)>0$ holds for any $\kappa \in K^{[\omega]}$ and any $d \in F$. Conversely, we can recover F from $K^{[\omega]}$; namely, F is the fundamental system of roots in the negative-definite sublattice $\{v \in S^{[\omega]} \mid (v,h)=0\}$ of $S^{[\omega]}$ associated with a linear map $v \mapsto (v,\kappa)$ given by any vector $\kappa \in K^{[\omega]}$.

We put

$$\mathcal{K} := \{\ ([\omega],x) \in (Kc\Omega)^{\sharp 0} \ \mid\ x \in K^{[\omega]}\ \}.$$

Because $c\Omega^\sharp$ is connected, the space $\mathcal{K}$ is also connected. Consider the connected subspace

$$\widetilde{\mathcal{Z}} := \tau_2^{-1}(\mathcal{K})$$

of $\mathcal{Y}_2$, where $\tau_2 : \mathcal{Y}_2 \xrightarrow{\sim} (K\Omega_L)^0$ is the refined period map. By the construction of the diagram (5.1), the image $\mathcal{Z}$ of $\widetilde{\mathcal{Z}}$ by the projection $\mathcal{Y}_2 \to \mathcal{Y}_1$ is a connected Hausdorff complex submanifold of $\mathcal{Y}_1$, and the period map $\tau_1 : \mathcal{Y}_1 \to \Omega_L$ maps $\mathcal{Z}$ to $c\Omega^\sharp \subset \Omega_L$ isomorphically. Let $(\pi : \mathcal{X} \to \mathcal{Z}, \Phi)$ be the pull-back of the universal family $(\pi_1 : \mathcal{X}_1 \to \mathcal{Y}_1, \Phi_1)$ of L-marked $K3$ surfaces by $\mathcal{Z} \hookrightarrow \mathcal{Y}_1$. For $z \in \mathcal{Z}$, we put $X_z := \pi^{-1}(z)$, and denote by $\phi_z : L_{X_z} \xrightarrow{\sim} L$ the isometry obtained from $\Phi : R^2\pi_*\mathbb{Z} \xrightarrow{\sim} L \times \mathcal{Z}$. The vector $h \in L$ gives rise to a section ζ of $R^2\pi_*\mathbb{Z}$. We have $\zeta(z) \in S_{X_z}$ for any $z \in \mathcal{Z}$, because $\phi_z(\omega_{X_z}) \in N \otimes \mathbb{C}$ and $h \perp N$. We denote by $\mathcal{L}_z$ the line bundle on X_z such that $[\mathcal{L}_z] = \zeta(z)$. By the definition of $\mathcal{Z}$, the isometry ϕ_z maps isomorphically the Kähler cone K_{X_z} of X_z to the fiber $K^{\tau_1(z)}$ of $\mathcal{K} \to c\Omega^\sharp$ over $\tau_1(z) = \phi_z([\omega_{X_z}])$. Since h is contained in the closure of $K^{\tau_1(z)}$, we see from Proposition 4.5 that $\mathcal{L}_z$ is nef. Since $\tau_1(z) = \phi_z([\omega_{X_z}]) \notin P(e)$ for any $e \in LE_h$, there are no vectors $v \in S_{X_z}$ such that $(v,v)=0$ and $(v,[\mathcal{L}_z])=1$. Therefore $|\mathcal{L}_z|$ has no fixed components by Proposition 4.7. Since $\tau_1(z) = \phi_z([\omega_{X_z}]) \notin P(d)$ for any $d \in LD_h \setminus {LD_h}^0$, the set of vectors $v \in S_{X_z}$ such that $(v,v)=-2$ and $(v,[\mathcal{L}_z])=0$ is equal to the set $\phi_z^{-1}({LD_h}^0)$ of roots of Dynkin type R. Therefore $(X_z,\mathcal{L}_z)$ is a polarized $K3$ surface of type (R, n) by Proposition 4.8. Thus $(\pi : \mathcal{X} \to \mathcal{Z}, \zeta)$ is a cohomological family of polarized $K3$ surfaces of type (R, n).

Let z be a point of $\mathcal{Z}$. Recall that $F_{(X_z, \mathcal{L}_z)}$ is the fundamental system of roots in the negative-definite sublattice $\{v \in S_{X_z} \mid (v, [\mathcal{L}_z]) = 0\}$ of S_{X_z} associated with a linear map $v \mapsto (v, \kappa_{X_z})$ given by a Kähler class $\kappa_{X_z} \in K_{X_z}$ of X_z, and F is the fundamental system of roots in the negative-definite sublattice $\{v \in S^{\tau_1(z)} \mid (v, h) = 0\}$ of $S^{\tau_1(z)}$ associated with a linear map $v \mapsto (v, \kappa^{\tau_1(z)})$ given by a vector $\kappa^{\tau_1(z)} \in K^{\tau_1(z)}$. Because ϕ_z maps S_{X_z} to $S^{\tau_1(z)}$, $[\mathcal{L}_z]$ to h, and K_{X_z} to $K^{\tau_1(z)}$, it induces a bijection from $F_{(X_z, \mathcal{L}_z)}$ to F. Moreover the period point $\tau_1(z)$ is a point of $c\Omega$ by definition. Therefore, by Remark 3.6, we have

$$\rho(X_z, \mathcal{L}_z) = [M, N, L, c\Omega].$$

Since $c\Omega^\sharp$ is non-empty, the surjectivity of $\widetilde{\rho}$ is proved.

Let $(X', \mathcal{L}')$ be another polarized $K3$ surface of type (R, n) such that $\rho(X', \mathcal{L}') = [M, N, L, c\Omega]$. Then there exists a marking $\phi' : L_{X'} \xrightarrow{\sim} L$ that maps $[\mathcal{L}']$ to h, $F_{(X', \mathcal{L}')}$ to F, and such that the period point

$$[\omega'] := [\phi'(\omega_{X'})] \quad \in \quad \Omega_L$$

of the L-marked $K3$ surface (X', ϕ') is a point of $c\Omega \subset \Omega_N$. Since $\mathcal{L}'$ is a polarization, there are no vectors $v \in L$ such that $(v, \omega') = 0$, $(v, h) = 1$ and $(v, v) = 0$ by Proposition 4.7. Since $(X', \mathcal{L}')$ is of type (R, n), the vectors $v \in L$ such that $(v, \omega') = 0$, $(v, h) = 0$ and $(v, v) = -2$ form a root system of type R by Corollary 4.9, and hence the set of these roots coincides with $L{D_h}^0$. Therefore $[\omega'] \in c\Omega$ is contained in $c\Omega^\sharp$. Since ϕ' maps $[\mathcal{L}']$ to h and $F_{(X', \mathcal{L}')}$ to F, it maps the Kähler cone $K_{X'}$ of X' to the fiber $K^{[\omega']}$ of $\mathcal{K} \to c\Omega^\sharp$ over $[\omega']$ by Corollary 4.10 and Remark 6.6. Therefore, if $\kappa_{X'} \in K_{X'}$ is a Kähler class of X', the point of $\mathcal{Y}_2$ corresponding to the triple $(X', \phi', \kappa_{X'})$ is contained in $\widetilde{\mathcal{Z}} = {\tau_2}^{-1}(\mathcal{K})$. Hence (X', ϕ') is isomorphic to a member of the family $(\pi : \mathcal{X} \to \mathcal{Z}, \Phi)$ of L-marked $K3$ surfaces, and consequently, $(X', [\mathcal{L}'])$ is isomorphic to a member of the cohomological family $(\pi : \mathcal{X} \to \mathcal{Z}, \zeta)$ of polarized $K3$ surfaces. Thus the injectivity of $\widetilde{\rho}$ is established by Lemma 6.4. □

During the proof, we have also proved the following corollary:

Corollary 6.7. *Let* $(X, \mathcal{L})$ *and* $(X', \mathcal{L}')$ *be polarized* $K3$ *surfaces of type* (R, n)*. If* $(X, \mathcal{L}) \sim_{conn} (X', \mathcal{L}')$*, then* $(X, \mathcal{L}) \approx_{cohom} (X', \mathcal{L}')$*.*

7. THE SET $Q(R, n)/\sim$

In this section, we explain how to calculate the set $Q(R, n)/\sim$.

The even overlattices of $M^0 = \Sigma_R^- \oplus \langle h \rangle$ are in one-to-one correspondence with the totally isotropic subgroups of the discriminant form (G_{M^0}, q_{M^0}). (See Nikulin [14, Proposition 1.4.1].) For an even overlattice M of M^0, we can see whether M satisfies the conditions (m1) and (m2) by the method described in [29]. Since G_{M^0} is finite, we can calculate the set Ms. The group $\mathrm{Aut}(F, h)$ is easy to calculate, because it is isomorphic to the automorphism group of the Dynkin diagram of type R. Hence the image of the natural homomorphism

$$\mathrm{Aut}(F, h) \quad \hookrightarrow \quad O(M^0) \quad \to \quad O(q_{M^0})$$

is also easy to calculate. From the results of the calculation, we see that the image is equal to the image of the natural homomorphism $O(\Sigma_R^-) \to O(q_{\Sigma_R^-})$, which is given in [28, Section 6.2]. Hence we can calculate the set

$$\overline{Ms} := \mathrm{Aut}(F, h)\backslash Ms$$

of the orbits of the action of $\mathrm{Aut}(F, h)$ on Ms. For an element M of Ms, let $[M] \in \overline{Ms}$ denote the orbit containing M. We also put

$$\mathrm{Aut}(F, h, M) := \{\, g \in \mathrm{Aut}(F, h) \;\mid\; g \text{ fixes } M \in Ms \,\}.$$

We have a natural map

$$\mathrm{pr} \;:\; Q(R, n)/\!\sim \;\;\to\;\; \overline{Ms}$$

that maps $[M, N, L, c\Omega]$ to $[M]$. We denote by $\overline{Ms}' \subset \overline{Ms}$ the image of the map $\mathrm{pr} : Q(R, n)/\!\sim\, \to \overline{Ms}$; that is, we put

$$\overline{Ms}' := \{\, [M] \in \overline{Ms} \;\mid\; Ns(M) \neq \emptyset \,\}.$$

For $[M] \in \overline{Ms}$, we can see whether $Ns(M)$ is empty or not by the criterion of Nikulin [14, Theorem 1.10.1]. Hence $\overline{Ms}'$ is calculated. For each $[M] \in \overline{Ms}'$, we have a natural map

$$\mathrm{pr}_{[M]} \;:\; \mathrm{pr}^{-1}([M]) \;\;\to\;\; Ns(M)$$

that maps $[M', N', L', c\Omega'] \in \mathrm{pr}^{-1}([M])$ to the lattice $N \in Ns(M)$ isometric to $N' \in Ns(M')$. (Note that, if $[M'] = [M]$, then M and M' are isometric, and hence $Ns(M)$ and $Ns(M')$ are canonically identified.) Let N be an element of $Ns(M)$. We can regard $\mathrm{Aut}(F, h, M)$ as a subgroup of $O(M)$. Hence the group

$$\Gamma(M, N) := \mathrm{Aut}(F, h, M) \times O(N)$$

acts on the set $Ls(M, N) \times c\Omega s(N)$ in the natural way. The fiber $\mathrm{pr}_{[M]}^{-1}(N)$ is, by definition, equal to the set of orbits of this action:

$$\mathrm{pr}_{[M]}^{-1}(N) = \Gamma(M, N)\backslash(Ls(M, N) \times c\Omega s(N)).$$

We explain how to calculate the set $Ns(M)$ for $[M] \in \overline{Ms}'$. By Nikulin [14, Corollary 1.9.4], every element of $Ns(M)$ is contained in the same genus. If $r := \mathrm{rank}(R) < 19$, the isomorphism class of an indefinite lattice N of signature $(2, 19 - r)$ is determined by the spinor genus by Eichler's theorem. (See, for example, Cassels [8].) The method of enumeration of spinor genera in a given genus is explained in Conway and Sloane [9, Chapter 15].

When $\mathrm{rank}(R) < 19$, the calculation of $Ns(M)$ is simplified by the following theorem of Nikulin [14, Theorem 1.14.2] in many cases. The *length* $l(G)$ of a finite abelian group G is the minimal number of generators of G. The finite quadratic forms $u_+^{(2)}(2)$ and $v_+^{(2)}(2)$ are the discriminant forms of the even lattices of rank 2 given by the intersection matrices

$$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix},$$

respectively. For a finite quadratic form (G, q) and a prime integer p, we denote by (G_p, q_p) the p-part of (G, q).

Theorem 7.1 (Theorem 1.14.2 in [14]). *Let T be an even indefinite lattice. Suppose that the discriminant form (G_T, q_T) of T satisfies the following conditions:*

(i) $\operatorname{rank}(T) \ge l((G_T)_p) + 2$ *for any odd prime* p, *and*

(ii) *if* $\operatorname{rank}(T) = l((G_T)_2)$, *then* $((G_T)_2, (q_T)_2)$ *contains* $u_+^{(2)}(2)$ *or* $v_+^{(2)}(2)$ *as an orthogonal direct-sum factor.*

Then the genus of T *contains only one isomorphism class, and the natural homomorphism* $O(T) \to O(q_T)$ *is surjective.*

We consider the following conditions (k1) and (k2) on $[M] \in \overline{Ms}'$. We put $r := \operatorname{rank}(R) = \operatorname{rank}(M) - 1$.

(k1) $21 - r \ge l((G_M)_p) + 2$ for any odd prime p, and

(k2) if $21 - r = l((G_M)_2)$, then $((G_M)_2, -(q_M)_2)$ contains $u_+^{(2)}(2)$ or $v_+^{(2)}(2)$ as an orthogonal direct-sum factor.

Proposition 7.2. *Suppose that* $\operatorname{rank}(R) < 19$. *If* $[M] \in \overline{Ms}'$ *satisfies the conditions* (k1) *and* (k2), *then* $Ns(M)$ *consists of a single element.*

Proof. Suppose that $N \in Ns(M)$. From the condition $\operatorname{rank}(R) < 19$, we see that N is indefinite by (n1). The discriminant form (G_N, q_N) is isomorphic to $(G_M, -q_M)$ by (n2). Since M satisfies (k1) and (k2), the lattice N of rank $21 - r$ satisfies the conditions in Theorem 7.1. All elements of $Ns(M)$ are contained in the same genus by Nikulin [14, Corollary 1.9.4]. Therefore the proposition follows from the first assertion of Theorem 7.1. □

Next we consider the set $Ls(M, N)$ and the action of the group $\Gamma(M, N)$ on $Ls(M, N) \times c\Omega s(N)$. By Nikulin [14, Proposition 1.6.1], there exists a natural bijection between the set $Ls(M, N)$ and the set of isomorphisms of finite quadratic forms from $(G_M, -q_M)$ to (G_N, q_N). In particular, each of the groups $O(q_M)$ and $O(q_N)$ acts on $Ls(M, N)$ simple-transitively. Since $G_M \cong G_N$ is a finite abelian group, we can calculate the set $Ls(M, N)$ easily.

When $\operatorname{rank}(R) < 19$, the lattice N is indefinite, and hence $O(N)$ is, in general, an infinite group, and it is difficult to calculate the orbits of the action of $\Gamma(M, N)$ on $Ls(M, N) \times c\Omega s(N)$ in a direct way. Using Theorem 7.1, however, we can show that the action is transitive in many cases. We consider the following condition on the pair (M, N):

Switching condition. There exist an isomorphism $\iota : G_M \xrightarrow{\sim} G_N$ and isometries $g_M \in \operatorname{Aut}(F, h, M) \subset O(M)$, $g_N \in O(N)$ with the following properties:

(i) $q_N \circ \iota = -q_M$, so that ι defines an element L_ι of $Ls(M, N)$,

(ii) $\bar{g}_N \circ \iota = \iota \circ \bar{g}_M$, so that $g_M \oplus g_N \in O(M \oplus N)$ fixes $L_\iota \in Ls(M, N)$, and

(iii) g_N interchanges the two connected components of Ω_N.

Proposition 7.3. *Suppose that* $\operatorname{rank}(R) < 19$, *that* $[M] \in \overline{Ms}'$ *satisfies the conditions* (k1) *and* (k2), *and that* $N \in Ns(M)$. *Then the cardinality of* $\operatorname{pr}^{-1}([M])$ *is at most two. If* (M, N) *satisfies the switching condition, then* $\operatorname{pr}^{-1}([M])$ *consists of a single element.*

Proof. By Proposition 7.2, we have $Ns(M) = \{N\}$. Hence $\operatorname{pr}^{-1}([M])$ is equal to $\operatorname{pr}_{[M]}^{-1}(N)$. As we have shown in the proof of Proposition 7.2, the lattice N satisfies the conditions in Theorem 7.1. Since $O(q_N)$ acts on $Ls(M, N)$ simple-transitively, the second assertion of Theorem 7.1 implies that $O(N)$ acts on $Ls(M, N)$ transitively. Hence the cardinality of $\operatorname{pr}_{[M]}^{-1}(N)$ is at most $|c\Omega s(N)| = 2$. Suppose that

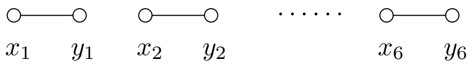

FIGURE 8.1. The Dynkin diagram of type $6A_2$

(M, N) satisfies the switching condition. Then there exists an element (g_M, g_N) of $\Gamma(M, N)$ that fixes $L_\iota \in Ls(M, N)$ and acts non-trivially on $c\Omega s(N)$. Hence the action of $\Gamma(M, N)$ on $Ls(M, N) \times c\Omega s(N)$ is transitive. □

8. SIX-CUSPIDAL SEXTIC CURVES

In this section, we prove Theorem 1.1 by showing $|Q(6A_2, 2)/{\sim}| = 2$.

First we fix some conventions about lattices. Let $v_1, \dots, v_r$ be a basis of an even lattice Λ. We denote by $v_1^\vee, \dots, v_r^\vee$ the basis of $\Lambda^\vee$ dual to $v_1, \dots, v_r$. We say that elements $\xi_1, \dots, \xi_s$ of G_Λ *form a basis of* G_Λ if

$$G_\Lambda = \langle \xi_1 \rangle \times \cdots \times \langle \xi_s \rangle$$

holds. We define a finite symmetric bilinear form $b_\Lambda : G_\Lambda \times G_\Lambda \to \mathbb{Q}/\mathbb{Z}$ by

$$b_\Lambda(\xi, \xi') := \frac{1}{2}\big(q_\Lambda(\xi + \xi') - q_\Lambda(\xi) - q_\Lambda(\xi')\big) \ \in \ \mathbb{Q}/\mathbb{Z}.$$

If $\xi_1, \dots, \xi_s$ form a basis of G_Λ, then we can express the discriminant form $q_\Lambda : G_\Lambda \to \mathbb{Q}/2\mathbb{Z}$ by the symmetric $s \times s$-matrix (a_{ij}) defined by

$$a_{ij} := \begin{cases} q_\Lambda(\xi_i) \ \in \ \mathbb{Q}/2\mathbb{Z} & \text{if } i = j, \\ b_\Lambda(\xi_i, \xi_j) \ \in \ \mathbb{Q}/\mathbb{Z} & \text{if } i \neq j. \end{cases}$$

For a positive integer l, let $U(l)$ be the even indefinite lattice of rank 2 given by the intersection matrix

$$\begin{bmatrix} 0 & l \\ l & 0 \end{bmatrix}.$$

We set $\langle h \rangle$ and $\langle m \rangle$ to be the lattices of rank 1 generated by h with $(h, h) = 2$ and by m with $(m, m) = -2$, respectively.

We define a basis $x_1, y_1, x_2, y_2, \dots, x_6, y_6$ of $\Sigma^-_{6A_2}$ by Figure 8.1, and put

$$F := \{x_1, y_1, x_2, y_2, \dots, x_6, y_6\},$$

which is a fundamental system of roots in $\Sigma^-_{6A_2}$. The vectors in F with h form a basis of $M^0 = \Sigma^-_{6A_2} \oplus \langle h \rangle$. We put

$$\xi_i := x_i^\vee \bmod M^0, \quad \eta_i := y_i^\vee \bmod M^0, \quad \text{and} \quad \gamma := h^\vee \bmod M^0.$$

The elements $\xi_1, \dots, \xi_6$ and γ form a basis of $G_{M^0} \cong {\mathbb{F}_3}^6 \oplus \mathbb{F}_2$, with respect to which q_{M^0} is expressed by the diagonal matrix

$$\operatorname{diag}[\, -2/3, \ \dots, \ -2/3, \ 1/2 \,].$$

We express $u_1\xi_1 + \cdots + u_6\xi_6 + v\gamma \in G_{M^0}$ with $u_i \in \mathbb{F}_3$ and $v \in \mathbb{F}_2$ by

$$(u|v) = (u_1, \dots, u_6 \,|\, v).$$

The automorphism group $\operatorname{Aut}(F, h)$ of the Dynkin diagram of type $6A_2$ (Figure 8.1) is generated by the permutations of the six connected components and the interchanging of x_i and y_i for $i = 1, \dots, 6$. Therefore $\operatorname{Aut}(F, h)$ is isomorphic to the semi-direct product $\{\pm 1\}^{\oplus 6} \rtimes S_6$, where S_6 is the full-symmetric group of degree 6. Since $\xi_i = -\eta_i$ in G_{M^0}, the group $\mathrm{Aut}(F, h)$ acts on G_{M^0} by the permutations of the first 6 coordinates $u_1, \dots, u_6$ and the multiplications by -1 on each of $u_1, \dots, u_6$. For $(u|v) = (u_1, \dots, u_6 \,|\, v) \in G_{M^0}$, we put

$$\mathrm{wt}(u) := |\,\{i \,|\, u_i \neq 0\}\,|.$$

Then $q_{M^0}(u|v) = 0$ holds if and only if $v = 0$ and $\mathrm{wt}(u) \equiv 0 \bmod 3$. For $(u|v) \in G_{M^0}$ satisfying $q_{M^0}(u|v) = 0$, we denote by $M_{(u|0)}$ the even overlattice of M^0 corresponding to the totally isotropic subgroup $\langle (u|0) \rangle$ of (G_{M^0}, q_{M^0}) generated by $(u|0)$. Suppose that $\mathrm{wt}(u) = 3$. Then $M_{(u|0)}$ does not satisfy the no-new-roots condition (m2). Indeed, the Dynkin type of the system of roots in the orthogonal complement of $\langle h \rangle$ in $M_{(u|0)}$ is $E_6 + 3A_2$. If $\mathrm{wt}(u) = 6$, then we can easily verify that $M_{(u|0)}$ satisfies the conditions (m1) and (m2). We put

$$M^1 := M_{(1,1,1,1,1,1\,|\,0)}.$$

Then we have

$$\overline{Ms} = \mathrm{Aut}(F, h) \backslash Ms = \{[M^0], [M^1]\}.$$

In order to prove $|Q(6A_2, 2)/\!\sim\!| = 2$, it is therefore enough to show the following:

$$|\,\mathrm{pr}^{-1}([M^0])| = 1 \quad \text{and} \quad |\,\mathrm{pr}^{-1}([M^1])| = 1.$$

We put

$$N^0 := U(3) \oplus U(3) \oplus \Sigma^-_{A_2} \oplus \Sigma^-_{A_2} \oplus \langle m \rangle.$$

The signature of N^0 is $(2, 7)$. Let $a_1, b_1, a_2, b_2, z_1, w_1, z_2, w_2, m$ be a basis of N^0 with respect to which the intersection matrix of N^0 is equal to

$$\mathrm{diag}[\, M_U, M_U, -M_A, -M_A, -2 \,],$$

where

$$M_U := \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \quad \text{and} \quad M_A := \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Then G_{N^0} is generated by

$$\begin{aligned} &\alpha_1 := a_1^\vee \bmod N^0, \quad \beta_1 := b_1^\vee \bmod N^0, \\ &\alpha_2 := a_2^\vee \bmod N^0, \quad \beta_2 := b_2^\vee \bmod N^0, \\ &\zeta_1 := z_1^\vee \bmod N^0, \quad \zeta_2 := z_2^\vee \bmod N^0, \quad \mu := m^\vee \bmod N^0, \end{aligned}$$

and these elements form a basis of G_{N^0}, with respect to which q_{N^0} is expressed by the matrix

$$\mathrm{diag}\left[\begin{bmatrix} 0 & 1/3 \\ 1/3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1/3 \\ 1/3 & 0 \end{bmatrix}, -2/3, -2/3, -1/2 \right].$$

We define an isomorphism $\iota^0 : G_{M^0} \xrightarrow{\sim} G_{N^0}$ by

$$\begin{aligned} &\iota^0(\xi_1) = \alpha_1 + \beta_1, \quad \iota^0(\xi_2) = \alpha_2 + \beta_2, \\ &\iota^0(\xi_3) = (\alpha_1 - \beta_1) + (\alpha_2 - \beta_2), \quad \iota^0(\xi_4) = (\alpha_1 - \beta_1) - (\alpha_2 - \beta_2), \\ &\iota^0(\xi_5) = \zeta_1 + \zeta_2, \quad \iota^0(\xi_6) = \zeta_1 - \zeta_2, \quad \iota^0(\gamma) = \mu. \end{aligned}$$

Then we can easily verify that ι^0 satisfies $q_{N^0} \circ \iota^0 = -q_{M^0}$. Hence N^0 satisfies the conditions (n1) and (n2). Since M^0 satisfies the conditions (k1) and (k2), we see that $Ns(M^0) = \{N^0\}$ by Proposition 7.2. We show that (M^0, N^0) satisfies the switching condition. We define $g_{M^0} \in \mathrm{Aut}(F, h, M^0) = \mathrm{Aut}(F, h)$ by the table

$$\begin{array}{c|ccccccccccccc} v & x_1 & y_1 & x_2 & y_2 & x_3 & y_3 & x_4 & y_4 & x_5 & y_5 & x_6 & y_6 & h \\ \hline g_{M^0}(v) & x_2 & y_2 & x_1 & y_1 & x_3 & y_3 & y_4 & x_4 & x_5 & y_5 & x_6 & y_6 & h \end{array},$$

and $g_{N^0} \in O(N^0)$ by the table

$$\begin{array}{c|ccccccccc} v & a_1 & b_1 & a_2 & b_2 & z_1 & w_1 & z_2 & w_2 & m \\ \hline g_{N^0}(v) & a_2 & b_2 & a_1 & b_1 & z_1 & w_1 & z_2 & w_2 & m \end{array}.$$

Then g_{M^0} acts on G_{M^0} as

$$\begin{array}{c|ccccccc} \sigma & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \gamma \\ \hline \bar{g}_{M^0}(\sigma) & \xi_2 & \xi_1 & \xi_3 & -\xi_4 & \xi_5 & \xi_6 & \gamma \end{array},$$

and g_{N^0} acts on G_{N^0} as

$$\begin{array}{c|ccccccc} \sigma & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \zeta_1 & \zeta_2 & \mu \\ \hline \bar{g}_{N^0}(\sigma) & \alpha_2 & \beta_2 & \alpha_1 & \beta_1 & \zeta_1 & \zeta_2 & \mu \end{array}.$$

It is straightforward to verify that ι^0, g_{M^0} and g_{N^0} satisfy the properties (i)-(iii) required in the switching condition. Therefore $\mathrm{pr}^{-1}([M^0])$ consists of a single element by Proposition 7.3.

Let $H^1 := \langle (1, 1, \dots, 1 \,|\, 0) \rangle$ be the cyclic subgroup of G_{M^0} generated by $(1, 1, \dots, 1 \,|\, 0) = \xi_1 + \cdots + \xi_6$. By [14, Proposition 1.4.1], the discriminant group G_{M^1} of M^1 is equal to $(H^1)^\perp / H^1$, where $(H^1)^\perp$ is the orthogonal complement of H^1 with respect to b_{M^0}. As a basis of G_{M^1}, we take the following five elements modulo H^1:

$$\tau_1 := \xi_1 - \xi_2,\ \ \tau_2 := \xi_1 + \xi_2 - \xi_3 - \xi_4,\ \ \tau_3 := \xi_3 - \xi_4,\ \ \tau_4 := \xi_5 - \xi_6,\ \ \gamma.$$

Then q_{M^1} is expressed by the diagonal matrix

$$\mathrm{diag}[\, 2/3,\ 4/3,\ 2/3,\ 2/3,\ 1/2 \,].$$

We put

$$N^1 := \Sigma^+_{A_2} \oplus \Sigma^-_{A_2} \oplus \Sigma^-_{A_2} \oplus \Sigma^-_{A_2} \oplus \langle m \rangle,$$

and let $w_1, z_1, w_2, z_2, w_3, z_3, w_4, z_4, m$ be a basis of N^1 with respect to which the intersection matrix of N^1 is given by

$$\mathrm{diag}[\, M_A, -M_A, -M_A, -M_A, -2 \,].$$

Then the signature of N^1 is $(2, 7)$. We put

$$\omega_i := w_i^\vee \bmod N^1 \quad \text{for} \quad i = 1, \dots, 4, \quad \text{and} \quad \mu := m^\vee \bmod N^1.$$

Then $\omega_1, \dots, \omega_4, \mu$ form a basis of G_{N^1}, with respect to which q_{N^1} is expressed by the diagonal matrix

$$\mathrm{diag}[\, 2/3,\ -2/3,\ -2/3,\ -2/3,\ -1/2 \,].$$

The isomorphism $\iota^1 : G_{M^1} \xrightarrow{\sim} G_{N^1}$ given by

$$\iota^1(\tau_1) = \omega_2,\ \ \iota^1(\tau_2) = \omega_1,\ \ \iota^1(\tau_3) = \omega_3,\ \ \iota^1(\tau_4) = \omega_4,\ \ \iota^1(\gamma) = \mu$$

satisfies $q_{N^1} \circ \iota^1 = -q_{M^1}$. Hence N^1 satisfies the conditions (n1) and (n2). Since M^1 satisfies the conditions (k1) and (k2), we see that $Ns(M^1) = \{N^1\}$ by Proposition 7.2. To show that (M^1, N^1) satisfies the switching condition, we define

$g_{M^0} \in \mathrm{Aut}(F, h)$ by the table

v	x_1	y_1	x_2	y_2	x_3	y_3	x_4	y_4	x_5	y_5	x_6	y_6	h
$g_{M^0}(v)$	y_1	x_1	y_2	x_2	y_3	x_3	y_4	x_4	y_5	x_5	y_6	x_6	h

.

Then g_{M^0} acts on G_{M^0} by

$$(u|v) \mapsto (-u|v).$$

In particular, this action maps the subgroup H^1 to itself and hence fixes $M^1 \in Ms$. Let $g_{M^1} \in O(M^1)$ be the isometry of M^1 induced by g_{M^0}. The action of g_{M^1} on G_{M^1} is the multiplication by -1 on the 3-part, and the identity on the 2-part. We define $g_{N^1} \in O(N^1)$ by the table

v	w_1	z_1	w_2	z_2	w_3	z_3	w_4	z_4	m
$g_{N^1}(v)$	z_1	w_1	z_2	w_2	z_3	w_3	z_4	w_4	m

.

The action of g_{N^1} on G_{N^1} is multiplication by -1 on the 3-part, and the identity on the 2-part. Then it is straightforward to verify that ι^1, g_{M^1} and g_{N^1} satisfy the properties (i)-(iii) required in the switching condition. Therefore $\mathrm{pr}^{-1}([M^1])$ consists of a single element by Proposition 7.3. □

Remark 8.1. Let $\mathcal{M}(6A_2)^0$ and $\mathcal{M}(6A_2)^1$ be the connected components of $\mathcal{M}(6A_2)$ corresponding to $[M^0]$ and $[M^1]$, respectively. A six-cuspidal sextic curve C is a member of $\mathcal{M}(6A_2)^1$ if and only if the six cusps of C are lying on a conic, or equivalently, $\pi_1(\mathbb{P}^2 \setminus C)$ is non-abelian, or equivalently, C is of torus type. Suppose that C is of torus type, and let $Q \subset \mathbb{P}^2$ be the conic passing through the six cusps of C. Let $(X, \mathcal{L})$ be the polarized $K3$ surface of type $(6A_2, 2)$ obtained as the double cover of $\mathbb{P}^2$ branched along C. Then the proper transform of Q in X consists of two effective divisors $Q_1^{\sim}$ and $Q_2^{\sim}$ that are interchanged by the involution of X over $\mathbb{P}^2$. The primitive closure $M_{(X,\mathcal{L})}$ of $M^0_{(X,\mathcal{L})}$ in L_X, which is an overlattice of $M^0_{(X,\mathcal{L})}$ with index 3, is obtained by adding the cohomology class $[Q_1^{\sim}]$ of $Q_1^{\sim}$ to $M^0_{(X,\mathcal{L})}$.

9. Maximizing sextics

In the following, we write the lattice of rank 2 given by the intersection matrix

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

by $T[a, b, c]$, or simply by $[a, b, c]$. The following is a part of the classical theory of binary forms due to Gauss [12]. (See also Conway and Sloane [9, Chapter 15].)

Proposition 9.1. *Let d be a positive integer. Then the set*

$$BF(d) := \{\ T[a, b, c] \ \mid\ ac - b^2 = d \quad \text{and} \quad 0 \le 2b \le a \le c\ \}$$

is a complete list of representatives of isomorphism classes of positive-definite lattices of rank 2 with discriminant d.

Let R be a Dynkin type of rank 19, $\langle h \rangle$ the lattice of rank 1 generated by h with $(h, h) = 2$, and M an even overlattice of $M^0 = \Sigma_R^- \oplus \langle h \rangle$ satisfying the conditions (m1) and (m2). From the list $BF(|G_M|)$, we select even lattices N such that the discriminant form (G_N, q_N) is isomorphic to $(G_M, -q_M)$, and make the set $Ns(M)$. Let N be an element of $Ns(M)$. Since N is positive-definite, the orthogonal group

$O(N)$ is finite, and we can calculate all the elements of $O(N)$ easily. Therefore we can calculate the set

$$\mathrm{pr}_{[M]}^{-1}(N) \;=\; (\mathrm{Aut}(F,h,M)\times O(N))\setminus(Ls(M,N)\times c\Omega s(N)).$$

Here is the explanation of the entries of Table MS of maximizing sextics. The Dynkin types R are sorted by a lexicographic order. The entry t is the total number of the connected components of the space $\mathcal{M}(R)$ of maximizing sextics of type R. The entry m is the cardinality of the set

$$\overline{Ms}' = \{\ [M]\in\overline{Ms}\ \mid\ Ns(M)\neq\emptyset\ \}.$$

For each $[M]\in\overline{Ms}'$, we present the set $Ns(M)$. The sets $Ns(M)$ for distinct $[M]$ are separated by short horizontal lines. Let N be a lattice in $Ns(M)$, and let $\tau: c\Omega s(N)\to c\Omega s(N)$ be the transposition of the two connected components of Ω_N. An orbit $\mathrm{Orb}\subset Ls(M,N)\times c\Omega s(N)$ of the action of $\Gamma(M,N)=\mathrm{Aut}(F,h,M)\times O(N)$ is called *real* if Orb is stable under τ. Let o_r and $2o_c$ be the numbers of real orbits and non-real orbits. The cardinality of $\mathrm{pr}_{[M]}^{-1}(N)$ is therefore o_r+2o_c.

Example 9.2. Let us consider the Dynkin type $A_{15}+A_4$. The discriminant form of $M^0=\Sigma^-_{A_{15}+A_4}\oplus\langle h\rangle$ is isomorphic to

$$(\ \mathbb{Z}/(16)\times\mathbb{Z}/(5)\times\mathbb{Z}/(2),\ \mathrm{diag}[-15/16,-4/5,1/2]\ ).$$

There are only two isotropic elements: $(0,0,0)$ and $(8,0,0)$. Let M^1 be the even overlattice of M^0 corresponding to the totally isotropic subgroup generated by $(8,0,0)$. Then M^1 satisfies the conditions (m1) and (m2). Hence we have $Ms=\{M^0,M^1\}$. We can easily confirm that

$$\mathrm{Aut}(F,h)=\mathrm{Aut}(F,h,M^0)=\mathrm{Aut}(F,h,M^1)\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}.$$

Therefore we have $\overline{Ms}=\{[M^0],[M^1]\}$. The set $Ns(M^0)$ consists of a single element $N^0:=T[8,4,22]$. The cardinality of $Ls(M^0,N^0)$ is 8. The group $O(N^0)$ is equal to $\{\mathrm{diag}[\pm1,\pm1]\}$. The group $\Gamma(M^0,N^0)$ acts on $Ls(M^0,N^0)\times c\Omega s(N^0)$, and decomposes it into two non-real orbits Orb^0 and $\tau(\mathrm{Orb}^0)$. The set $Ns(M^1)$ consists of a single element $N^1:=T[2,0,20]$. The cardinality of $Ls(M^1,N^1)$ is 4. The group $O(N^1)$ is equal to $\{\mathrm{diag}[\pm1,\pm1]\}$. The group $\Gamma(M^1,N^1)$ acts on $Ls(M^1,N^1)\times c\Omega(N^1)$ transitively, and hence we have a single real orbit Orb^1. Therefore $\mathcal{M}(A_{15}+A_4)$ consists of three connected components.

According to Yang's list [33], there exist two configurations of irreducible components of maximizing sextics of type $A_{15}+A_4$. (See [33, p.209] for the precise definition of the configuration of irreducible components of a sextic curve with at most simple singularities.) Let C^0, $C^{0\prime}$ and C^1 be members of the connected components corresponding to Orb^0, $\tau(\mathrm{Orb}^0)$ and Orb^1, respectively. Then C^0 and $C^{0\prime}$ have the same configuration, while C^0 and C^1 have different configurations. The two connected components of $\mathcal{M}(A_{15}+A_4)$ discovered by Artal Bartolo, Carmona Ruber and Cogolludo Agustín [2, Theorem 5.8 (i)] cannot be distinguished by the configurations of irreducible components, because their members can be defined by equations conjugated by the Galois group of $\mathbb{Q}(i)/\mathbb{Q}$. Therefore these two connected components must be corresponding to the two non-real orbits.

Example 9.3. We consider the maximizing sextics of type $A_{16}+A_2+A_1$. In [2, Theorem 5.8 (ii)], it is shown that $\mathcal{M}(A_{16}+A_2+A_1)$ consists of three connected components, and that members of these connected components can be defined by equations that are conjugate under the action of the Galois group over $\mathbb{Q}$ of the equation

$$17s^3 - 18s^2 - 228s + 536 = 0.$$

This equation has two non-real roots α, $\bar{\alpha}$ and a real root β. Let $C_\alpha, C_{\bar{\alpha}}, C_\beta$ be the corresponding maximizing sextics, and $X_\alpha, X_{\bar{\alpha}}, X_\beta$ the corresponding $K3$ surfaces. The set $\overline{Ms}'$ consists of a single element $[M^0]$, and the set $Ns(M^0)$ consists of two element $N^0 := T[10, 4, 22]$ and $N^1 := T[6, 0, 34]$. The action of $\Gamma(M^0, N^0)$ decomposes $Ls(M^0, N^0) \times c\Omega s(N^0)$ into two non-real orbits, while $\Gamma(M^0, N^1)$ acts on $Ls(M^0, N^1) \times c\Omega s(N^1)$ transitively. The complex conjugation induces a diffeomorphism between X_α and $X_{\bar{\alpha}}$ that maps algebraic cycles to algebraic cycles. Hence the transcendental lattices of X_α and $X_{\bar{\alpha}}$ are isometric. Therefore C_α and $C_{\bar{\alpha}}$ are members of the connected components corresponding to the two non-real orbits in $Ls(M^0, N^0) \times c\Omega s(N^0)$, and C_β is a member of the connected component corresponding to the single real orbit in $Ls(M^0, N^1) \times c\Omega s(N^1)$.

Example 9.4. We consider the Dynkin type $2E_6 + A_5 + A_2$. Oka and Pho [22, nt. 99] proved that the moduli of sextic curves of *torus type* with the singularities of type $[[2E_6 + A_5], [A_2]]$ has exactly two irreducible components, and that members of these components can be defined by equations that are conjugate under the action of the Galois group of $\mathbb{Q}(\sqrt{3})/\mathbb{Q}$. According to Table MS, we see that the moduli $\mathcal{M}(2E_6 + A_5 + A_2)$ has exactly two connected components. We have $\overline{Ms}' = \{[M^1]\}$ with $[M^1 : M^0] = 3$, $Ns(M^1) = \{N^1\}$ with $N^1 := T[6, 0, 6]$, and $Ls(M^1, N^1) \times c\Omega s(N^1)$ is the union of two real orbits under the action of $\Gamma(M^1, N^1)$.

Table MS

No.	R	t	m	N	o_r	o_c
1	$2E_8+A_3$	1	1	$[2,0,4]$	1	0
2	$2E_8+A_2+A_1$	1	1	$[2,0,6]$	1	0
3	$E_8+E_7+D_4$	1	1	$[2,0,2]$	1	0
4	$E_8+E_7+A_4$	2	1	$[4,2,6]$	0	1
5	$E_8+E_7+A_3+A_1$	1	1	$[2,0,4]$	1	0
6	$E_8+E_7+2A_2$	1	1	$[6,0,6]$	1	0
7	$E_8+E_6+D_5$	1	1	$[2,0,12]$	1	0
8	$E_8+E_6+A_5$	2	1	$[6,0,6]$	0	1
9	$E_8+E_6+A_4+A_1$	1	1	$[2,0,30]$	1	0
10	$E_8+E_6+A_3+A_2$	1	1	$[6,0,12]$	1	0
11	E_8+D_{11}	1	1	$[2,0,4]$	1	0
12	$E_8+D_{10}+A_1$	1	1	$[2,0,2]$	1	0
13	$E_8+D_9+A_2$	1	1	$[4,0,6]$	1	0
14	$E_8+D_8+A_2+A_1$	1	1	$[2,0,6]$	1	0
15	$E_8+D_7+A_4$	1	1	$[2,0,20]$	1	0
16	$E_8+D_7+A_3+A_1$	1	1	$[4,0,4]$	1	0
17	$E_8+D_6+D_5$	1	1	$[2,0,4]$	1	0
18	$E_8+D_6+A_5$	1	1	$[4,2,4]$	1	0
19	$E_8+D_6+A_3+A_2$	1	1	$[4,0,6]$	1	0
20	$E_8+D_5+A_6$	2	1	$[6,2,10]$	0	1
21	$E_8+D_5+A_5+A_1$	1	1	$[2,0,12]$	1	0
22	$E_8+D_5+A_4+A_2$	1	1	$[6,0,20]$	1	0
23	E_8+A_{11}	2	1	$[4,0,6]$	0	1
24	$E_8+A_{10}+A_1$	3	1	$[6,2,8]$	0	1
				$[2,0,22]$	1	0
25	$E_8+A_9+A_2$	2	2	$[6,0,10]$	1	0
				$[4,1,4]$	1	0
26	$E_8+A_9+2A_1$	1	1	$[2,0,10]$	1	0
27	$E_8+A_8+A_3$	1	1	$[4,0,18]$	1	0
28	$E_8+A_8+A_2+A_1$	3	1	$[6,0,18]$	1	1
29	$E_8+A_7+A_4$	2	1	$[6,2,14]$	0	1
30	$E_8+A_7+A_3+A_1$	2	1	$[4,0,8]$	0	1
31	$E_8+A_7+2A_2$	1	1	$[6,0,24]$	1	0
32	$E_8+A_7+A_2+2A_1$	1	1	$[2,0,24]$	1	0
33	$E_8+A_6+A_5$	2	1	$[4,2,22]$	0	1
34	$E_8+A_6+A_4+A_1$	3	1	$[8,2,18]$	0	1
				$[2,0,70]$	1	0
35	$E_8+A_6+A_3+A_2$	1	1	$[4,0,42]$	1	0
36	$E_8+A_6+2A_2+A_1$	1	1	$[6,0,42]$	1	0
37	$E_8+A_5+A_4+A_2$	2	1	$[6,0,30]$	2	0
38	$E_8+A_5+A_4+2A_1$	1	1	$[8,2,8]$	1	0

39	$E_8 + A_5 + 2A_3$	1	1	$[4, 0, 12]$	1	0
40	$E_8 + A_5 + A_3 + A_2 + A_1$	1	1	$[6, 0, 12]$	1	0
41	$E_8 + A_4 + 2A_3 + A_1$	1	1	$[4, 0, 20]$	1	0
42	$E_8 + A_4 + A_3 + 2A_2$	1	1	$[12, 0, 30]$	1	0
43	$2E_7 + D_5$	2	2	$[2, 0, 4]$	1	0
				$[2, 0, 4]$	1	0
44	$2E_7 + D_4 + A_1$	1	1	$[2, 0, 2]$	1	0
45	$2E_7 + A_5$	1	1	$[4, 2, 4]$	1	0
46	$2E_7 + A_3 + A_2$	2	2	$[4, 0, 6]$	1	0
				$[4, 0, 6]$	1	0
47	$2E_7 + A_2 + 3A_1$	1	1	$[2, 0, 6]$	1	0
48	$E_7 + 2E_6$	1	1	$[6, 0, 6]$	1	0
49	$E_7 + E_6 + D_6$	1	1	$[4, 2, 4]$	1	0
50	$E_7 + E_6 + D_5 + A_1$	1	1	$[2, 0, 12]$	1	0
51	$E_7 + E_6 + A_6$	2	1	$[4, 2, 22]$	0	1
52	$E_7 + E_6 + A_4 + A_2$	2	1	$[6, 0, 30]$	2	0
53	$E_7 + E_6 + A_4 + 2A_1$	1	1	$[8, 2, 8]$	1	0
54	$E_7 + E_6 + 2A_3$	1	1	$[4, 0, 12]$	1	0
55	$E_7 + E_6 + A_3 + A_2 + A_1$	1	1	$[6, 0, 12]$	1	0
56	$E_7 + D_{12}$	2	2	$[2, 0, 2]$	1	0
				$[2, 0, 2]$	1	0
57	$E_7 + D_{11} + A_1$	1	1	$[2, 0, 4]$	1	0
58	$E_7 + D_{10} + A_2$	3	3	$[2, 0, 6]$	1	0
				$[2, 0, 6]$	1	0
				$[2, 1, 2]$	1	0
59	$E_7 + D_{10} + 2A_1$	1	1	$[2, 0, 2]$	1	0
60	$E_7 + D_9 + A_3$	1	1	$[4, 0, 4]$	1	0
61	$E_7 + D_9 + A_2 + A_1$	1	1	$[4, 0, 6]$	1	0
62	$E_7 + D_8 + A_4$	1	1	$[4, 2, 6]$	1	0
63	$E_7 + D_8 + A_3 + A_1$	3	3	$[2, 0, 4]$	1	0
				$[2, 0, 4]$	1	0
				$[2, 0, 4]$	1	0
64	$E_7 + D_8 + 2A_2$	1	1	$[6, 0, 6]$	1	0
65	$E_7 + D_8 + A_2 + 2A_1$	1	1	$[2, 0, 6]$	1	0
66	$E_7 + D_8 + 4A_1$	1	1	$[2, 0, 2]$	1	0
67	$E_7 + D_7 + D_5$	1	1	$[4, 0, 4]$	1	0
68	$E_7 + D_7 + D_4 + A_1$	1	1	$[2, 0, 4]$	1	0
69	$E_7 + D_7 + A_5$	3	3	$[2, 0, 12]$	1	0
				$[2, 0, 12]$	1	0
				$[2, 0, 12]$	1	0
70	$E_7 + D_7 + A_4 + A_1$	1	1	$[2, 0, 20]$	1	0
71	$E_7 + 2D_6$	2	2	$[2, 0, 2]$	1	0
				$[2, 0, 2]$	1	0
72	$E_7 + D_6 + D_5 + A_1$	1	1	$[2, 0, 4]$	1	0
73	$E_7 + D_6 + D_4 + A_2$	1	1	$[2, 0, 6]$	1	0

74	$E_7 + D_6 + A_4 + 2A_1$	1	1	$[4,2,6]$	1	0
75	$E_7 + D_6 + 2A_3$	1	1	$[4,0,4]$	1	0
76	$E_7 + D_6 + A_3 + A_2 + A_1$	1	1	$[4,0,6]$	1	0
77	$E_7 + D_6 + A_3 + 3A_1$	1	1	$[2,0,4]$	1	0
78	$E_7 + D_6 + 2A_2 + 2A_1$	1	1	$[6,0,6]$	1	0
79	$E_7 + 2D_5 + 2A_1$	1	1	$[4,0,4]$	1	0
80	$E_7 + D_5 + D_4 + A_2 + A_1$	1	1	$[4,0,6]$	1	0
81	$E_7 + D_5 + A_7$	2	1	$[4,0,8]$	0	1
82	$E_7 + D_5 + A_6 + A_1$	2	1	$[6,2,10]$	0	1
83	$E_7 + D_5 + A_5 + A_2$	3	3	$[6,0,12]$	1	0
				$[6,0,12]$	1	0
				$[6,0,12]$	1	0
84	$E_7 + D_5 + A_4 + A_3$	2	1	$[4,0,20]$	2	0
85	$E_7 + D_5 + A_4 + A_2 + A_1$	1	1	$[6,0,20]$	1	0
86	$E_7 + D_5 + A_3 + 2A_2$	1	1	$[12,0,12]$	1	0
87	$E_7 + D_4 + A_8$	1	1	$[2,0,18]$	1	0
88	$E_7 + D_4 + A_6 + A_2$	1	1	$[2,0,42]$	1	0
89	$E_7 + D_4 + A_5 + A_3$	1	1	$[2,0,12]$	1	0
90	$E_7 + D_4 + A_5 + 3A_1$	1	1	$[4,2,4]$	1	0
91	$E_7 + D_4 + 2A_4$	1	1	$[10,0,10]$	1	0
92	$E_7 + D_4 + A_4 + A_3 + A_1$	1	1	$[2,0,20]$	1	0
93	$E_7 + D_4 + 2A_3 + 2A_1$	1	1	$[4,0,4]$	1	0
94	$E_7 + A_{12}$	2	1	$[4,2,14]$	0	1
95	$E_7 + A_{11} + A_1$	2	1	$[4,0,6]$	0	1
96	$E_7 + A_{10} + A_2$	2	1	$[6,0,22]$	2	0
97	$E_7 + A_9 + A_3$	3	3	$[4,0,10]$	1	0
				$[4,0,10]$	1	0
				$[4,0,10]$	1	0
98	$E_7 + A_9 + A_2 + A_1$	4	3	$[4,2,16]$	0	1
				$[6,0,10]$	1	0
				$[6,0,10]$	1	0
99	$E_7 + A_9 + 3A_1$	1	1	$[2,0,10]$	1	0
100	$E_7 + A_8 + A_4$	2	1	$[4,2,46]$	0	1
101	$E_7 + A_8 + A_3 + A_1$	1	1	$[4,0,18]$	1	0
102	$E_7 + A_7 + A_4 + A_1$	4	2	$[6,2,14]$	0	1
				$[6,2,14]$	0	1
103	$E_7 + A_7 + A_3 + A_2$	2	1	$[4,0,24]$	2	0
104	$E_7 + A_7 + A_3 + 2A_1$	1	1	$[4,0,8]$	1	0
105	$E_7 + A_7 + 2A_2 + A_1$	2	2	$[6,0,24]$	1	0
				$[6,0,24]$	1	0
106	$E_7 + A_7 + A_2 + 3A_1$	1	1	$[2,0,24]$	1	0
107	$E_7 + 2A_6$	2	1	$[14,0,14]$	0	1
108	$E_7 + A_6 + A_4 + A_2$	2	1	$[6,0,70]$	2	0
109	$E_7 + A_6 + A_3 + A_2 + A_1$	1	1	$[4,0,42]$	1	0
110	$E_7 + A_6 + 2A_2 + 2A_1$	1	1	$[12,6,24]$	1	0

111	$E_7 + 2A_5 + 2A_1$	2	2	$[6, 0, 6]$	1	0
				$[6, 0, 6]$	1	0
112	$E_7 + A_5 + A_4 + A_3$	3	3	$[4, 0, 30]$	1	0
				$[4, 0, 30]$	1	0
				$[4, 0, 30]$	1	0
113	$E_7 + A_5 + A_4 + 3A_1$	1	1	$[2, 0, 30]$	1	0
114	$E_7 + A_5 + 2A_3 + A_1$	3	3	$[4, 0, 12]$	1	0
				$[4, 0, 12]$	1	0
				$[4, 0, 12]$	1	0
115	$E_7 + A_5 + A_3 + 4A_1$	1	1	$[2, 0, 12]$	1	0
116	$E_7 + 2A_4 + 2A_2$	1	1	$[30, 0, 30]$	1	0
117	$E_7 + A_4 + A_3 + 2A_2 + A_1$	1	1	$[12, 0, 30]$	1	0
118	$E_7 + 2A_3 + 2A_2 + 2A_1$	1	1	$[12, 0, 12]$	1	0
119	$3E_6 + A_1$	1	1	$[2, 0, 6]$	1	0
120	$2E_6 + A_7$	1	1	$[6, 0, 24]$	1	0
121	$2E_6 + A_6 + A_1$	1	1	$[6, 0, 42]$	1	0
122	$2E_6 + A_5 + A_2$	2	1	$[6, 0, 6]$	2	0
123	$2E_6 + A_4 + A_3$	1	1	$[12, 0, 30]$	1	0
124	$2E_6 + A_3 + 2A_2$	1	1	$[6, 0, 12]$	1	0
125	$E_6 + D_{13}$	1	1	$[2, 0, 12]$	1	0
126	$E_6 + D_{11} + A_2$	1	1	$[6, 0, 12]$	1	0
127	$E_6 + D_{10} + A_3$	1	1	$[2, 0, 12]$	1	0
128	$E_6 + D_{10} + 3A_1$	1	1	$[4, 2, 4]$	1	0
129	$E_6 + D_9 + A_4$	1	1	$[4, 0, 30]$	1	0
130	$E_6 + D_9 + A_3 + A_1$	1	1	$[4, 0, 12]$	1	0
131	$E_6 + D_8 + A_5$	1	1	$[6, 0, 6]$	1	0
132	$E_6 + D_8 + A_4 + A_1$	1	1	$[2, 0, 30]$	1	0
133	$E_6 + D_8 + A_3 + 2A_1$	1	1	$[2, 0, 12]$	1	0
134	$E_6 + D_7 + D_6$	1	1	$[2, 0, 12]$	1	0
135	$E_6 + D_7 + D_5 + A_1$	1	1	$[4, 0, 12]$	1	0
136	$E_6 + D_7 + A_6$	1	1	$[2, 0, 84]$	1	0
137	$E_6 + D_6 + D_5 + A_2$	1	1	$[6, 0, 12]$	1	0
138	$E_6 + D_6 + A_5 + 2A_1$	1	1	$[6, 0, 6]$	1	0
139	$E_6 + D_6 + A_4 + A_3$	1	1	$[4, 0, 30]$	1	0
140	$E_6 + D_6 + 2A_3 + A_1$	1	1	$[4, 0, 12]$	1	0
141	$E_6 + D_5 + A_8$	3	1	$[12, 0, 18]$	1	1
142	$E_6 + D_5 + A_6 + A_2$	2	1	$[12, 0, 42]$	1	0
				$[6, 0, 84]$	1	0
143	$E_6 + D_5 + A_5 + A_3$	2	1	$[12, 0, 12]$	2	0
144	$E_6 + D_5 + 2A_4$	1	1	$[20, 0, 30]$	1	0
145	$E_6 + D_5 + A_4 + A_3 + A_1$	1	1	$[12, 0, 20]$	1	0
146	$E_6 + A_{13}$	2	1	$[4, 2, 22]$	0	1
147	$E_6 + A_{12} + A_1$	2	1	$[10, 2, 16]$	0	1
148	$E_6 + A_{11} + A_2$	1	1	$[4, 0, 6]$	1	0

149	$E_6+A_{11}+2A_1$	2	2	$[6,0,12]$	1	0
				$[2,0,4]$	1	0
150	$E_6+A_{10}+A_3$	2	1	$[12,0,22]$	1	0
				$[4,0,66]$	1	0
151	$E_6+A_{10}+A_2+A_1$	3	1	$[18,6,24]$	0	1
				$[6,0,66]$	1	0
152	$E_6+A_9+A_4$	4	2	$[10,0,30]$	1	1
				$[10,5,10]$	1	0
153	$E_6+A_9+A_3+A_1$	2	2	$[10,0,12]$	1	0
				$[10,0,12]$	1	0
154	$E_6+A_9+A_2+2A_1$	1	1	$[12,6,18]$	1	0
155	$E_6+A_8+A_4+A_1$	3	1	$[18,0,30]$	1	1
156	$E_6+A_8+A_3+A_2$	1	1	$[4,0,18]$	1	0
157	$E_6+A_8+2A_2+A_1$	3	1	$[6,0,18]$	1	1
158	$E_6+A_7+A_6$	2	1	$[16,4,22]$	0	1
159	$E_6+A_7+A_5+A_1$	4	2	$[6,0,24]$	0	1
				$[6,0,24]$	0	1
160	$E_6+A_7+A_4+A_2$	2	1	$[24,0,30]$	1	0
				$[6,0,120]$	1	0
161	$E_6+A_7+A_4+2A_1$	1	1	$[2,0,120]$	1	0
162	$E_6+A_7+A_3+A_2+A_1$	2	1	$[12,0,24]$	2	0
163	$E_6+A_7+A_3+3A_1$	1	1	$[8,0,12]$	1	0
164	$E_6+A_6+A_5+2A_1$	2	1	$[12,6,24]$	2	0
165	$E_6+A_6+A_4+A_3$	1	1	$[12,0,70]$	1	0
166	$E_6+A_6+A_4+A_2+A_1$	3	1	$[30,0,42]$	1	0
				$[18,6,72]$	0	1
167	$E_6+A_6+2A_3+A_1$	1	1	$[4,0,84]$	1	0
168	$E_6+2A_5+A_3$	2	2	$[4,0,6]$	1	0
				$[4,0,6]$	1	0
169	$E_6+2A_5+3A_1$	1	1	$[2,0,6]$	1	0
170	$E_6+A_5+2A_4$	2	1	$[30,0,30]$	2	0
171	$E_6+A_5+A_4+A_3+A_1$	2	1	$[12,0,30]$	2	0
172	$E_6+A_5+A_4+2A_2$	2	1	$[6,0,30]$	2	0
173	$E_6+A_5+2A_3+2A_1$	1	1	$[12,0,12]$	1	0
174	$E_6+A_5+A_3+2A_2+A_1$	1	1	$[6,0,12]$	1	0
175	$E_6+A_5+4A_2$	1	1	$[6,0,6]$	1	0
176	$E_6+A_4+2A_3+A_2+A_1$	1	1	$[24,12,36]$	1	0
177	D_{19}	1	1	$[2,0,4]$	1	0
178	$D_{18}+A_1$	1	1	$[2,0,2]$	1	0
179	$D_{17}+A_2$	1	1	$[4,0,6]$	1	0
180	$D_{16}+A_3$	1	1	$[2,0,4]$	1	0
181	$D_{16}+A_2+A_1$	2	2	$[2,0,6]$	1	0
				$[2,0,6]$	1	0
182	$D_{15}+A_4$	1	1	$[2,0,20]$	1	0
183	$D_{15}+A_3+A_1$	1	1	$[4,0,4]$	1	0
184	$D_{14}+D_5$	1	1	$[2,0,4]$	1	0

185	$D_{14}+D_4+A_1$	1	1	$[2,0,2]$	1	0
186	$D_{14}+A_5$	1	1	$[4,2,4]$	1	0
187	$D_{14}+A_4+A_1$	2	1	$[4,2,6]$	0	1
188	$D_{14}+A_3+A_2$	1	1	$[4,0,6]$	1	0
189	$D_{14}+A_3+2A_1$	1	1	$[2,0,4]$	1	0
190	$D_{14}+2A_2+A_1$	1	1	$[6,0,6]$	1	0
191	$D_{13}+D_6$	1	1	$[2,0,4]$	1	0
192	$D_{13}+A_6$	2	1	$[6,2,10]$	0	1
193	$D_{13}+A_5+A_1$	1	1	$[2,0,12]$	1	0
194	$D_{13}+A_4+A_2$	1	1	$[6,0,20]$	1	0
195	$D_{12}+D_7$	1	1	$[2,0,4]$	1	0
196	$D_{12}+D_6+A_1$	1	1	$[2,0,2]$	1	0
197	$D_{12}+D_5+A_2$	1	1	$[4,0,6]$	1	0
198	$D_{12}+D_5+2A_1$	1	1	$[2,0,4]$	1	0
199	$D_{12}+D_4+A_2+A_1$	1	1	$[2,0,6]$	1	0
200	$D_{12}+A_5+2A_1$	1	1	$[4,2,4]$	1	0
201	$D_{12}+A_4+A_3$	1	1	$[2,0,20]$	1	0
202	$D_{12}+2A_3+A_1$	1	1	$[4,0,4]$	1	0
203	$D_{12}+A_3+A_2+2A_1$	1	1	$[4,0,6]$	1	0
204	$D_{11}+D_7+A_1$	1	1	$[4,0,4]$	1	0
205	$D_{11}+D_6+A_2$	1	1	$[4,0,6]$	1	0
206	$D_{11}+A_8$	1	1	$[4,0,18]$	1	0
207	$D_{11}+A_7+A_1$	2	1	$[4,0,8]$	0	1
208	$D_{11}+A_6+A_2$	1	1	$[4,0,42]$	1	0
209	$D_{11}+A_5+A_3$	2	2	$[4,0,12]$	1	0
				$[4,0,12]$	1	0
210	$D_{11}+A_5+A_2+A_1$	1	1	$[6,0,12]$	1	0
211	$D_{11}+A_4+A_3+A_1$	2	1	$[4,0,20]$	2	0
212	$D_{11}+A_4+2A_2$	1	1	$[12,0,30]$	1	0
213	$D_{10}+D_9$	1	1	$[2,0,4]$	1	0
214	$D_{10}+D_8+A_1$	2	2	$[2,0,2]$	1	0
				$[2,0,2]$	1	0
215	$D_{10}+D_7+A_2$	1	1	$[4,0,6]$	1	0
216	$D_{10}+D_7+2A_1$	1	1	$[2,0,4]$	1	0
217	$D_{10}+D_6+A_3$	2	2	$[2,0,4]$	1	0
				$[2,0,4]$	1	0
218	$D_{10}+D_6+A_2+A_1$	2	2	$[2,0,6]$	1	0
				$[2,0,6]$	1	0
219	$D_{10}+D_6+3A_1$	1	1	$[2,0,2]$	1	0
220	$D_{10}+D_5+A_4$	1	1	$[2,0,20]$	1	0
221	$D_{10}+D_5+A_3+A_1$	2	2	$[4,0,4]$	1	0
				$[4,0,4]$	1	0
222	$D_{10}+D_5+A_2+2A_1$	1	1	$[4,0,6]$	1	0
223	$D_{10}+D_4+A_5$	1	1	$[4,2,4]$	1	0
224	$D_{10}+D_4+A_4+A_1$	1	1	$[4,2,6]$	1	0

225	$D_{10}+D_4+A_3+2A_1$	1	1	$[2,0,4]$	1	0
226	$D_{10}+D_4+2A_2+A_1$	1	1	$[6,0,6]$	1	0
227	$D_{10}+A_9$	3	2	$[2,0,10]$	1	0
				$[2,0,10]$	2	0
228	$D_{10}+A_8+A_1$	2	1	$[2,0,18]$	2	0
229	$D_{10}+A_7+A_2$	2	2	$[2,0,24]$	1	0
				$[2,0,24]$	1	0
230	$D_{10}+A_6+A_3$	2	1	$[6,2,10]$	0	1
231	$D_{10}+A_6+A_2+A_1$	2	1	$[2,0,42]$	2	0
232	$D_{10}+A_5+A_4$	4	4	$[2,0,30]$	1	0
				$[8,2,8]$	1	0
				$[2,0,30]$	1	0
				$[2,1,8]$	1	0
233	$D_{10}+A_5+A_3+A_1$	4	4	$[2,0,12]$	1	0
				$[2,0,12]$	1	0
				$[2,0,12]$	1	0
				$[2,0,12]$	1	0
234	$D_{10}+2A_4+A_1$	1	1	$[10,0,10]$	1	0
235	$D_{10}+A_4+A_3+A_2$	1	1	$[6,0,20]$	1	0
236	$D_{10}+A_4+A_3+2A_1$	1	1	$[2,0,20]$	1	0
237	$D_9+D_8+2A_1$	1	1	$[2,0,4]$	1	0
238	$D_9+D_6+A_4$	1	1	$[2,0,20]$	1	0
239	$D_9+D_6+A_3+A_1$	1	1	$[4,0,4]$	1	0
240	$D_9+D_5+A_5$	2	2	$[4,0,12]$	1	0
				$[4,0,12]$	1	0
241	$D_9+D_5+2A_2+A_1$	1	1	$[12,0,12]$	1	0
242	$D_9+D_4+A_5+A_1$	1	1	$[2,0,12]$	1	0
243	D_9+A_{10}	1	1	$[4,0,22]$	1	0
244	$D_9+A_9+A_1$	2	2	$[4,0,10]$	1	0
				$[4,0,10]$	1	0
245	$D_9+A_7+A_3$	1	1	$[2,0,8]$	1	0
246	$D_9+A_7+A_2+A_1$	2	1	$[4,0,24]$	2	0
247	$D_9+A_6+A_4$	1	1	$[4,0,70]$	1	0
248	$D_9+A_5+A_4+A_1$	1	1	$[4,0,30]$	1	0
249	$D_9+A_5+A_3+2A_1$	1	1	$[4,0,12]$	1	0
250	$D_9+2A_4+A_2$	1	1	$[10,0,60]$	1	0
251	$2D_8+A_3$	1	1	$[2,0,4]$	1	0
252	$2D_8+A_2+A_1$	2	2	$[2,0,6]$	1	0
				$[2,0,6]$	1	0
253	$2D_8+3A_1$	1	1	$[2,0,2]$	1	0
254	$D_8+D_7+A_3+A_1$	1	1	$[4,0,4]$	1	0
255	$D_8+D_7+A_2+2A_1$	1	1	$[4,0,6]$	1	0
256	$D_8+D_6+D_5$	1	1	$[2,0,4]$	1	0
257	$D_8+D_6+D_4+A_1$	2	2	$[2,0,2]$	1	0
				$[2,0,2]$	1	0
258	$D_8+D_6+A_5$	1	1	$[4,2,4]$	1	0

259	$D_8 + D_6 + A_4 + A_1$	2	1	$[4, 2, 6]$	0	1
260	$D_8 + D_6 + A_3 + A_2$	1	1	$[4, 0, 6]$	1	0
261	$D_8 + D_6 + A_3 + 2A_1$	3	3	$[2, 0, 4]$	1	0
				$[2, 0, 4]$	1	0
				$[2, 0, 4]$	1	0
262	$D_8 + D_6 + 2A_2 + A_1$	1	1	$[6, 0, 6]$	1	0
263	$D_8 + D_6 + A_2 + 3A_1$	1	1	$[2, 0, 6]$	1	0
264	$D_8 + D_5 + D_4 + 2A_1$	1	1	$[2, 0, 4]$	1	0
265	$D_8 + D_5 + A_5 + A_1$	3	3	$[2, 0, 12]$	1	0
				$[2, 0, 12]$	1	0
				$[2, 0, 12]$	1	0
266	$D_8 + D_5 + A_4 + 2A_1$	1	1	$[2, 0, 20]$	1	0
267	$D_8 + D_5 + A_3 + 3A_1$	1	1	$[4, 0, 4]$	1	0
268	$D_8 + D_4 + A_5 + 2A_1$	1	1	$[4, 2, 4]$	1	0
269	$D_8 + D_4 + 2A_3 + A_1$	1	1	$[4, 0, 4]$	1	0
270	$D_8 + D_4 + A_3 + A_2 + 2A_1$	1	1	$[4, 0, 6]$	1	0
271	$D_8 + A_{11}$	1	1	$[4, 0, 6]$	1	0
272	$D_8 + A_{10} + A_1$	3	1	$[6, 2, 8]$	0	1
				$[2, 0, 22]$	1	0
273	$D_8 + A_9 + A_2$	2	2	$[6, 0, 10]$	1	0
				$[4, 2, 16]$	1	0
274	$D_8 + A_9 + 2A_1$	2	1	$[2, 0, 10]$	2	0
275	$D_8 + A_8 + A_2 + A_1$	3	1	$[6, 0, 18]$	1	1
276	$D_8 + A_7 + A_4$	2	1	$[6, 2, 14]$	0	1
277	$D_8 + A_7 + A_3 + A_1$	3	2	$[4, 0, 8]$	1	0
				$[4, 0, 8]$	0	1
278	$D_8 + A_7 + 2A_2$	1	1	$[6, 0, 24]$	1	0
279	$D_8 + A_7 + A_2 + 2A_1$	2	2	$[2, 0, 24]$	1	0
				$[2, 0, 24]$	1	0
280	$D_8 + A_6 + A_5$	1	1	$[4, 2, 22]$	1	0
281	$D_8 + A_6 + A_4 + A_1$	3	1	$[8, 2, 18]$	0	1
				$[2, 0, 70]$	1	0
282	$D_8 + A_6 + A_3 + 2A_1$	2	1	$[6, 2, 10]$	0	1
283	$D_8 + A_6 + 2A_2 + A_1$	1	1	$[6, 0, 42]$	1	0
284	$D_8 + 2A_5 + A_1$	3	2	$[6, 0, 6]$	1	0
				$[6, 0, 6]$	0	1
285	$D_8 + A_5 + A_4 + A_2$	1	1	$[6, 0, 30]$	1	0
286	$D_8 + A_5 + A_4 + 2A_1$	3	2	$[8, 2, 8]$	0	1
				$[2, 0, 30]$	1	0
287	$D_8 + A_5 + 2A_3$	1	1	$[4, 0, 12]$	1	0
288	$D_8 + A_5 + A_3 + A_2 + A_1$	3	3	$[6, 0, 12]$	1	0
				$[6, 0, 12]$	1	0
				$[6, 0, 12]$	1	0
289	$D_8 + A_5 + A_3 + 3A_1$	1	1	$[2, 0, 12]$	1	0
290	$D_8 + A_4 + 2A_3 + A_1$	1	1	$[4, 0, 20]$	1	0
291	$D_8 + A_4 + A_3 + A_2 + 2A_1$	1	1	$[6, 0, 20]$	1	0

292	$2D_7 + A_5$	1	1	$[4, 0, 12]$	1	0
293	$2D_7 + A_4 + A_1$	1	1	$[4, 0, 20]$	1	0
294	$D_7 + 2D_6$	1	1	$[2, 0, 4]$	1	0
295	$D_7 + D_6 + D_5 + A_1$	1	1	$[4, 0, 4]$	1	0
296	$D_7 + D_6 + A_6$	2	1	$[6, 2, 10]$	0	1
297	$D_7 + D_6 + A_5 + A_1$	1	1	$[2, 0, 12]$	1	0
298	$D_7 + D_6 + A_4 + A_2$	1	1	$[6, 0, 20]$	1	0
299	$D_7 + D_5 + A_7$	1	1	$[2, 0, 8]$	1	0
300	$D_7 + D_5 + A_5 + 2A_1$	1	1	$[4, 0, 12]$	1	0
301	$D_7 + D_4 + A_7 + A_1$	1	1	$[4, 0, 8]$	1	0
302	$D_7 + D_4 + A_5 + A_2 + A_1$	1	1	$[6, 0, 12]$	1	0
303	$D_7 + A_{12}$	3	1	$[6, 2, 18]$	0	1
				$[2, 0, 52]$	1	0
304	$D_7 + A_{11} + A_1$	2	2	$[2, 0, 6]$	1	0
				$[2, 0, 6]$	1	0
305	$D_7 + A_{10} + A_2$	2	1	$[14, 4, 20]$	0	1
306	$D_7 + A_9 + A_3$	2	1	$[8, 4, 12]$	0	1
307	$D_7 + A_9 + A_2 + A_1$	2	2	$[2, 0, 60]$	1	0
				$[2, 0, 60]$	1	0
308	$D_7 + A_8 + A_4$	2	1	$[18, 0, 20]$	1	0
				$[2, 0, 180]$	1	0
309	$D_7 + A_8 + A_3 + A_1$	2	1	$[8, 4, 20]$	0	1
310	$D_7 + A_7 + A_5$	2	1	$[8, 0, 12]$	0	1
311	$D_7 + A_7 + A_4 + A_1$	2	1	$[8, 0, 20]$	0	1
312	$D_7 + A_7 + A_3 + A_2$	1	1	$[6, 0, 8]$	1	0
313	$D_7 + A_7 + A_3 + 2A_1$	1	1	$[2, 0, 8]$	1	0
314	$D_7 + 2A_6$	2	1	$[14, 0, 28]$	0	1
315	$D_7 + A_6 + A_5 + A_1$	1	1	$[2, 0, 84]$	1	0
316	$D_7 + A_6 + A_4 + A_2$	1	1	$[20, 0, 42]$	1	0
317	$D_7 + A_5 + A_4 + A_3$	2	2	$[12, 0, 20]$	1	0
				$[12, 0, 20]$	1	0
318	$D_7 + 2A_4 + A_3 + A_1$	1	1	$[20, 0, 20]$	1	0
319	$3D_6 + A_1$	1	1	$[2, 0, 2]$	1	0
320	$2D_6 + D_5 + A_2$	1	1	$[4, 0, 6]$	1	0
321	$2D_6 + D_5 + 2A_1$	1	1	$[2, 0, 4]$	1	0
322	$2D_6 + D_4 + A_3$	1	1	$[2, 0, 4]$	1	0
323	$2D_6 + D_4 + A_2 + A_1$	1	1	$[2, 0, 6]$	1	0
324	$2D_6 + D_4 + 3A_1$	2	2	$[2, 0, 2]$	1	0
				$[2, 0, 2]$	1	0
325	$2D_6 + A_5 + 2A_1$	1	1	$[4, 2, 4]$	1	0
326	$2D_6 + A_4 + A_3$	1	1	$[2, 0, 20]$	1	0
327	$2D_6 + 2A_3 + A_1$	1	1	$[4, 0, 4]$	1	0
328	$2D_6 + A_3 + A_2 + 2A_1$	1	1	$[4, 0, 6]$	1	0
329	$2D_6 + A_3 + 4A_1$	1	1	$[2, 0, 4]$	1	0
330	$D_6 + D_5 + D_4 + A_3 + A_1$	1	1	$[4, 0, 4]$	1	0

331	$D_6 + D_5 + A_8$	1	1	$[4, 0, 18]$	1	0
332	$D_6 + D_5 + A_7 + A_1$	2	1	$[4, 0, 8]$	0	1
333	$D_6 + D_5 + A_6 + A_2$	1	1	$[4, 0, 42]$	1	0
334	$D_6 + D_5 + A_5 + A_3$	4	4	$[4, 0, 12]$	1	0
				$[4, 0, 12]$	1	0
				$[4, 0, 12]$	1	0
				$[4, 0, 12]$	1	0
335	$D_6 + D_5 + A_5 + A_2 + A_1$	1	1	$[6, 0, 12]$	1	0
336	$D_6 + D_5 + A_5 + 3A_1$	1	1	$[2, 0, 12]$	1	0
337	$D_6 + D_5 + A_4 + A_3 + A_1$	2	1	$[4, 0, 20]$	2	0
338	$D_6 + D_5 + A_4 + 2A_2$	1	1	$[12, 0, 30]$	1	0
339	$D_6 + D_5 + A_3 + 2A_2 + A_1$	1	1	$[12, 0, 12]$	1	0
340	$D_6 + 2D_4 + A_3 + 2A_1$	1	1	$[2, 0, 4]$	1	0
341	$D_6 + D_4 + A_9$	1	1	$[2, 0, 10]$	1	0
342	$D_6 + D_4 + A_7 + A_2$	1	1	$[2, 0, 24]$	1	0
343	$D_6 + D_4 + A_5 + A_4$	2	2	$[2, 0, 30]$	1	0
				$[8, 2, 8]$	1	0
344	$D_6 + D_4 + A_5 + A_3 + A_1$	3	3	$[2, 0, 12]$	1	0
				$[2, 0, 12]$	1	0
				$[2, 0, 12]$	1	0
345	$D_6 + D_4 + 2A_3 + 3A_1$	1	1	$[4, 0, 4]$	1	0
346	$D_6 + A_{11} + 2A_1$	1	1	$[4, 0, 6]$	1	0
347	$D_6 + A_{10} + A_3$	1	1	$[4, 0, 22]$	1	0
348	$D_6 + A_9 + A_3 + A_1$	2	2	$[4, 0, 10]$	1	0
				$[4, 0, 10]$	1	0
349	$D_6 + A_9 + A_2 + 2A_1$	2	2	$[6, 0, 10]$	1	0
				$[4, 2, 16]$	1	0
350	$D_6 + A_7 + A_4 + 2A_1$	2	1	$[6, 2, 14]$	0	1
351	$D_6 + A_7 + 2A_3$	1	1	$[2, 0, 8]$	1	0
352	$D_6 + A_7 + A_3 + A_2 + A_1$	2	1	$[4, 0, 24]$	2	0
353	$D_6 + A_7 + A_3 + 3A_1$	1	1	$[4, 0, 8]$	1	0
354	$D_6 + A_7 + 2A_2 + 2A_1$	1	1	$[6, 0, 24]$	1	0
355	$D_6 + A_6 + A_5 + 2A_1$	1	1	$[4, 2, 22]$	1	0
356	$D_6 + A_6 + A_4 + A_3$	1	1	$[4, 0, 70]$	1	0
357	$D_6 + 2A_5 + 3A_1$	1	1	$[6, 0, 6]$	1	0
358	$D_6 + A_5 + A_4 + A_3 + A_1$	1	1	$[4, 0, 30]$	1	0
359	$D_6 + A_5 + A_4 + A_2 + 2A_1$	1	1	$[6, 0, 30]$	1	0
360	$D_6 + A_5 + 2A_3 + 2A_1$	2	2	$[4, 0, 12]$	1	0
				$[4, 0, 12]$	1	0
361	$D_6 + A_5 + A_3 + A_2 + 3A_1$	1	1	$[6, 0, 12]$	1	0
362	$D_6 + 2A_4 + A_3 + A_2$	1	1	$[10, 0, 60]$	1	0
363	$2D_5 + A_9$	1	1	$[8, 4, 12]$	1	0
364	$2D_5 + A_7 + A_2$	1	1	$[6, 0, 8]$	1	0
365	$2D_5 + A_7 + 2A_1$	1	1	$[2, 0, 8]$	1	0
366	$2D_5 + A_6 + A_2 + A_1$	1	1	$[8, 4, 44]$	1	0

367	$2D_5 + A_5 + A_4$	1	1	$[12, 0, 20]$	1	0
368	$2D_5 + 2A_3 + A_2 + A_1$	1	1	$[8, 4, 8]$	1	0
369	$D_5 + D_4 + A_9 + A_1$	1	1	$[4, 0, 10]$	1	0
370	$D_5 + D_4 + A_7 + A_2 + A_1$	1	1	$[4, 0, 24]$	1	0
371	$D_5 + D_4 + A_5 + A_4 + A_1$	1	1	$[4, 0, 30]$	1	0
372	$D_5 + D_4 + A_5 + A_3 + 2A_1$	2	2	$[4, 0, 12]$	1	0
				$[4, 0, 12]$	1	0
373	$D_5 + A_{14}$	2	1	$[10, 0, 12]$	0	1
374	$D_5 + A_{13} + A_1$	4	2	$[6, 2, 10]$	0	1
				$[6, 2, 10]$	0	1
375	$D_5 + A_{12} + A_2$	1	1	$[6, 0, 52]$	1	0
376	$D_5 + A_{11} + A_3$	2	2	$[4, 0, 6]$	1	0
				$[4, 0, 6]$	1	0
377	$D_5 + A_{11} + A_2 + A_1$	6	2	$[12, 0, 12]$	0	2
				$[6, 0, 6]$	2	0
378	$D_5 + A_{10} + A_4$	3	1	$[20, 0, 22]$	1	0
				$[12, 4, 38]$	0	1
379	$D_5 + A_{10} + 2A_2$	1	1	$[12, 0, 66]$	1	0
380	$D_5 + A_9 + A_5$	3	3	$[10, 0, 12]$	1	0
				$[10, 0, 12]$	1	0
				$[10, 0, 12]$	1	0
381	$D_5 + A_9 + A_4 + A_1$	6	2	$[10, 0, 20]$	1	1
				$[10, 0, 20]$	1	1
382	$D_5 + A_9 + A_3 + A_2$	2	2	$[4, 0, 60]$	1	0
				$[4, 0, 60]$	1	0
383	$D_5 + A_9 + A_3 + 2A_1$	2	1	$[8, 4, 12]$	0	1
384	$D_5 + A_9 + 2A_2 + A_1$	2	2	$[6, 0, 60]$	1	0
				$[6, 0, 60]$	1	0
385	$D_5 + A_8 + A_6$	2	1	$[10, 4, 52]$	0	1
386	$D_5 + A_8 + A_5 + A_1$	3	1	$[12, 0, 18]$	1	1
387	$D_5 + A_8 + A_4 + A_2$	3	1	$[6, 0, 180]$	1	1
388	$D_5 + A_8 + 3A_2$	1	1	$[12, 0, 18]$	1	0
389	$D_5 + 2A_7$	1	1	$[2, 0, 4]$	1	0
390	$D_5 + A_7 + A_6 + A_1$	4	1	$[12, 4, 20]$	0	2
391	$D_5 + A_7 + A_5 + A_2$	2	1	$[12, 0, 24]$	2	0
392	$D_5 + A_7 + A_5 + 2A_1$	3	2	$[8, 0, 12]$	0	1
				$[8, 0, 12]$	1	0
393	$D_5 + A_7 + A_4 + A_2 + A_1$	2	1	$[20, 0, 24]$	2	0
394	$D_5 + A_7 + 2A_3 + A_1$	3	2	$[4, 0, 8]$	1	0
				$[4, 0, 8]$	2	0
395	$D_5 + A_7 + A_3 + A_2 + 2A_1$	1	1	$[6, 0, 8]$	1	0
396	$D_5 + A_6 + A_5 + A_3$	2	1	$[4, 0, 84]$	2	0
397	$D_5 + A_6 + A_5 + A_2 + A_1$	2	1	$[12, 0, 42]$	1	0
				$[6, 0, 84]$	1	0
398	$D_5 + A_6 + 2A_4$	2	1	$[20, 0, 70]$	1	0
				$[10, 0, 140]$	1	0

399	$D_5 + A_6 + A_4 + 2A_2$	1	1	$[30, 0, 84]$	1	0
400	$D_5 + A_6 + A_3 + 2A_2 + A_1$	1	1	$[12, 0, 84]$	1	0
401	$D_5 + 2A_5 + A_4$	3	2	$[12, 0, 30]$	1	0
				$[12, 0, 30]$	2	0
402	$D_5 + 2A_5 + A_3 + A_1$	3	2	$[12, 0, 12]$	1	0
				$[12, 0, 12]$	2	0
403	$D_5 + 2A_5 + 2A_2$	2	2	$[6, 0, 12]$	1	0
				$[6, 0, 12]$	1	0
404	$D_5 + A_5 + 2A_4 + A_1$	1	1	$[20, 0, 30]$	1	0
405	$D_5 + A_5 + A_4 + A_3 + A_2$	2	1	$[24, 12, 36]$	0	1
406	$D_5 + A_5 + A_4 + A_3 + 2A_1$	1	1	$[12, 0, 20]$	1	0
407	$4D_4 + 3A_1$	1	1	$[2, 0, 2]$	1	0
408	$2D_4 + 2A_5 + A_1$	1	1	$[6, 0, 6]$	1	0
409	$D_4 + A_{15}$	1	1	$[2, 0, 4]$	1	0
410	$D_4 + A_{13} + A_2$	1	1	$[2, 0, 42]$	1	0
411	$D_4 + A_{11} + A_4$	1	1	$[6, 0, 20]$	1	0
412	$D_4 + A_{11} + A_3 + A_1$	1	1	$[2, 0, 6]$	1	0
413	$D_4 + A_{11} + 2A_2$	1	1	$[2, 0, 12]$	1	0
414	$D_4 + A_{10} + A_5$	1	1	$[2, 0, 66]$	1	0
415	$D_4 + A_9 + A_6$	1	1	$[4, 2, 36]$	1	0
416	$D_4 + A_9 + A_4 + A_2$	1	1	$[20, 10, 20]$	1	0
417	$D_4 + A_9 + A_3 + A_2 + A_1$	1	1	$[2, 0, 60]$	1	0
418	$D_4 + A_8 + A_5 + A_2$	1	1	$[2, 0, 18]$	1	0
419	$D_4 + 2A_7 + A_1$	1	1	$[4, 0, 4]$	1	0
420	$D_4 + A_7 + A_5 + A_3$	1	1	$[8, 0, 12]$	1	0
421	$D_4 + A_7 + A_4 + A_3 + A_1$	1	1	$[8, 0, 20]$	1	0
422	$D_4 + A_7 + 2A_3 + 2A_1$	1	1	$[2, 0, 8]$	1	0
423	$D_4 + A_6 + A_5 + A_4$	1	1	$[20, 10, 26]$	1	0
424	$D_4 + A_6 + A_5 + A_3 + A_1$	1	1	$[2, 0, 84]$	1	0
425	$D_4 + 3A_5$	3	3	$[2, 0, 6]$	1	0
				$[2, 0, 6]$	1	0
				$[2, 1, 2]$	1	0
426	A_{19}	2	1	$[2, 0, 20]$	2	0
427	$A_{18} + A_1$	3	1	$[8, 2, 10]$	0	1
				$[2, 0, 38]$	1	0
428	$A_{17} + A_2$	2	2	$[2, 0, 6]$	1	0
				$[2, 1, 2]$	1	0
429	$A_{17} + 2A_1$	2	2	$[4, 2, 10]$	1	0
				$[2, 0, 2]$	1	0
430	$A_{16} + A_3$	2	1	$[4, 0, 34]$	1	0
				$[2, 0, 68]$	1	0
431	$A_{16} + A_2 + A_1$	3	1	$[10, 4, 22]$	0	1
				$[6, 0, 34]$	1	0
432	$A_{15} + A_4$	3	2	$[8, 4, 22]$	0	1
				$[2, 0, 20]$	1	0

433	$A_{15}+A_3+A_1$	2	2	$[4,0,4]$	1	0
				$[2,0,2]$	1	0
434	$A_{15}+A_2+2A_1$	3	2	$[10,2,10]$	0	1
				$[4,0,6]$	1	0
435	$A_{14}+A_4+A_1$	6	1	$[10,0,30]$	0	3
436	$A_{14}+A_3+A_2$	1	1	$[4,0,10]$	1	0
437	$A_{14}+2A_2+A_1$	1	1	$[6,0,10]$	1	0
438	$A_{13}+A_6$	4	1	$[14,0,14]$	0	2
439	$A_{13}+A_5+A_1$	2	1	$[4,2,22]$	0	1
440	$A_{13}+A_4+A_2$	2	1	$[6,0,70]$	2	0
441	$A_{13}+A_4+2A_1$	3	1	$[8,2,18]$	0	1
				$[2,0,70]$	1	0
442	$A_{13}+A_3+A_2+A_1$	2	2	$[4,0,42]$	1	0
				$[4,0,42]$	1	0
443	$A_{13}+2A_2+2A_1$	2	2	$[12,6,24]$	1	0
				$[6,0,42]$	1	0
444	$A_{12}+A_7$	2	1	$[14,4,16]$	0	1
445	$A_{12}+A_6+A_1$	3	1	$[8,2,46]$	0	1
				$[2,0,182]$	1	0
446	$A_{12}+A_5+2A_1$	2	1	$[12,6,16]$	1	0
				$[4,2,40]$	1	0
447	$A_{12}+A_4+A_3$	1	1	$[4,0,130]$	1	0
448	$A_{12}+A_4+A_2+A_1$	3	1	$[24,6,34]$	0	1
				$[6,0,130]$	1	0
449	$A_{12}+2A_3+A_1$	1	1	$[4,0,52]$	1	0
450	$A_{11}+A_7+A_1$	4	1	$[4,0,24]$	0	2
451	$A_{11}+A_6+2A_1$	1	1	$[4,0,42]$	1	0
452	$A_{11}+A_5+A_3$	3	3	$[4,0,4]$	1	0
				$[2,0,2]$	1	0
				$[2,0,2]$	1	0
453	$A_{11}+A_5+A_2+A_1$	1	1	$[4,0,6]$	1	0
454	$A_{11}+A_5+3A_1$	2	2	$[6,0,12]$	1	0
				$[2,0,4]$	1	0
455	$A_{11}+2A_4$	2	1	$[10,0,60]$	2	0
456	$A_{11}+A_4+2A_2$	1	1	$[4,0,30]$	1	0
457	$A_{11}+A_4+A_2+2A_1$	2	1	$[12,0,30]$	0	1
458	$A_{11}+2A_3+2A_1$	1	1	$[4,0,6]$	1	0
459	$A_{11}+A_3+2A_2+A_1$	1	1	$[4,0,12]$	1	0
460	$A_{11}+3A_2+2A_1$	1	1	$[6,0,12]$	1	0
461	$A_{10}+A_9$	4	2	$[10,0,22]$	1	0
				$[2,0,110]$	1	0
				$[8,3,8]$	1	0
				$[2,1,28]$	1	0
462	$A_{10}+A_8+A_1$	3	1	$[18,0,22]$	1	0
				$[10,2,40]$	0	1

463	$A_{10}+A_7+A_2$	2	1	$[22,0,24]$	1	0
				$[6,0,88]$	1	0
464	$A_{10}+A_7+2A_1$	3	1	$[10,2,18]$	0	1
				$[2,0,88]$	1	0
465	$A_{10}+A_6+A_3$	2	1	$[10,2,62]$	0	1
466	$A_{10}+A_6+A_2+A_1$	3	1	$[22,0,42]$	1	0
				$[16,2,58]$	0	1
467	$A_{10}+A_5+A_4$	2	1	$[22,0,30]$	2	0
468	$A_{10}+A_5+A_3+A_1$	2	1	$[12,0,22]$	1	0
				$[4,0,66]$	1	0
469	$A_{10}+2A_4+A_1$	3	1	$[30,10,40]$	0	1
				$[10,0,110]$	1	0
470	$A_{10}+A_4+A_3+A_2$	1	1	$[4,0,330]$	1	0
471	$A_{10}+A_4+2A_2+A_1$	2	1	$[30,0,66]$	1	0
				$[6,0,330]$	1	0
472	$2A_9+A_1$	3	2	$[10,0,10]$	2	0
				$[2,0,2]$	1	0
473	$A_9+A_8+2A_1$	1	1	$[10,0,18]$	1	0
474	$A_9+A_7+A_3$	2	1	$[4,0,40]$	0	1
475	$A_9+A_7+A_2+A_1$	3	3	$[10,0,24]$	1	0
				$[10,0,24]$	1	0
				$[10,0,24]$	1	0
476	$A_9+A_6+A_4$	6	2	$[10,0,70]$	1	1
				$[10,5,20]$	1	1
477	$A_9+A_6+A_3+A_1$	2	2	$[2,0,140]$	1	0
				$[2,0,140]$	1	0
478	$A_9+A_6+A_2+2A_1$	1	1	$[10,0,42]$	1	0
479	$A_9+A_5+A_4+A_1$	6	2	$[10,0,30]$	1	1
				$[10,0,30]$	1	1
480	$A_9+A_5+A_3+2A_1$	1	1	$[10,0,12]$	1	0
481	$A_9+A_5+A_2+3A_1$	1	1	$[12,6,18]$	1	0
482	$A_9+2A_4+A_2$	2	2	$[6,0,10]$	1	0
				$[4,1,4]$	1	0
483	$A_9+2A_4+2A_1$	1	1	$[2,0,10]$	1	0
484	$A_9+2A_3+A_2+2A_1$	1	1	$[4,0,60]$	1	0
485	$2A_8+A_3$	1	1	$[2,0,36]$	1	0
486	$A_8+A_7+A_4$	2	1	$[16,4,46]$	0	1
487	$A_8+A_7+A_3+A_1$	2	1	$[4,0,72]$	0	1
488	$A_8+A_7+A_2+2A_1$	3	1	$[18,0,24]$	1	1
489	$A_8+A_6+A_4+A_1$	3	1	$[22,4,58]$	0	1
				$[18,0,70]$	1	0
490	$A_8+A_5+A_4+A_2$	2	1	$[4,2,46]$	0	1
491	$A_8+A_5+A_4+2A_1$	3	1	$[12,6,48]$	1	1
492	$A_8+A_5+A_3+A_2+A_1$	1	1	$[4,0,18]$	1	0
493	$A_8+A_4+2A_3+A_1$	1	1	$[4,0,180]$	1	0
494	$A_8+A_4+3A_2+A_1$	1	1	$[18,0,30]$	1	0

495	$2A_7 + A_5$	1	1	$[4, 0, 12]$	1	0
496	$2A_7 + A_4 + A_1$	5	2	$[16, 8, 24]$	0	2
				$[4, 0, 20]$	1	0
497	$2A_7 + A_3 + A_2$	1	1	$[4, 0, 6]$	1	0
498	$2A_7 + A_3 + 2A_1$	2	2	$[2, 0, 4]$	1	0
				$[2, 0, 4]$	1	0
499	$2A_7 + 2A_2 + A_1$	1	1	$[24, 0, 24]$	1	0
500	$A_7 + 2A_6$	2	1	$[14, 0, 56]$	0	1
501	$A_7 + A_6 + A_5 + A_1$	4	2	$[16, 4, 22]$	0	1
				$[16, 4, 22]$	0	1
502	$A_7 + A_6 + A_4 + A_2$	2	1	$[24, 0, 70]$	1	0
				$[6, 0, 280]$	1	0
503	$A_7 + A_6 + A_4 + 2A_1$	3	1	$[18, 4, 32]$	0	1
				$[2, 0, 280]$	1	0
504	$A_7 + A_6 + 2A_2 + 2A_1$	1	1	$[24, 0, 42]$	1	0
505	$A_7 + 2A_5 + 2A_1$	3	2	$[6, 0, 24]$	1	0
				$[6, 0, 24]$	0	1
506	$A_7 + A_5 + A_4 + A_3$	2	1	$[4, 0, 120]$	2	0
507	$A_7 + A_5 + A_4 + A_2 + A_1$	4	2	$[24, 0, 30]$	1	0
				$[6, 0, 120]$	1	0
				$[24, 0, 30]$	1	0
				$[6, 0, 120]$	1	0
508	$A_7 + A_5 + A_4 + 3A_1$	1	1	$[2, 0, 120]$	1	0
509	$A_7 + A_5 + A_3 + A_2 + 2A_1$	1	1	$[12, 0, 24]$	1	0
510	$A_7 + 2A_4 + 2A_2$	1	1	$[30, 0, 120]$	1	0
511	$3A_6 + A_1$	1	1	$[2, 0, 14]$	1	0
512	$2A_6 + A_4 + A_2 + A_1$	2	1	$[42, 0, 70]$	2	0
513	$A_6 + 2A_5 + 3A_1$	1	1	$[6, 0, 42]$	1	0
514	$A_6 + A_5 + 2A_4$	2	1	$[30, 0, 70]$	2	0
515	$A_6 + A_5 + A_4 + A_3 + A_1$	1	1	$[12, 0, 70]$	1	0
516	$3A_5 + A_3 + A_1$	1	1	$[4, 0, 6]$	1	0
517	$3A_5 + 2A_2$	1	1	$[4, 2, 4]$	1	0
518	$3A_5 + A_2 + 2A_1$	1	1	$[6, 0, 6]$	1	0
519	$3A_5 + 4A_1$	1	1	$[2, 0, 6]$	1	0

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