ON CERTAIN DUALITY OF NÉRON-SEVERI LATTICES OF SUPERSINGULAR $K^3$ SURFACES AND ITS APPLICATION TO GENERIC SUPERSINGULAR $K^3$ SURFACES

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ABSTRACT. Let $X$ and $Y$ be supersingular $K^3$ surfaces defined over an algebraically closed field. Suppose that the sum of their Artin invariants is $11$. Then there exists a certain duality between their Néron-Severi lattices. We investigate geometric consequences of this duality. As an application, we classify genus one fibrations on supersingular $K^3$ surfaces with Artin invariant $10$ in characteristic $2$ and $3$, and give a set of generators of the automorphism group of the nef cone of these supersingular $K^3$ surfaces. The difference between the automorphism group of a supersingular $K^3$ surface $X$ and the automorphism group of its nef cone is determined by the period of $X$. We define the notion of genericity for supersingular $K^3$ surfaces in terms of the period, and prove the existence of generic supersingular $K^3$ surfaces in odd characteristics for each Artin invariant larger than $1$.

1. INTRODUCTION

A $K^3$ surface defined over an algebraically closed field $k$ is said to be supersingular (in the sense of Shioda) if its Picard number is $22$. Supersingular $K^3$ surfaces exist only when $k$ is of positive characteristic. Let $X$ be a supersingular $K^3$ surface in characteristic $p > 0$, and let $S_X$ denote its Néron-Severi lattice. Artin [1] showed that the discriminant group of $S_X$ is a $p$-elementary abelian group of rank $2\sigma$, where $\sigma$ is an integer such that $1 \leq \sigma \leq 10$. This integer $\sigma$ is called the Artin invariant of $X$. The isomorphism class of the lattice $S_X$ depends only on $p$ and $\sigma$ (Rudakov and Shafarevich [27]). Moreover supersingular $K^3$ surfaces with Artin invariant $\sigma$ form a $(\sigma - 1)$-dimensional family, and a supersingular $K^3$ surface with Artin invariant $1$ in characteristic $p$ is unique up to isomorphisms (Ogus [24], [25], Rudakov and Shafarevich [27]).

Recently many studies of supersingular $K^3$ surfaces in small characteristics with Artin invariant $1$ have appeared. For example, for $p = 2$, Dolgachev and Kondo [8], Katsura and Kondo [12], Elkies and Schütt [11]; for $p = 3$, Katsura and Kondo [13], Kondo and Shimada [18], Sengupta [28]; and for $p = 5$, Shimada [33]. On the other hand, geometric properties of supersingular $K^3$ surfaces with big Artin invariant are not so much known (e.g. Rudakov and Shafarevich [26], [27], Shioda [35], Shimada [31], [32]).

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In this paper, we present some methods to investigate supersingular $K_3$ surfaces with big Artin invariant by means of the following simple observation. Let $X_{p,\sigma}$ be a supersingular $K_3$ surface in characteristic $p$ with Artin invariant $\sigma$, and let $S_{p,\sigma}$ denote its Néron-Severi lattice.

**Lemma 1.1.** Suppose that $\sigma + \sigma' = 11$. Then $S_{p,\sigma}$ is isomorphic to $S_{p,\sigma'}^\vee(p)$, where $S_{p,\sigma}^\vee(p)$ is the lattice obtained from the dual lattice $S_{p,\sigma}^{\vee}$ of $S_{p,\sigma}$ by multiplying the symmetric bilinear form with $p$.

Lemma 1.1 is proved in Section 3. We use this duality between $S_{p,\sigma}$ and $S_{p,\sigma'}$ in the study of genus one fibrations and the automorphism groups of supersingular $K_3$ surfaces.

First, we apply Lemma 1.1 to the classification of genus one fibrations. Note that the Néron-Severi lattice $S_Y$ of a $K_3$ surface $Y$ is a hyperbolic lattice. The orthogonal group $O(S_Y)$ contains the stabilizer subgroup $O^+(S_Y)$ of a positive cone of $S_Y \otimes \mathbb{R}$ as a subgroup of index 2.

**Definition 1.2.** Let $Y$ be a $K_3$ surface, and let $\phi : Y \rightarrow \mathbb{P}^1$ be a genus one fibration. We denote by $f_\phi \in S_Y$ the class of a fiber of $\phi$. Let $\psi : Y \rightarrow \mathbb{P}^1$ be another genus one fibration on $Y$. We say that $\phi$ and $\psi$ are Aut-equivalent if there exist $g \in \text{Aut}(Y)$ and $\bar{g} \in \text{Aut}(\mathbb{P}^1)$ such that $\phi \circ g = \bar{g} \circ \psi$ holds, while we say that $\phi$ and $\psi$ are lattice equivalent if there exists $g \in O^+(S_Y)$ such that $f_\phi g = f_\psi$. We denote by $E(Y)$ the set of lattice equivalence classes of genus one fibrations on $Y$, and by $[\phi] \in E(Y)$ the lattice equivalence class containing $\phi$.

Many combinatorial properties of a genus one fibration $\phi : Y \rightarrow \mathbb{P}^1$ depend only on the lattice equivalence class $[\phi]$. See Proposition 4.1. Moreover, when $\sigma = 10$, the classification of genus one fibrations by Aut-equivalence seems to be too fine, as is suggested by Proposition 9.2. Therefore, we concentrate upon the study of lattice equivalence classes.

Using Lemma 1.1, we prove the following:

**Theorem 1.3.** Suppose that $\sigma + \sigma' = 11$. Then there exists a canonical one-to-one correspondence

$$[\phi] \mapsto [\phi']$$

between $E(X_{p,\sigma})$ and $E(X_{p,\sigma'})$.

We say that a genus one fibration is Jacobian if it admits a section.

**Theorem 1.4.** Suppose that a genus one fibration $\phi : X_{p,\sigma} \rightarrow \mathbb{P}^1$ is a Jacobian fibration, and let $\phi' : X_{p,\sigma'} \rightarrow \mathbb{P}^1$ be a genus one fibration on $X_{p,\sigma'}$ with $\sigma' = 11 - \sigma$ such that $[\phi'] \in E(X_{p,\sigma'})$ corresponds to $[\phi] \in E(X_{p,\sigma})$ by Theorem 1.3. Then $\phi'$ does not admit a section.

Elkies and Schütt [11] proved the following:

**Theorem 1.5** ([11]). Any genus one fibration on $X_{p,1}$ admits a section.

Therefore we obtain the following:
Corollary 1.6. There exist no Jacobian fibrations on $X_{p,10}$.

By an $ADE$-type, we mean a finite formal sum of the symbols $A_i$ ($i \geq 1$), $D_j$ ($j \geq 4$) and $E_k$ ($k = 6, 7, 8$) with non-negative integer coefficients. For a genus one fibration $\phi : Y \to \mathbb{P}^1$ on a $K3$ surface $Y$, we have the $ADE$-type of reducible fibers of $\phi$. This $ADE$-type depends only on the lattice equivalence class $[\phi] \in \mathbb{E}(Y)$ (see Proposition 4.1). Therefore we can use $R_{[\phi]}$ to denote the $ADE$-type of the reducible fibers of $\phi$.

From the classification of lattice equivalence classes of genus one fibrations of $X_{2,1}$ by Elkies and Schütt [11], and that of $X_{3,1}$ by Sengupta [28], we obtain the classification of lattice equivalence classes of genus one fibrations on $X_{2,10}$ and $X_{3,10}$. In particular, we obtain the list of $ADE$-types $R_{[\phi']}$ of the reducible fibers of genus one fibrations $\phi'$ on $X_{2,10}$ and $X_{3,10}$. See Theorems 4.8 and 4.9.

In Elkies and Schütt [11] and Sengupta [28] mentioned above, they also obtained explicit defining equations of the Jacobian fibrations. Besides [11] and [28], there have been many works on the classification of $Aut$-equivalence classes and lattice equivalence classes of Jacobian fibrations on a $K3$ surface (e.g. Oguiso [23], Nishiyama [22], Shimada and Zhang [34], Shimada [29], Kloosterman [16]). In particular, the lattice equivalence classes of all extremal (quasi-) elliptic fibrations (i.e., Jacobian fibrations with Mordell-Weil rank zero) on supersingular $K3$ surfaces are classified in Shimada [30].

As the second application of Lemma 1.1, we investigate the automorphism group of the nef cone of a supersingular $K3$ surface. For a $K3$ surface $Y$, let $\text{Nef}(Y) \subset S_Y \otimes \mathbb{R}$ denote the nef cone. We denote by $Aut(\text{Nef}(Y)) \subset O^+(S_Y)$ the group of isometries of $S_Y$ that preserve $\text{Nef}(Y)$. Since $Aut(X_{p,\sigma})$ acts on $S_{p,\sigma}$ faithfully (Rudakov and Shafarevich [27, Section 8, Proposition 3]), we have

\begin{equation}
Aut(X_{p,\sigma}) \subset Aut(\text{Nef}(X_{p,\sigma})) \subset O^+(S_{p,\sigma}).
\end{equation}

Using the description of $Aut(X_{2,1})$ by Dolgachev and Kondo [8], and that of $Aut(X_{3,1})$ by Kondo and Shimada [18], we give a set of generators of $Aut(\text{Nef}(X_{2,10}))$ and $Aut(\text{Nef}(X_{3,10}))$ in Theorems 6.4 and 6.9, respectively.

Suppose that $p$ is odd. We fix a lattice $N$ isomorphic to $S_{p,\sigma}$. Then a quadratic space $(N_0, q_0)$ of dimension $2\sigma$ over $\mathbb{F}_p$ is defined by

\begin{equation}
N_0 := pN^\vee / pN \quad \text{and} \quad q_0(p x \mod pN) := px^2 \mod p \quad (x \in N^\vee).
\end{equation}

We fix a marking $\eta : N \cong S_{p,\sigma}$ for a supersingular $K3$ surface $X := X_{p,\sigma}$ defined over $k$. Then $Aut(\text{Nef}(X))$ acts on $(N_0, q_0)$, and the period $K_{(X,\eta)} \subset N_0 \otimes k$ of the marked supersingular $K3$ surface $(X, \eta)$ is defined as the Frobenius pull-back of the kernel of the natural homomorphism

\[ N \otimes k \to S_X \otimes k \to H^2_{\text{DR}}(X/k) \]
(see Section 7). In virtue of Torelli theorem for supersingular $K3$ surfaces by Ogus [24], [25], the subgroup $\text{Aut}(X)$ of $\text{Aut}(\text{Nef}(X))$ is equal to the stabilizer subgroup of the period $K_{(X, \eta)}$. In particular, the index of $\text{Aut}(X_{p, \sigma})$ in $\text{Aut}(\text{Nef}(X_{p, \sigma}))$ is finite. On the other hand, the classification of 2-reflective lattices due to Nikulin [21] implies that $\text{Aut}(\text{Nef}(X_{p, \sigma}))$ is infinite. Hence, at least when $p$ is odd, $\text{Aut}(X_{p, \sigma})$ is an infinite group. See Sections 5 and 7 for details. Moreover, Lieblich and Maulik [19] proved that, if $p > 2$, then $\text{Aut}(X_{p, \sigma})$ is finitely generated and its action on $\text{Nef}(X_{p, \sigma})$ has a rational polyhedral fundamental domain.

We say that a supersingular $K3$ surface $X$ is generic if there exists a marking $\eta : N \cong S_X$ such that the isometries of $(N_0, q_0)$ that preserve the period $K_{(X, \eta)} \subset N_0 \otimes k$ are only scalar multiplications (see Definition 7.5). Using the surjectivity of the period mapping proved by Ogus [25], we prove the following:

**Theorem 1.7.** Suppose that $p$ is odd and $\sigma > 1$. Then there exist an algebraically closed field $k$ and a supersingular $K3$ surface $X$ with Artin invariant $\sigma$ defined over $k$ that is generic.

Suppose that $X_{3, 10}$ is generic. From the generators of $\text{Aut}(\text{Nef}(X_{3, 10}))$ given in Theorem 6.9, we can obtain a finite set of generators of $\text{Aut}(X_{3, 10})$. However, the computation would be very heavy. See Remarks 7.7 and 7.8.

As the third application, we show by an example that a lattice equivalence class of genus one fibrations on $X_{3, 10}$ can contain a very large number of $\text{Aut}$-equivalence classes, provided that $X_{3, 10}$ is generic. An analogous result for a generic complex Enriques surface was obtained by Barth and Peters [2].

This paper is organized as follows. In Section 2, we fix notation and terminologies about lattices and $K3$ surfaces. In Section 3, Lemma 1.1 is proved by means of the fundamental results of Rudakov and Shafarevich [27] on the Néron-Severi lattices of supersingular $K3$ surfaces. In Section 4, we study genus one fibrations on supersingular $K3$ surfaces, and prove Theorems 1.3 and 1.4. Moreover, the bijections $\mathcal{E}(X_{p, 1}) \cong \mathcal{E}(X_{p, 10})$ for $p = 2$ and 3 are given explicitly in Tables 4.1 and 4.2. In Section 5, we review the classical method to investigate the orthogonal group of a hyperbolic lattice by means of a chamber decomposition of the associated hyperbolic space, and fix some notation and terminologies. We then apply this method to the nef cone of a supersingular $K3$ surface. In Section 6, we give a set of generators of $\text{Aut}(\text{Nef}(X_{2, 10}))$ and $\text{Aut}(\text{Nef}(X_{3, 10}))$. In Section 7, we review the theory of the period mapping and Torelli theorem for supersingular $K3$ surfaces in odd characteristics due to Ogus [24], [25], and describe the relation between $\text{Aut}(X_{p, \sigma})$ and $\text{Aut}(\text{Nef}(X_{p, \sigma}))$. In Section 8, we prove Theorem 1.7. In the last section, we illustrate that the number of $\text{Aut}$-equivalence classes of genus one fibrations on $X_{3, 10}$ is intractably large if $X_{3, 10}$ is generic.

**Convention.** We use $\text{Aut}$ to denote automorphism groups of lattice theoretic objects, and $\text{Aut}$ to denote automorphism groups of geometric objects on $K3$ surfaces.
2. Preliminaries

2.1. Lattices. A \( \mathbb{Q} \)-lattice is a free \( \mathbb{Z} \)-module \( L \) of finite rank equipped with a non-degenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle_L : L \times L \to \mathbb{Q} \). We omit the subscript \( L \) in \( \langle \cdot, \cdot \rangle_L \) if no confusions will occur. If \( \langle \cdot, \cdot \rangle_L \) takes values in \( \mathbb{Z} \), we say that \( L \) is a lattice. For \( x \in L \otimes \mathbb{R} \), we call \( x^2 := \langle x, x \rangle \) the norm of \( x \). A vector in \( L \otimes \mathbb{R} \) of norm \( n \) is sometimes called an \( n \)-vector. A lattice \( L \) is said to be even if \( x^2 \in 2\mathbb{Z} \) holds for any \( x \in L \).

Let \( L \) be a free \( \mathbb{Z} \)-module of finite rank. A submodule \( M \) of \( L \) is primitive if \( L/M \) is torsion free. A non-zero vector \( v \in L \) is primitive if the submodule of \( L \) generated by \( v \) is primitive.

Let \( L \) be a \( \mathbb{Q} \)-lattice of rank \( r \). For a non-zero rational number \( m \), we denote by \( L(m) \) the free \( \mathbb{Z} \)-module \( L \) with the symmetric bilinear form \( \langle x, y \rangle_{L(m)} := m \langle x, y \rangle_L \). The signature of \( L \) is the signature of the real quadratic space \( L \otimes \mathbb{R} \). We say that \( L \) is negative definite if the signature of \( L \) is \((0, r)\), and \( L \) is hyperbolic if the signature is \((1, r-1)\). A Gram matrix of \( L \) is an \( r \times r \) matrix with entries \( \langle e_i, e_j \rangle \), where \( \{e_1, \ldots, e_r\} \) is a basis of \( L \). The determinant of a Gram matrix of \( L \) is called the discriminant of \( L \).

For an even lattice \( L \), the set of \((-2)\)-vectors is denoted by \( \mathcal{R}(L) \). A negative definite even lattice \( L \) is called a root lattice if \( L \) is generated by \( \mathcal{R}(L) \). Let \( R \) be an ADE-type. The root lattice of type \( R \) is the root lattice whose Gram matrix is the Cartan matrix of type \( R \). Suppose that \( L \) is negative definite. By the ADE-type of \( \mathcal{R}(L) \), we mean the ADE-type of the root sublattice \( \langle \mathcal{R}(L) \rangle \) of \( L \) generated by \( \mathcal{R}(L) \). (See, for example, Ebeling [10] for the classification of root lattices.)

Let \( L \) be an even lattice and let \( L^\vee := \text{Hom}(L, \mathbb{Z}) \) be identified with a submodule of \( L \otimes \mathbb{Q} \) with the extended symmetric bilinear form. We call this \( \mathbb{Q} \)-lattice \( L^\vee \) the dual lattice of \( L \). The discriminant group of \( L \) is defined to be the quotient \( L^\vee/L \), and is denoted by \( A_L \). We define the discriminant quadratic form of \( L \)

\[ q_L : A_L \to \mathbb{Q}/2\mathbb{Z} \]

by \( q_L(x \mod L) := x^2 \mod 2\mathbb{Z} \). The order of \( A_L \) is equal to the discriminant of \( L \) up to sign. We say that \( L \) is unimodular if \( A_L \) is trivial, while \( L \) is \( p \)-elementary if \( A_L \) is \( p \)-elementary. An even 2-elementary lattice \( L \) is said to be of type I if \( q_L(x \mod L) \in \mathbb{Z}/2\mathbb{Z} \) holds for any \( x \in L^\vee \). Note that \( L \) is \( p \)-elementary if and only if \( pG_L^{-1} \) is an integer matrix, where \( G_L \) is a Gram matrix of \( L \).

Let \( O(L) \) denote the orthogonal group of a lattice \( L \), that is, the group of isomorphisms of \( L \) preserving \( \langle \cdot, \cdot \rangle_L \). We assume that \( O(L) \) acts on \( L \) from right, and the action of \( g \in O(L) \) on \( v \in L \otimes \mathbb{R} \) is denoted by \( v \mapsto gv^g \). Similarly \( O(q_L) \) denotes the group of isomorphisms of \( A_L \) preserving \( q_L \). There is a natural homomorphism \( O(L) \to O(q_L) \).

Let \( L \) be a hyperbolic lattice. A positive cone of \( L \) is one of the two connected components of

\[ \{ x \in L \otimes \mathbb{R} \mid x^2 > 0 \} . \]
Let $P_L$ be a positive cone of $L$. We denote by $O^+(L)$ the group of isometries of $L$ that preserve $P_L$. We have $O(L) = O^+(L) \times \{ \pm 1 \}$. For a vector $v \in L \otimes \mathbb{R}$ with $v^2 < 0$, we put
\[(v^\perp) := \{ x \in P_L \mid \langle x, v \rangle = 0 \},\]
which is a real hyperplane of $P_L$. An isometry $g \in O^+(L)$ is called a reflection with respect to $v$ or a reflection into $(v^\perp)$ if $g$ is of order 2 and fixes each point of $(v^\perp)$. An element $r$ of $R(L)$ defines a reflection $s_r : x \mapsto x + \langle x, r \rangle r$ with respect to $r$. We denote by $W(L)$ the subgroup of $O^+(L)$ generated by the set of these reflections \(\{ s_r | r \in R(L) \} \). It is obvious that $W(L)$ is normal in $O^+(L)$.

2.2. $K3$ surfaces. Let $Y$ be a $K3$ surface, and let $S_Y$ denote the Néron-Severi lattice of $Y$. A smooth rational curve on $Y$ is called a $(-2)$-curve. We denote by $\mathcal{P}(Y) \subset S_Y \otimes \mathbb{R}$ the positive cone containing an ample class of $Y$. Recall that the nef cone $\text{Nef}(Y)$ of $Y$ is defined by
\[\text{Nef}(Y) := \{ x \in S_Y \otimes \mathbb{R} \mid \langle x, [C] \rangle \geq 0 \text{ for any curve } C \text{ on } Y \},\]
where $[C] \in S_Y$ is the class of a curve $C \subset Y$. Then $\text{Nef}(Y)$ is contained in the closure $\overline{\mathcal{P}(Y)}$ of $\mathcal{P}(Y)$ in $S_Y \otimes \mathbb{R}$. We put
\[\text{Nef}^\circ(Y) := \text{Nef}(Y) \cap \mathcal{P}(Y) = \{ x \in \text{Nef}(Y) \mid x^2 > 0 \}.\]
The following is well-known. See, for example, Rudakov and Shafarevich [27, Section 3].

**Proposition 2.1.** (1) We have
\[\text{Nef}(Y) = \{ x \in S_Y \otimes \mathbb{R} \mid \langle x, [C] \rangle \geq 0 \text{ for any } (-2)\text{-curve } C \text{ on } Y \}.\]
(2) If $v \in S_Y$ is contained in $\overline{\mathcal{P}(Y)}$, then there exists $g \in W(S_Y)$ such that $v^g \in \text{Nef}(Y)$.

3. Néron-Severi lattices of supersingular $K3$ surfaces

Let $X_{p,\sigma}$ be a supersingular $K3$ surface with Artin invariant $\sigma$ in characteristic $p > 0$. Then the isomorphism class of the Néron-Severi lattice $S_{p,\sigma}$ of $X_{p,\sigma}$ depends only on $p$ and $\sigma$, and is characterized as follows (see Rudakov-Shafarevich [27, Sections 3,4 and 5] for the proof).

**Theorem 3.1 ([27]).** (1) The lattice $S_{p,\sigma}$ is an even hyperbolic $p$-elementary lattice of rank 22 with discriminant $-p^{2\sigma}$. Moreover, $S_{2,\sigma}$ is of type I.
(2) Suppose that $N$ is an even hyperbolic $p$-elementary lattice of rank 22 with discriminant $-p^{2\sigma}$. When $p = 2$, we further assume that $N$ is of type I. Then $N$ is isomorphic to $S_{p,\sigma}$.

Using this theorem, we can prove Lemma 1.1 easily.
Proof of Lemma 1.1. It is enough to show that $S_{p,\sigma}^\vee(p)$ is an even $p$-elementary lattice of discriminant $-p^{2\sigma'}$, and that $S_{2,\sigma}^\vee(2)$ is of type I. Since $S_{p,\sigma}$ is $p$-elementary, we have $pS_{p,\sigma}^\vee \subset S_{p,\sigma}$. Therefore $S_{p,\sigma}^\vee(p)$ is a lattice. Let $G_{p,\sigma}$ be a Gram matrix of $S_{p,\sigma}$. Then the determinant of the Gram matrix $pG_{p,\sigma}^{-1}$ of $S_{p,\sigma}^\vee(p)$ is equal to $p^{22} \cdot \det(G_{p,\sigma})^{-1} = -p^{2\sigma'}$. Therefore the discriminant of $S_{p,\sigma}^\vee(p)$ is $-p^{2\sigma'}$. Since $p(pG_{p,\sigma}^{-1})^{-1} = G_{p,\sigma}$ is an integer matrix, $S_{p,\sigma}^\vee(p)$ is $p$-elementary. Suppose that $p$ is odd. Then, for any $\xi \in S_{p,\sigma}^\vee$, we have $p\xi \in S_{p,\sigma}$ and hence $\langle p\xi, p\xi \rangle_{S_{p,\sigma}} = p\langle \xi, \xi \rangle_{S_{p,\sigma}^\vee(p)}$ is even. Therefore $S_{p,\sigma}^\vee(p)$ is even. Suppose that $p = 2$. Then, for any $\xi \in S_{2,\sigma}^\vee$, we have $\langle \xi, \xi \rangle_{S_{2,\sigma}^\vee(2)} \in \mathbb{Z}$, because $S_{2,\sigma}$ is of type I. Therefore $S_{2,\sigma}^\vee(2)$ is even. Moreover, for any $\eta \in (S_{2,\sigma}^\vee(2))^\vee = S_{2,\sigma}(1/2)$, we have $\langle \eta, \eta \rangle_{S_{2,\sigma}(1/2)} \in \mathbb{Z}$, because $S_{2,\sigma}$ is even. Therefore $S_{2,\sigma}^\vee(2)$ is of type I. \hfill \Box

Corollary 3.2. Suppose that $\sigma + \sigma' = 11$. Then there exists an embedding of $\mathbb{Z}$-modules

$$j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$$

that induces an isomorphism of lattices $S_{p,\sigma}^\vee(p) \cong S_{p,\sigma'}$. This embedding induces an isomorphism

$$j_* : O(S_{p,\sigma}) \cong O(S_{p,\sigma'}).$$

Moreover such an embedding $j$ is unique up to compositions with elements of $O(S_{p,\sigma'})$.

Remark 3.3. Suppose that $v \in S_{p,\sigma}$ satisfies $v^2 \geq 0$. Then, by Proposition 2.1(2), we can choose $j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$ in Corollary 3.2 in such a way that $j(v)$ is contained in Nef$(X_{p,\sigma})$.

4. Genus one fibrations

Let $Y$ be a $K3$ surface defined over an algebraically closed field of arbitrary characteristic. Recall that $f_\phi \in S_Y$ is the class of a fiber of a genus one fibration $\phi : Y \to \mathbb{P}^1$, $E(Y)$ is the set of lattice equivalence classes of genus one fibrations on $Y$, and $[\phi] \in E(Y)$ is the class containing $\phi$. We summarize properties of a genus one fibration $\phi : Y \to \mathbb{P}^1$ that depends only on the class $[\phi]$. See Sections 3 and 4 of Rudakov and Shafarevich [27], and Shioda [36] for the proof.

(1) The fibration $\phi$ admits a section if and only if there exists a $(-2)$-vector $z \in S_Y$ such that $\langle f_\phi, z \rangle = 1$.

(2) Note that $f_\phi \in S_Y$ is primitive of norm 0, and that $\langle f_\phi \rangle / \langle f_\phi \rangle$ is an even negative definite lattice, where $\langle f_\phi \rangle$ is the orthogonal complement in $S_Y$ of the lattice $\langle f_\phi \rangle$ of rank 1 generated by $f_\phi$. The $ADE$-type of the reducible fibers of $\phi$ is equal to the $ADE$-type of the set $R(\langle f_\phi \rangle / \langle f_\phi \rangle)$ of $(-2)$-vectors in $\langle f_\phi \rangle$.

(3) Suppose that $\phi$ admits a section $Z \subset Y$. Then $f_\phi$ and $[Z] \in S_Y$ generate an even unimodular hyperbolic lattice $U_\phi$ of rank 2 in $S_Y$. Let $K_\phi$ denote the orthogonal complement of $U_\phi$ in $S_Y$. We have an orthogonal direct-sum decomposition

$$S_Y = U_\phi \oplus K_\phi,$$
and the lattice $\langle f_\psi \rangle / \langle f_\phi \rangle$ is isomorphic to $K_\phi$. Then the Mordell-Weil group of $\phi$ is isomorphic to $K_\phi / \langle \mathcal{R}(K_\phi) \rangle$, where $\langle \mathcal{R}(K_\phi) \rangle$ is the root sublattice of $K_\phi$ generated by the $(-2)$-vectors in $K_\phi$.

(4) In characteristic $2$ or $3$, $\phi$ is quasi-elliptic if and only if $\langle \mathcal{R}(K_\phi) \rangle$ is $p$-elementary of rank $20$.

As a corollary, we obtain the following:

**Proposition 4.1.** Suppose that genus one fibrations $\phi : Y \to \mathbb{P}^1$ and $\psi : Y \to \mathbb{P}^1$ on $Y$ are lattice-equivalent. Then the following hold:

1. The fibration $\phi$ admits a section if and only if so does $\psi$.
2. The ADE-type of the reducible fibers of $\phi$ is equal to that of $\psi$.
3. Suppose that $\phi$ and $\psi$ admit a section. Then the Mordell-Weil groups for $\phi$ and for $\psi$ are isomorphic.
4. In characteristic $2$ or $3$, the fibration $\phi$ is quasi-elliptic if and only if so is $\psi$.

**Definition 4.2.** For a hyperbolic lattice $S$, we put

$$\tilde{E}(S) := \{ v \in S \otimes \mathbb{Q} \mid v \neq 0, v^2 = 0 \} / \mathbb{Q}^\times$$

and

$$\mathcal{E}(S) := \tilde{E}(S) / \mathcal{O}(S).$$

**Remark 4.3.** Let a positive cone $\mathcal{P}_S$ of $S$ be fixed. Then each element of $\tilde{E}(S)$ is represented by a unique non-zero primitive vector $v \in S$ of norm $0$ that is contained in the closure $\mathcal{P}_S$ of $\mathcal{P}_S$ in $S \otimes \mathbb{R}$.

In Sections 3 and 4 of Rudakov and Shafarevich [27], the following is proved:

**Proposition 4.4.** Let $v$ be a non-zero vector of $S_Y$. Then there exists a genus one fibration $\phi : Y \to \mathbb{P}^1$ such that $v = f_\phi$ if and only if $v$ is primitive, $v^2 = 0$, and $v \in \text{Nef}(Y)$.

Combining Propositions 2.1, 4.4 and Remark 4.3, we obtain the following:

**Corollary 4.5.** The map $\phi \mapsto f_\phi$ induces a bijection from $\mathcal{E}(Y)$ to $\mathcal{E}(S_Y)$. 

From now on, we work over an algebraically closed field of characteristic $p > 0$.

**Proof of Theorem 1.3.** Consider the embedding $j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$ in Corollary 3.2. Then $j$ is unique up to $\mathcal{O}(S_{p,\sigma'})$, induces a bijection from $\tilde{E}(S_{p,\sigma})$ to $\tilde{E}(S_{p,\sigma'})$, and induces an isomorphism $\mathcal{O}(S_{p,\sigma}) \cong \mathcal{O}(S_{p,\sigma'})$. Hence it induces a canonical bijection from $\mathcal{E}(S_{p,\sigma})$ to $\mathcal{E}(S_{p,\sigma'})$. \hfill $\square$

We denote this canonical one-to-one correspondence from $\mathcal{E}(X_{p,\sigma})$ to $\mathcal{E}(X_{p,\sigma'})$ by $[\phi] \mapsto [\phi']$.

**Remark 4.6.** Let a genus one fibration $\phi : X_{p,\sigma} \to \mathbb{P}^1$ be given, and let $\phi' : X_{p,\sigma'} \to \mathbb{P}^1$ be a representative of $[\phi']$. Then we can choose the embedding $j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$ inducing $S_{p,\sigma}'(p) \cong S_{p,\sigma'}$, in such a way that $j(f_\phi)$ is a scalar multiple of $f_{\phi'}$ by a positive integer.

**Theorem 4.7.** Suppose that a genus one fibration $\phi : X_{p,\sigma} \to \mathbb{P}^1$ admits a section. Then the corresponding genus one fibration $\phi' : X_{p,\sigma'} \to \mathbb{P}^1$ does not admit a section. Moreover the ADE-type of the reducible fibers of $\phi'$ is equal to the ADE-type of $\mathcal{R}(K'_{\phi}(p))$. 

Table 4.1. Genus one fibrations on $X_{2,1}$ and $X_{2,10}$

<table>
<thead>
<tr>
<th>No.</th>
<th>$R_N$</th>
<th>$\sigma = 1$</th>
<th>$\sigma = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$R_{[\phi]}$</td>
<td>$\text{MW}_{\text{tor}}$</td>
</tr>
<tr>
<td>1</td>
<td>$4A_5 + D_4$</td>
<td>$4A_5$</td>
<td>[3, 6]</td>
</tr>
<tr>
<td>2</td>
<td>$6D_4$</td>
<td>$5D_4$</td>
<td>[2, 2, 2, 2]</td>
</tr>
<tr>
<td>3</td>
<td>$2A_7 + 2D_5$</td>
<td>$2A_7 + D_8$</td>
<td>[8]</td>
</tr>
<tr>
<td>4</td>
<td>$2A_9 + D_6$</td>
<td>$2A_1 + 2A_9$</td>
<td>[10]</td>
</tr>
<tr>
<td>5</td>
<td>$4D_6$</td>
<td>$2A_1 + 3D_6$</td>
<td>[2, 2, 2]</td>
</tr>
<tr>
<td>6</td>
<td>$A_{11} + D_7 + E_6$</td>
<td>$A_{11} + D_7$</td>
<td>[4]</td>
</tr>
<tr>
<td>7</td>
<td>$A_{11} + D_7 + E_6$</td>
<td>$A_3 + A_{11} + E_6$</td>
<td>[6]</td>
</tr>
<tr>
<td>8</td>
<td>$4E_6$</td>
<td>$3E_6$</td>
<td>[3]</td>
</tr>
<tr>
<td>9</td>
<td>$3D_8$</td>
<td>$D_4 + 2D_8$</td>
<td>[2, 2]</td>
</tr>
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<td>10</td>
<td>$A_{15} + D_9$</td>
<td>$A_{15} + D_5$</td>
<td>[4]</td>
</tr>
<tr>
<td>11</td>
<td>$A_{17} + E_7$</td>
<td>$3A_1 + A_{17}$</td>
<td>[6]</td>
</tr>
<tr>
<td>12</td>
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<td>$3A_1 + D_{10} + E_7$</td>
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</tr>
<tr>
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<td>[2]</td>
</tr>
<tr>
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<td>15</td>
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<td>$D_4 + D_{16}$</td>
<td>[2]</td>
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<tr>
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<td>[1]</td>
</tr>
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<td>$3E_8$</td>
<td>$D_4 + 2E_8$</td>
<td>[1]</td>
</tr>
<tr>
<td>18</td>
<td>$D_{24}$</td>
<td>$D_{20}$</td>
<td>[1]</td>
</tr>
</tbody>
</table>

**Proof.** Let $z \in S_{p,\sigma}$ be the class of a section of $\phi$. We choose $j : S_{p,\sigma} \rightarrow S_{p,\sigma'}$ as in Remark 4.6. Since $U_{\phi}^\vee = U_{\phi}$, we see that $j(f_\phi)$ is primitive in $S_{p,\sigma'}$ and hence $j(f_\phi) = f_{\phi'}$. We have an isomorphism $S_{p,\phi'} \cong U_{\phi}(p) \oplus K_{\phi}^\vee(p)$ such that $f_{\phi'}$ and $j(z)$ form a basis of $U_{\phi}(p)$. Since there are no vectors $v \in U_{\phi}(p) \oplus K_{\phi}^\vee(p)$ with $\langle v, f_{\phi'} \rangle = 1$, the fibration $\phi'$ does not admit a section. Moreover the lattice $\langle f_{\phi'} \rangle^+ / \langle f_{\phi'} \rangle$ is isomorphic to $K_{\phi}^\vee(p)$. \hfill $\square$

The list of lattice equivalence classes of genus one fibrations on $X_{2,1}$ and $X_{3,1}$ were obtained by Elkies and Schütt [11] and by Sengupta [28], respectively. From their results, we obtain the following results on supersingular $K3$ surfaces with Artin invariant 10:

**Theorem 4.8.** There exist 18 lattice equivalence classes of genus one fibrations on $X_{2,10}$. The ADE-type $R_{[\phi]}$ of the reducible fibers of each $[\phi'] \in E(X_{2,10})$ is given at the last column of Table 4.1.

**Theorem 4.9.** There exist 52 lattice equivalence classes of genus one fibrations on $X_{3,10}$. The ADE-type $R_{[\phi]}$ of the reducible fibers of each $[\phi'] \in E(X_{3,10})$ is given at the last column of Table 4.2.
<table>
<thead>
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<th>No.</th>
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<td>$\text{MW}_{\text{tor}}$</td>
<td>rank($\text{MW}$)</td>
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<td>2</td>
<td>$6A_3$</td>
<td>$[4, 4]$</td>
<td>2</td>
</tr>
<tr>
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<td>2</td>
</tr>
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<td>4</td>
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<tr>
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<tr>
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<tr>
<td>7</td>
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<td>2</td>
</tr>
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<td>29</td>
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<tr>
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<td>36</td>
<td>$A_{12} + D_6$</td>
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</tbody>
</table>

(to be continued)
In Table 4.1 (resp. Table 4.2), the lists $\mathbb{E}(X_{2,1})$ and $\mathbb{E}(X_{2,10})$ (resp. $\mathbb{E}(X_{3,1})$ and $\mathbb{E}(X_{3,10})$) are presented. Two lattice equivalence classes in the same row are the pair of $[\phi] \in \mathbb{E}(X_{p,1})$ and its corresponding partner $[\phi'] \in \mathbb{E}(X_{p,10})$. The $ADE$-type $R_\phi$ of the reducible fibers of $\phi$, and the torsion $MW_{tor}$ and the rank of the Mordell-Weil group of $\phi$ are also given. (Recall that $\phi$ is Jacobian for any $[\phi] \in \mathbb{E}(X_{p,1})$ by Elkies and Schütt [11].) The meaning of the entry $R_N$ is explained in the proof of Theorems 4.8 and 4.9.

Proof of Theorems 4.8 and 4.9. By Theorem 4.7, it is enough to calculate the $ADE$-type of $R(K_\phi^\vee(p))$ for $p = 2, 3$ and $[\phi] \in \mathbb{E}(X_{p,1})$. The lattices $K_\phi$ are calculated in Elkies and Schütt [11] and Sen-gupta [28] by Nishiyama’s method [22]. We put

$$T := \text{the root lattice of type} \begin{cases} D_4 & \text{if } p = 2, \\ 2A_2 & \text{if } p = 3. \end{cases}$$

Then, for each $[\phi] \in \mathbb{E}(X_{p,1})$, there exist a Niemeier lattice $N_\phi$ and a primitive embedding of $T$ into $N_\phi$ such that $K_\phi$ is isomorphic to the orthogonal complement of $T$ in $N_\phi$. The entry $R_N$ in Tables 4.1 and 4.2 indicates the $ADE$-type of $R(N_\phi)$. From a Gram matrix of $K_\phi$, we can calculate the $ADE$-type of $R(K_\phi^\vee(p))$ by the algorithm described in [32, Section 4] or [33, Section 3].

Corollary 4.10. There exist no quasi-elliptic fibrations on $X_{3,10}$. 

<table>
<thead>
<tr>
<th>No.</th>
<th>$R_N$</th>
<th>$\sigma = 1$</th>
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</tr>
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</tr>
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<td>$2D_9$</td>
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</tr>
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<td>52</td>
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<td>$[1]$ 2</td>
</tr>
</tbody>
</table>

Table 4.2. Genus one fibrations on $X_{3,1}$ and $X_{3,10}$.
Remark 4.11. Rudakov and Shafarevich [27, Section 5] showed that there exists a quasi-elliptic fibration on $X_{2,\sigma}$ for any $\sigma$. The quasi-elliptic fibration on $X_{2,10}$ (No. 18 of Table 4.1) was discovered by Rudakov and Shafarevich [26, Section 4].

5. Chamber decomposition of a positive cone

Let $S$ be an even hyperbolic lattice, and let $\mathcal{P}_S \subset S \otimes \mathbb{R}$ be a positive cone. In this section, we review a general method to find a set of generators of a subgroup of $O^+(S)$ by means of a chamber decomposition of $\mathcal{P}_S$, which was developed by Vinberg [37, 38], Conway [7] and Borcherds [3, 4].

Any real hyperplane in $\mathcal{P}_S$ is written in the form $(v)^\perp$ by some vector $v \in S \otimes \mathbb{R}$ with negative norm. We denote by $H_S$ the set of real hyperplanes in $\mathcal{P}_S$, which is canonically identified with

\[ \{ v \in S \otimes \mathbb{R} \mid v^2 < 0 \} / \mathbb{R}^\times. \]

For a subset $V$ of $\{ v \in S \otimes \mathbb{R} \mid v^2 < 0 \}$, we denote by $V^* \subset H_S$ the image of $V$ by $v \mapsto (v)^\perp$. A closed subset $D$ of $\mathcal{P}_S$ is called a chamber if the interior $D^\circ$ of $D$ is non-empty and there exists a set $\Delta_D$ of vectors $v \in S \otimes \mathbb{R}$ with $v^2 < 0$ such that

\[ D = \{ x \in \mathcal{P}_S \mid \langle x, v \rangle \geq 0 \text{ for all } v \in \Delta_D \}. \]

A hyperplane $(v)^\perp$ of $\mathcal{P}_S$ is called a wall of $D$ if $D^\circ \cap (v)^\perp = \emptyset$ and $D \cap (v)^\perp$ contains an open subset of $(v)^\perp$. When $D$ is a chamber, we always assume that the set $\Delta_D$ is minimal in the sense that, for any $v \in \Delta_D$, there exists a point $x \in \mathcal{P}_S$ such that $\langle x, v \rangle < 0$ and $\langle x, v' \rangle \geq 0$ for any $v' \in \Delta_D \setminus \{v\}$, that is, the projection $\Delta_D \to \Delta_D^*$ is bijective and every hyperplane $(v)^\perp \in \Delta_D^*$ is a wall of $D$.

For a chamber $D$, we put

\[ Aut(D) := \{ g \in O^+(S) \mid D^g = D \}. \]

A chamber $D$ is said to be fundamental if the following hold:

(i) $\mathcal{P}_S$ is the union of all $D^g$, where $g$ runs through $O^+(S)$, and

(ii) if $D^g \cap D^g \neq \emptyset$, then $g \in Aut(D)$.

Let $\mathcal{F}$ be a family of hyperplanes in $\mathcal{P}_S$ with the following properties:

(a) $\mathcal{F}$ is locally finite in $\mathcal{P}_S$, and

(b) $\mathcal{F}$ is invariant under the action of $O^+(S)$ on $H_S$.

Then the closure of each connected component of

\[ \mathcal{P}_S \setminus \bigcup_{\mathcal{F}} (v)^\perp \]

is a chamber, which we call an $\mathcal{F}$-chamber.
Proposition 5.1. An $F$ satisfies the property (i) if and only if every $(b)$ of $F$. The ‘only if’ part is obvious. We prove the ‘if’ part. It is enough to show that, for an arbitrary $N > D$ such that $P$ is locally finite in $D$, there exists a unique $F$-chamber $D'$ distinct from $D$ such that $D \cap D' \cap (v)^{\perp}$ contains an open subset of $(v)^{\perp}$. We say that $D'$ is adjacent to $D$ along $(v)^{\perp}$, and that $(v)^{\perp}$ is the wall between the adjacent chambers $D$ and $D'$.

Proposition 5.2. An $F$-chamber $D$ is fundamental if and only if, for each $v \in \Delta_D$, there exists $g_v \in O^{+}(S)$ such that $D^{g_v}$ is adjacent to $D$ along $(v)^{\perp}$.

Proof. The ‘only if’ part is obvious. We prove the ‘if’ part. It is enough to show that, for an arbitrary $F$-chamber $D'$, there exists $g \in O^{+}(S)$ such that $D' = D^{g}$. Since the family $F$ of hyperplanes is locally finite in $\mathcal{P}_S$, there exists a finite chain of $F$-chambers $D_0 = D, D_1, \ldots, D_N = D'$ such that $D_i$ and $D_{i+1}$ are adjacent. We show, by induction on $N$, that there exists a sequence of vectors $v_1, \ldots, v_N$ in $\Delta_D$ such that $D_i = D^{g_{v_i} \cdots g_{v_1}}$ holds for $i = 1, \ldots, N$. The case $N = 0$ is trivial. Suppose that $N > 0$. Let $(w)^{\perp}$ be the wall between $D_{N-1}$ and $D_N$, and let $v_N \in \Delta_D$ be the vector such that the wall $(v_N)^{\perp}$ of $D$ is mapped to the wall $(w)^{\perp}$ of $D_{N-1}$ by $g_{v_{N-1}} \cdots g_{v_1}$. Then we have $D_N = D^{g_{v_{N-1}} \cdots g_{v_1}}$. □

Remark 5.2. If an $F$-chamber is fundamental, then any $F$-chamber is fundamental.

Let $G$ be a subset of $\mathcal{F}$ that is invariant under the action of $O^{+}(S)$. Then $G$ is locally finite, and any $G$-chamber is a union of $F$-chambers. If an $F$-chamber is fundamental, then any $G$-chamber is also fundamental.

Proposition 5.3. Let $D$ be an $F$-chamber and let $C$ be a $G$-chamber such that $D \subset C$. Suppose that $D$ is fundamental. For $v \in \Delta_D$, let $g_v \in O^{+}(S)$ be an isometry such that $D^{g_v}$ is adjacent to $D$ along $(v)^{\perp}$. We put

$$\Gamma := \{ g_v \mid v \in \Delta_D, (v)^{\perp} \notin G \}.$$ 

Then $\text{Aut}(C)$ is generated by $\text{Aut}(D)$ and $\Gamma$.

Proof. If $g_v \in \Gamma$, then $D^{g_v}$ is contained in $C$ because the wall $(v)^{\perp}$ between $D$ and $D^{g_v}$ does not belong to $G$, and hence $g_v \in \text{Aut}(C)$. Therefore the subgroup $\langle \text{Aut}(D), \Gamma \rangle$ of $O^{+}(S)$ generated by $\text{Aut}(D)$ and $\Gamma$ is contained in $\text{Aut}(C)$. To prove $\text{Aut}(C) \subset \langle \text{Aut}(D), \Gamma \rangle$, it is enough to show that, for any $g \in \text{Aut}(C)$, there exists a sequence $\gamma_1, \ldots, \gamma_N$ of elements of $\Gamma$ such that $D^g = D^{\gamma_N \cdots \gamma_1}$. There exists a sequence of $F$-chambers $D_0 = D, D_1, \ldots, D_N = D^g$ such that each $D_i$ is contained in $C$ and that $D_{i+1}$ is adjacent to $D_i$ for $i = 0, \ldots, N-1$. Suppose that we have constructed $\gamma_1, \ldots, \gamma_i \in \Gamma$ such that $D_i = D^{\gamma_i \cdots \gamma_1}$. The wall $(w)^{\perp}$ between $D_i$ and $D_{i+1}$ does not belong to $G$. Let $v_{i+1}$ be an element of $\Delta_D$ such that the wall $(v_{i+1})^{\perp}$ of $D$ is mapped to the wall $(w)^{\perp}$ of $D_i$ by $\gamma_i \cdots \gamma_1$. Since $G$ is invariant under the action of $O^{+}(S)$, we have $(v_{i+1})^{\perp} \notin G$ and hence $\gamma_{i+1} := g_{v_{i+1}}$ is an element of $\Gamma$. Then $D_{i+1} = D^{\gamma_{i+1} \cdots \gamma_1}$ holds. □
Remark 5.4. Let $D$ and $C$ be as in Proposition 5.3. Let $v$ and $v'$ be elements of $\Delta_D$. Suppose that the wall $(v)^\perp$ of $D$ is mapped to the wall $(v')^\perp$ of $D$ by $h \in Aut(D)$. Then $D^{bg_v,h^{-1}}$ is adjacent to $D$ along $(v)^\perp$. Let $\Delta_D$ be a subset of $\Delta_D$ such that the subset $\Delta_D^\perp$ of $\Delta_D$ is a complete set of representatives of the orbit decomposition of $\Delta_D$ by the action of $Aut(D)$. Then $Aut(C)$ is generated by $Aut(D)$ and $\{g_v \mid v \in \Delta_D \cap (v)^\perp \notin G\}$.

Considering the case $G = \emptyset$, we obtain the following:

Corollary 5.5. Let $D$ be an $\mathcal{F}$-chamber. If $D$ is fundamental, then $O^+(S)$ is generated by $Aut(D)$ and the isometries $g_v$ that map $D$ to its adjacent chambers.

Example 5.6. Recall that $W(S) \subset O^+(S)$ is the subgroup generated by $\{s_r \mid r \in R(S)\}$. Any $R(S)^*$-chamber is fundamental, because every $r \in R(S)$ defines a reflection $s_r$. It follows that $O^+(S)$ is equal to the semi-direct product of $W(S)$ and the automorphism group $Aut(D)$ of an $R(S)^*$-chamber $D$. In particular, we have

$$Aut(D) \cong O^+(S)/W(S).$$

Let $L$ be an even unimodular hyperbolic lattice, and let $\iota : S \hookrightarrow L$ be a primitive embedding. Let $\mathcal{P}_L$ be the positive cone of $L$ that contains $\iota(\mathcal{P}_S)$. We denote by $T_\iota$ the orthogonal complement of $S$ in $L$, and by

$$v \mapsto v_S$$

the orthogonal projection $L \otimes \mathbb{R} \to S \otimes \mathbb{R}$. Since $L$ is a submodule of $S^\vee \oplus T_\iota^\vee$, the image of $L$ by $v \mapsto v_S$ is contained in $S^\vee$. We assume the following:

(5.1) the natural homomorphism $O(T_\iota) \to O(q_{T_\iota})$ is surjective.

Then we have the following:

Proposition 5.7. For any $g \in O^+(S)$, there exists $\tilde{g} \in O^+(L)$ such that $\iota(v^g) = \iota(v)^{\tilde{g}}$ holds for any $v \in S \otimes \mathbb{R}$.

Proof. See Nikulin [20, Proposition 1.6.1].

A hyperplane $(r)^\perp$ of $\mathcal{P}_L$ defined by a $(-2)$-vector $r \in R(L)$ intersects $\iota(\mathcal{P}_S)$ if and only if $r_S^2 < 0$. We put

$$R(L,\iota) := \{ r_S \mid r \in R(L) \text{ and } r_S^2 < 0 \} \subset S^\vee.$$

Since $T_\iota$ is negative definite, we have $-2 \leq r_S^2$ for any $r \in R(L)$. Since $S^\vee$ is discrete in $S \otimes \mathbb{R}$, the family of hyperplanes $R(L,\iota)^*$ is locally finite in $\mathcal{P}_S$. By Proposition 5.7, if $r \in R(L)$ satisfies $r_S \in R(L,\iota)$, then, for any $g \in O^+(S)$, we have $r_S^g = (r^g)_S \in R(L,\iota)$. Therefore $R(L,\iota)$ is invariant under the action of $O^+(S)$. Note that $R(L,\iota)$ contains $R(S)$, and that $R(S)$ is obviously invariant under the action of $O^+(S)$. Therefore, by Proposition 5.3, we can obtain a set of generators of the automorphism group $Aut(C)$ of an $R(S)^*$-chamber $C$ if we find an $R(L,\iota)^*$-chamber $D$ contained
in $C$, show that $D$ is fundamental, calculate the group $\text{Aut}(D)$, and find isometries of $S$ that map $D$ to its adjacent chambers.

Let $L_{26}$ denote an even hyperbolic unimodular lattice of rank 26, which is unique up to isomorphisms by Eichler’s theorem (see, for example, Cassels [6]). The walls of an $\mathcal{R}(L_{26})^+$-chamber $D \subset L_{26} \otimes \mathbb{R}$ and the group $\text{Aut}(D) \subset O^+(L_{26})$ were determined by Conway [7]. Then Borcherds [3], [4] determined the structure of $O^+(S)$ for some even hyperbolic lattices $S$ of rank $< 26$ by embedding $S$ into $L_{26}$ in such a way that $T_1$ is a root lattice.


We say that an even hyperbolic lattice $S$ is 2-reflective if the index of $W(S)$ in $O^+(S)$ is finite, or equivalently, if the automorphism group of an $\mathcal{R}(S)^+$-chamber is finite (see Example 5.6). Nikulin [21] classified all 2-reflective lattices of rank $\geq 5$. It turns out that there are no 2-reflective lattices of rank $> 19$.

Let $Y$ be a K3 surface with the Néron-Severi lattice $S_Y$ and the positive cone $\mathcal{P}(Y)$ containing an ample class. Then the closed subset $\text{Nef}(Y) = \text{Nef}(Y) \cap \mathcal{P}(Y)$ of $\mathcal{P}(Y)$ is an $\mathcal{R}(S_Y)^+$-chamber by Proposition 2.1(1), and hence we have

$$\text{Aut}(\text{Nef}(Y)) = \text{Aut}(\text{Nef}^0(Y)) \cong O^+(S_Y)/W(S_Y).$$

Combining this fact with Nikulin’s classification of 2-reflective lattices, we obtain the following:

**Corollary 5.8.** For any supersingular K3 surface $X_{p,\sigma}$, the group $\text{Aut}(\text{Nef}(X_{p,\sigma}))$ is infinite.

### 6. The Groups $\text{Aut}(\text{Nef}(X_{2,10}))$ and $\text{Aut}(\text{Nef}(X_{3,10}))$

#### 6.1. The group $\text{Aut}(\text{Nef}(X_{2,10}))$

By Lemma 1.1, the result of Dolgachev and Kondo [8], and the method of the previous section, we obtain a set of generators of $\text{Aut}(\text{Nef}(X_{2,10}))$.

First we recall the results of [8]. As a projective model of $X_{2,1}$, we consider the minimal resolution $X$ of the inseparable double cover $Y \to \mathbb{P}^2$ of $\mathbb{P}^2$ defined by

$$w^2 = x_0x_1x_2(x_0^3 + x_1^3 + x_2^3).$$

Note that the projective plane $\mathbb{P}^2(\mathbb{F}_4)$ defined over $\mathbb{F}_4$ contains 21 points $p_1, \ldots, p_{21}$ and 21 lines $\ell_1, \ldots, \ell_{21}$. The inseparable double cover $Y$ has 21 ordinary nodes over the 21 points in $\mathbb{P}^2(\mathbb{F}_4)$ and hence $X$ has 21 disjoint $(-2)$-curves. We denote by $e_1, \ldots, e_{21} \in S_{2,1}$ the classes of these $(-2)$-curves, by $h \in S_{2,1}$ the class of the pullback of a line on $\mathbb{P}^2$, and by $f_1, \ldots, f_{21} \in S_{2,1}$ the
classes of the proper transforms of the 21 lines in \( \mathbb{P}^2(\mathbb{F}_4) \). Then \( S_{2,1} \) is generated by the \((-2)\)-vectors \( e_1, \ldots, e_{21}, f_1, \ldots, f_{21} \). The vector

\[
  w_M := \frac{1}{3} \sum_{i=1}^{21} (e_i + f_i)
\]

has the property

\[
  w_M \in S_X, \quad w^2_M = 14, \quad \langle w_M, e_i \rangle = \langle w_M, f_i \rangle = 1.
\]

The complete linear system associated with the line bundle corresponding to \( w_M \) defines an embedding of \( X \) into \( \mathbb{P}^2 \times \mathbb{P}^2 \), and its image \( X_M \subset \mathbb{P}^2 \times \mathbb{P}^2 \) is defined by

\[
  \begin{cases}
    x_0 y_0^2 + x_1 y_1^2 + x_2 y_2^2 &= 0, \\
    x_0^2 y_0 + x_1^2 y_1 + x_2^2 y_2 &= 0.
  \end{cases}
\]

Six points on \( \mathbb{P}^2(\mathbb{F}_4) \) are said to be general if no three points of them are collinear. There exist 168 sets of general six points in \( \mathbb{P}^2(\mathbb{F}_4) \). If \( I = \{p_{i_1}, \ldots, p_{i_6}\} \) is a set of general six points, then the \((-1)\)-vector

\[
  c_I := h - \frac{1}{2}(e_{i_1} + \cdots + e_{i_6})
\]

is contained in \( S_{2,1}^\vee \). Note that each \( c_I \) defines a reflection

\[
  x \mapsto x + 2(\langle x, c_I \rangle c_I)
\]

in \( O^+(S_{2,1}) \) because \( c_I \in S_{2,1}^\vee \). Let \( P(X_{2,1}) \) be the positive cone of \( S_{2,1} \) containing an ample class, and let \( \Delta(X_{2,1}) \) be the set consisting of \( e_1, \ldots, e_{21}, f_1, \ldots, f_{21} \) and the \((-1)\)-vectors \( c_I \) defined above. We define a chamber \( D(X_{2,1}) \) in \( P(X_{2,1}) \) by

\[
  D(X_{2,1}) := \{ x \in P(X_{2,1}) \mid \langle x, v \rangle \geq 0 \text{ for all } v \in \Delta(X_{2,1}) \}.
\]

Then, for each \( v \in \Delta(X_{2,1}) \), the hyperplane \( \langle v \rangle^\perp \) is a wall of \( D(X_{2,1}) \). Moreover the ample class \( w_M \) is contained in the interior of \( D(X_{2,1}) \). Recall that \( L_{26} \) is the even unimodular hyperbolic lattice of rank 26. There exists a primitive embedding \( \iota : S_{2,1} \hookrightarrow L_{26} \), which is unique up to \( O(L_{26}) \). The orthogonal complement \( T_i \) of \( S_{2,1} \) in \( L_{26} \) is isomorphic to the root lattice of type \( D_4 \), and hence satisfies the hypothesis (5.1).

**Proposition 6.1.** The chamber \( D(X_{2,1}) \) is an \( \mathcal{R}(L_{26}, \iota)^* \)-chamber contained in the \( \mathcal{R}(S_{2,1})^* \)-chamber \( \text{Nef}^\circ(X_{2,1}) \). An isometry \( g \in O^+(S_{2,1}) \) belongs to \( \text{Aut}(D(X_{2,1})) \) if and only if \( w^g_M = w_M \).

Thus we can apply Proposition 5.3 to the pair of chambers \( D(X_{2,1}) \) and \( \text{Nef}^\circ(X_{2,1}) \) for the study of \( \text{Aut}(\text{Nef}(X_{2,1})) \) and \( \text{Aut}(X_{2,1}) \).

We have the following elements in \( \text{Aut}(X_{2,1}) \) and \( O^+(S_{2,1}) \). Since \( \text{Aut}(X_{2,1}) \) is naturally embedded in \( O^+(S_{2,1}) \), we use the same letter to denote an element of \( \text{Aut}(X_{2,1}) \) and its image in \( O^+(S_{2,1}) \).
The action of $\text{PGL}(3, \mathbb{F}_4)$ on $\mathbb{P}^2$ induces automorphisms of the inseparable double cover $Y$ of $\mathbb{P}^2$, and hence automorphisms of $X_{2,1}$. Their action on $S_{2,1}$ preserves $D(X_{2,1})$.

The interchange of the two factors of $\mathbb{P}^2 \times \mathbb{P}^2$ preserves $X_M \subset \mathbb{P}^2 \times \mathbb{P}^2$, and hence it induces an involution $sw \in \text{Aut}(X_{2,1})$, which we call the switch. Its action on $S_{2,1}$ preserves $D(X_{2,1})$.

For each set $I$ of general six points in $\mathbb{P}^2(\mathbb{F}_4)$, the linear system of plane curves of degree 5 that pass through the points of $I$ and are singular at each point of $I$ defines a birational involution of $\mathbb{P}^2$, and this involution lifts to an involution of $Y$. Hence we obtain an involution $Cr_I \in \text{Aut}(X_{2,1})$, which we call a Cremona automorphism of $X_{2,1}$. The action of $Cr_I$ on $S_{2,1}$ is the reflection with respect to $c_I \in S_{2,1}^\vee$.

The Frobenius action of $\text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ on $X_M$ induces an isometry $\text{Fr}$ of $S_{2,1}$, which preserves $D(X_{2,1})$.

We have the reflections $s_{e_i}$ and $s_{f_i}$ with respect to the $(-2)$-vectors $e_i$ and $f_i$.

By the reflections $Cr_I$, $s_{e_i}$, and $s_{f_i}$, we see that the chamber $D(X_{2,1})$ is fundamental.

**Theorem 6.2 ([8]).** (1) The projective automorphism group $\text{Aut}(X_{2,1}, w_M)$ of $X_M \subset \mathbb{P}^2 \times \mathbb{P}^2$ is generated by $\text{PGL}(3, \mathbb{F}_4)$ and the switch $sw$.

(2) The group $\text{Aut}(D(X_{2,1}))$ is generated by $\text{Aut}(X_{2,1}, w_M)$ and $\text{Fr}$.

(3) The automorphism group $\text{Aut}(X_{2,1})$ is generated by $\text{Aut}(X_{2,1}, w_M)$ and the 168 Cremona automorphisms $Cr_I$.

(4) The group $\text{Aut}(\text{Nef}(X_{2,1}))$ is generated by $\text{Aut}(X_{2,1})$ and $\text{Fr}$.

(5) The group $\text{O}^+(S_{2,1})$ is generated by $\text{Aut}(\text{Nef}(X_{2,1}))$ and the $21 + 21$ reflections $s_{e_i}$ and $s_{f_i}$.

We then study $\text{Aut}(\text{Nef}(X_{2,10}))$. By Corollary 3.2, we have an embedding

$$j : S_{2,1} \hookrightarrow S_{2,10}$$

that induces $S_{2,10}^\vee(2) \cong S_{2,10}$. Composing $j$ with some element of $W(S_{2,10}) \times \{ \pm 1 \}$, we can assume that $j(w_M)$ is contained in $\text{Nef}(X_{2,10})$ (Proposition 2.1(2)). The isomorphism $j_* : \text{O}^+(S_{2,1}) \cong \text{O}^+(S_{2,10})$ induced by $j$ is denoted by

$$g \mapsto g'.$$

The $j(\mathcal{R}(L_{26}, \iota))^*$-chamber $j(D(X_{2,1}))$ is fundamental, and we have

$$\text{Aut}(j(D(X_{2,1}))) = \text{Aut}(D(X_{2,1})).$$

**Lemma 6.3.** The set $j(\mathcal{R}(L_{26}, \iota))$ contains $\mathcal{R}(S_{2,10})$. Hence the $j(\mathcal{R}(L_{26}, \iota))^*$-chamber $j(D(X_{2,1}))$ is contained in the $\mathcal{R}(S_{2,10})^*$-chamber $\text{Nef}^\vee(X_{2,10})$.

**Proof.** It is enough to show that, if $v \in S_{2,10}^\vee$ satisfies $v^2 = -1$, then $v \in \mathcal{R}(L_{26}, \iota)$, that is, there exists $u \in T^\vee$ such that $u^2 = -1$ and that $u + v$ is contained in the submodule $L_{26}$ of $S_{2,10}^\vee \oplus T^\vee$. By
Nikulin [20, Proposition 1.4.1], the submodule \( L_{26}/(S_{2,1} \oplus T_i) \) of \( (S_{2,1} \oplus T_i)/T_i = A_{S_{2,1}} \oplus A_{T_i} \) is the graph of an isomorphism

\[
q_{S_{2,1}} \cong -q_{T_i}.
\]

Hence it is enough to show that, for any \( \bar{u} \in A_{T_i} \) with \( q_{T_i}(\bar{u}) = 1 \), there exists \( u \in T_i \) such that \( u^2 = -1 \) and \( u \mod T_i = \bar{u} \). Since \( T_i \) is a root lattice of type \( D_4 \), we can confirm this fact by direct computation. The set of \((-1)\)-vectors in \( T_i \) consists of 24 vectors, and its image by the natural projection \( T_i \rightarrow A_{T_i} \) is the set of all non-zero elements of \( A_{T_i} \cong \mathbb{P}^2 \).

The set of walls of \( j(D(X_{2,1})) \) is equal to

\[
\{(j(e_i))^\perp | i = 1, \ldots, 21\} \cup \{(j(f_i))^\perp | i = 1, \ldots, 21\} \cup
\{(j(c_i))^\perp | I \text{ is a set of general six points}\}.
\]

Note that the 21 + 21 vectors \( j(e_i) \) and \( j(f_i) \) are of norm \(-4\) and the 168 vectors \( j(c_i) \) are of norm \(-2\). Note also that neither \( (j(e_i))^\perp \) nor \( (j(f_i))^\perp \) are contained in \( \mathcal{R}(S_{2,10})^\ast \), because there are no rational numbers \( \lambda \) such that \((-4)\lambda^2 = -2\). By Proposition 5.3, Theorem 6.2 and Lemma 6.3, we obtain the following:

**Theorem 6.4.** The group \( \text{Aut}(\text{Nef}(X_{2,10})) \) is generated by \( \text{PGL}(3, \mathbb{F}_4)^\vee \), \( \text{sw}^4 \), \( F_2^\vee \), \( s_e^\vee \) and \( s_f^\vee \).

6.2. **The group** \( \text{Aut}(\text{Nef}(X_{3,10})) \). By the same argument as above, we obtain a set of generators of \( \text{Aut}(\text{Nef}(X_{3,10})) \) from the result of Kondo and Shimada [18].

We consider the Fermat quartic surface

\[
X_{\mathbb{F}_4} : x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0
\]

in characteristic 3. Then \( X_{\mathbb{F}_4} \) is isomorphic to \( X_{3,1} \). The surface \( X_{\mathbb{F}_4} \) contains 112 lines, and their classes \( l_1, \ldots, l_{112} \) span \( S_{3,1} \). We denote by \( h_{\mathbb{F}_4} \in S_{3,1} \) the class of a hyperplane section of \( X_{\mathbb{F}_4} \).

There exists a primitive embedding \( \iota : S_{3,1} \hookrightarrow L_{26} \), which is unique up to \( \text{O}(L_{26}) \). The orthogonal complement \( T_i \) is isomorphic to the root lattice of type \( 2A_2 \), and hence satisfies the hypothesis (5.1). We calculated an \( \mathcal{R}(L_{26}, \iota)^\ast\)-chamber \( D(X_{3,1}) \) that contains \( h_{\mathbb{F}_4} \) in its interior, and found 648 vectors \( u_j \in S_{3,1}^\vee \) of norm \(-4/3\), and 5184 vectors \( w_k \in S_{3,1}^\vee \) of norm \(-2/3\) such that the walls of \( D(X_{3,1}) \) consist of the 112 hyperplanes \( (l_i)^\perp \), the 648 hyperplanes \( (u_j)^\perp \) and the 5184 hyperplanes \( (w_k)^\perp \). Note that the \( \mathcal{R}(L_{26}, \iota)^\ast\)-chamber \( D(X_{3,1}) \) is contained in the \( \mathcal{R}(S_{3,1})^\ast\)-chamber \( \text{Nef}^0(X_{3,1}) \), because \( h_{\mathbb{F}_4} \in D(X_{3,1})^0 \). Moreover, since \( 28 h_{\mathbb{F}_4} = \sum l_i \), the following holds:

**Proposition 6.5.** An isometry \( g \in O^+(S_{3,1}) \) belongs to \( \text{Aut}(D(X_{3,1})) \) if and only if \( h_{\mathbb{F}_4}^g = h_{\mathbb{F}_4} \).

We have the following elements in \( \text{Aut}(X_{3,1}) \) and \( O^+(S_{3,1}) \). Note that, for a polarization \( h \in S_{3,1} \) of degree 2, we have the deck transformation \( \tau(h) \in \text{Aut}(X_{3,1}) \) of the generically finite morphism \( X_{3,1} \rightarrow \mathbb{P}^2 \) of degree 2 induced by the the complete linear system associated with \( h \).
• The subgroup PGU(4,F9) of PGL(4,k) = Aut(ℙ3) acts on X_{FQ}. Its action on S_{3,1} preserves D(X_{3,1}). Moreover, the action of PGU(4,F9) on S'_{3,1} is transitive on each of the set of 112 vectors l_i, the set of 648 vectors u_j and the set of 5184 vectors w_k.

• There exists a polarization h_{648} ∈ S_{3,1} of degree 2 such that the deck transformation τ(h_{648}) ∈ Aut(X_{3,1}) maps D(X_{3,1}) to an ℛ(L_{26},ι)*-chamber adjacent to D(X_{3,1}) along one of the 648 walls (u_j)\tilde{\!}.

• There exists a polarization h_{5184} ∈ S_{3,1} of degree 2 such that the deck transformation τ(h_{5184}) ∈ Aut(X_{3,1}) maps D(X_{3,1}) to an ℛ(L_{26},ι)*-chamber adjacent to D(X_{3,1}) along one of the 5184 walls (w_k)\tilde{\!}.

• The Frobenius action of Gal(F9/F3) on X_{FQ} gives rise to an element Fr ∈ Aut(D(X_{3,1})) of order 2.

• We have the reflections s_\ell_i with respect to the classes l_i of the 112 lines on X_{FQ}.

Remark 6.6. The actions of the involutions τ(h_{648}) and τ(h_{5184}) on S_{3,1} are not reflections.

Thus D(X_{3,1}) is fundamental, and hence we have the following:

Theorem 6.7 ([18]). (1) The projective automorphism group Aut(X, h_{FQ}) of the Fermat quartic surface X_{FQ} ⊂ ℙ^3 is equal to PGU(4,F9).

(2) The group Aut(D(X_{3,1})) is generated by Aut(X, h_{FQ}) and Fr.

(3) The automorphism group Aut(X_{3,1}) is generated by Aut(X, h_{FQ}) and the two involutions τ(h_{648}) and τ(h_{5184}).

(4) The group Aut(Nef(X_{3,1})) is generated by Aut(X_{3,1}) and Fr.

(5) The group O^+(S_{3,1}) is generated by Aut(Nef(X_{3,1})) and the 112 reflections s_{\ell_i}.

By Corollary 3.2, we have an embedding

\[ j : S_{3,1} \hookrightarrow S_{3,10} \]

that induces S'_{3,1}(3) ∼= S_{3,10}. By Proposition 2.1(2), we can assume that j(h_{FQ}) is contained in Nef(X_{3,10}). The isomorphism j_* : O^+(S_{3,1}) ∼= O^+(S_{3,10}) induced by j is denoted by g ↣ g'. The j(ℛ(L_{26},ι))*-chamber j(D(X_{3,1})) is fundamental, and Aut(j(D(X_{3,1}))) is equal to Aut(D(X_{3,1})).

Lemma 6.8. The set j(ℛ(L_{26},ι)) contains ℛ(S_{3,10}). Hence the j(ℛ(L_{26},ι))*-chamber j(D(X_{3,1})) is contained in the ℛ(S_{3,10})*-chamber Nef^o(X_{3,10}).

Proof. It is enough to show that, if v ∈ S'_{3,1} satisfies v^2 = -2/3, then there exists u ∈ T_{\ell_i}' such that u^2 = -4/3 and that u + v is contained in L_{26} ⊂ S'_{3,1} \oplus T_{\ell_i}'. For this, it suffices to prove that, for any \bar{u} ∈ A_{\ell_i} with q_{T_{\ell_i}}(\bar{u}) = -4/3, there exists u ∈ T_{\ell_i}' such that u^2 = -4/3 and u mod T_{\ell_i} = \bar{u}. Since T_{\ell_i} is a root lattice of type 2A_2, we can confirm this fact by direct computation.

□
The set of walls of \( j(D(X_{3,1})) \) is equal to
\[
\{(j(l_i))^\perp \mid i = 1, \ldots, 112\} \cup \{(j(u_j))^\perp \mid j = 1, \ldots, 648\} \cup
\{(j(w_k))^\perp \mid k = 1, \ldots, 5184\}.
\]

Note that the vectors \( j(l_i) \) are of norm \(-6\), the vectors \( j(u_j) \) are of norm \(-4\), and the vectors \( j(w_k) \) are of norm \(-2\). Note also that neither \( (j(l_i))^\perp \) nor \( (j(u_j))^\perp \) are contained in \( R(S_{3,10})^* \). By Proposition 5.3, Theorem 6.7 and Lemma 6.8, we obtain the following:

**Theorem 6.9.** The group \( \text{Aut}(\text{Nef}(X_{3,10})) \) is generated by \( \text{PGU}(4, \mathbb{F}_9)' \), \( \text{Fr}' \), \( \text{sl}'_i \) and \( \tau(h_{648})' \).

7. Torelli Theorem for Supersingular \( K3 \) Surfaces

We review the theory of period mapping and Torelli theorem for supersingular \( K3 \) surfaces in odd characteristics by Ogus [24], [25]. Throughout this section, we assume that \( p \) is odd.

We summarize results on quadratic spaces over finite fields. See, for example, Kitaoka [15, Section 1.3]. Let \( \mathbb{F}_q \) be a finite extension of \( \mathbb{F}_p \). There exist exactly two isomorphism classes of non-degenerate quadratic forms in \( 2\sigma \) variables \( x_1, \ldots, x_{2\sigma} \) over \( \mathbb{F}_q \). They are represented by
\[
\begin{align*}
f_+ & := x_1 x_2 + \cdots + x_{2\sigma-1} x_{2\sigma}, \quad \text{and} \\
f_- & := x_1^2 + cx_1 x_2 + x_2^2 + x_3 x_4 + \cdots + x_{2\sigma-1} x_{2\sigma},
\end{align*}
\]
where \( c \) is an element of \( \mathbb{F}_q \) such that \( t^2 + ct + 1 \in \mathbb{F}_q[t] \) is irreducible. The quadratic form \( f_+ \) (resp. \( f_- \)) is called neutral (resp. non-neutral). The group \( \text{O}(\mathbb{F}_q^{2\sigma}, f_\epsilon) \) of the self-isometries of the quadratic space \( (\mathbb{F}_q^{2\sigma}, f_\epsilon) \), where \( \epsilon = \pm 1 \), is of order
\[
2 q^{\sigma(\sigma-1)} (q^{\sigma} - \epsilon) \prod_{i=1}^{\sigma-1} (q^{2i} - 1).
\]

Let \( N \) be an even hyperbolic \( p \)-elementary lattice of rank 22 with discriminant \(-p^{2\sigma}\). We define a quadratic space \( (N_0, q_0) \) over \( \mathbb{F}_p \) by (1.2). It is known that \( q_0 \) is non-degenerate and non-neutral. We denote by \( \text{O}(N_0, q_0) \) the group of the self-isometries of \( (N_0, q_0) \). Note that the scalar multiplications in \( \text{O}(N_0, q_0) \) are only \( \pm 1 \). Let \( k \) be a field of characteristic \( p \). We put
\[
\varphi := \text{id}_{N_0} \otimes F_k : N_0 \otimes k \to N_0 \otimes k,
\]
where \( F_k \) is the Frobenius map of \( k \).

**Definition 7.1.** A subspace \( K \) of \( N_0 \otimes k \) with \( \dim K = \sigma \) is said to be a characteristic subspace of \( (N_0, q_0) \) if \( K \) is totally isotropic with respect to the quadratic form \( q_0 \otimes k \) and \( \dim(K \cap \varphi(K)) = \sigma - 1 \) holds.
Suppose that $k$ is algebraically closed. Let $X$ be a supersingular $K3$ surface with Artin invariant $\sigma$ defined over $k$. An isomorphism 
$$\eta : N \cong S_X$$
of lattices is called a marking of $X$. We fix a marking $\eta$ of $X$. The composite of the marking $\eta$ and the Chern class map $S_X \to H^2_{\mathrm{DR}}(X/k)$ defines a linear homomorphism 
$$\tilde{\eta} : N \otimes k \to H^2_{\mathrm{DR}}(X/k).$$
It is known that $\ker \tilde{\eta}$ is contained in $N_0 \otimes k$, and is totally isotropic with respect to $q_0 \otimes k$. We put 
$$K_{(X, \eta)} := \varphi^{-1}(\ker \tilde{\eta}),$$
and call $K_{(X, \eta)}$ the period of the marked supersingular $K3$ surface $(X, \eta)$. Then it is proved by Ogus [24], [25] that $K_{(X, \eta)}$ is a characteristic subspace of $(N_0, q_0)$. We denote by $\eta^* : O(S_X) \cong O(N)$ the isomorphism induced by the marking $\eta$, and let 
$$\tilde{\eta}^* : O(S_X) \to O(N_0, q_0)$$
be the composite of $\eta^*$ with the natural homomorphism $O(N) \to O(N_0, q_0)$. As a corollary of Torelli theorem by Ogus [25, Corollary of Theorem II′], we have the following:

**Corollary 7.2.** Let $\eta$ be a marking of $X$. Then, as a subgroup of $O^+(S_X)$, the automorphism group $\mathrm{Aut}(X)$ of $X$ is equal to 
$$\{ g \in \mathrm{Aut}(\mathrm{Nef}(X)) \mid K_{(X, \eta)}^{\eta^*}(g) = K_{(X, \eta)} \}.$$
In particular, the index of $\mathrm{Aut}(X)$ in $\mathrm{Aut}(\mathrm{Nef}(X))$ is at most $|O(N_0, q_0)/\{\pm 1\}|$.

Combining Corollaries 5.8 and 7.2, we obtain the following:

**Corollary 7.3.** The automorphism group $\mathrm{Aut}(X)$ is infinite.

**Remark 7.4.** When $p = 3$ and $\sigma = 1$, the group $O(N_0, q_0)$ is of order 8, while the index of $\mathrm{Aut}(X_{3,1})$ in $\mathrm{Aut}(\mathrm{Nef}(X_{3,1}))$ is 2 by Theorem 6.7.

**Definition 7.5.** We say that a supersingular $K3$ surface $X$ with Artin invariant $\sigma$ is generic if there exists a marking $\eta$ for $X$ such that the subgroup 

\begin{equation}
\{ \gamma \in O(N_0, q_0) \mid K_{(X, \eta)}^{\gamma} = K_{(X, \eta)} \}
\end{equation}

of $O(N_0, q_0)$ consists of only scalar multiplications $\pm 1$.

If $X$ is generic, then the subgroup (7.3) consists of only scalar multiplications for any marking $\eta$. The existence of generic supersingular $K3$ surfaces with Artin invariant $> 1$ (Theorem 1.7) is proved in the next section.
Recall that $A_{S_X}$ is the discriminant group of $S_X$, and $q_{S_X} : A_{S_X} \to \mathbb{Q}/2\mathbb{Z}$ is the discriminant quadratic form. We will regard $A_{S_X}$ as a $2\sigma$-dimensional vector space over $\mathbb{F}_p$. Note that the image of $q_{S_X}$ is contained in $(2/p)\mathbb{Z}/2\mathbb{Z}$. We define $\bar{q}_{S_X} : A_{S_X} \to \mathbb{F}_p$ by

\[ \bar{q}_{S_X}(x \mod S_X) := p \cdot q_{S_X}(x) \mod p. \]

Then we obtain a quadratic space $(A_{S_X}, \bar{q}_{S_X})$ over $\mathbb{F}_p$. Note that we can recover $q_{S_X}$ from $\bar{q}_{S_X}$. We have natural homomorphisms

\[ (7.4) \quad O(S_X) \to O(q_{S_X}) \cong O(A_{S_X}, \bar{q}_{S_X}) \to \text{PO}(A_{S_X}, \bar{q}_{S_X}) := O(A_{S_X}, \bar{q}_{S_X})/\{\pm 1\}. \]

Let $\eta : N^\vee \cong S_X^\vee$ be the isomorphism induced by a marking $\eta$. Then the map

\[ px \mod pN_0 \in N_0 \mapsto \eta(x) \mod S_X \in A_{S_X} \quad (x \in N^\vee) \]

induces an isomorphism of quadratic spaces from $(N_0, q_0)$ to $(A_{S_X}, \bar{q}_{S_X})$. By Corollary 7.2, we obtain the following:

**Corollary 7.6.** Suppose that $X$ is generic. Then $\text{Aut}(X)$ is equal to the kernel of the homomorphism

\[ \Phi : \text{Aut}(\text{Nef}(X)) \to \text{PO}(A_{S_X}, \bar{q}_{S_X}) \]

obtained by restricting (7.4) to $\text{Aut}(\text{Nef}(X)) \subset O(S_X)$.

**Remark 7.7.** Suppose that $X$ is generic, and that we are given a subset $\{g_1, \ldots, g_n\}$ of $\text{Aut}(\text{Nef}(X))$ that generate $\text{Aut}(\text{Nef}(X))$. Then a finite set of generators of $\text{Aut}(X)$ is obtained by the following procedure. We construct a finite directed graph $(V, E)$ as follows. The set $V$ of vertices is the image of $\Phi$, that is, the subgroup of $\text{PO}(A_{S_X}, \bar{q}_{S_X})$ generated by $\Phi(g_1), \ldots, \Phi(g_n)$. The set $E$ of directed edges is the set of triples

\[ \alpha = (s_\alpha, g_\alpha, t_\alpha), \]

where $s_\alpha, t_\alpha \in V$ and $s_\alpha \Phi(g_\alpha) = t_\alpha$. The edge $\alpha$ is directed from $s_\alpha$ to $t_\alpha$ and labelled with a generator $g_\alpha$. We put $\alpha^{-1} := (t_\alpha, g_\alpha^{-1}, s_\alpha)$. We use the identity element $e \in V$ as a base point of the 1-dimensional CW-complex $\Gamma$ associated with $(V, E)$. Then the fundamental group $\pi_1(\Gamma, e)$ is a free group of finite rank, and its generators are calculated from the graph $(V, E)$. Consider a loop

\[ \gamma = \alpha^{\varepsilon_0}_0 \ldots \alpha^{\varepsilon_m}_m \]

of $\Gamma$ from $e$ to $e$, where $\varepsilon_i = \pm 1$ and $\alpha^{\varepsilon_j}_j = (v_j, g^{\varepsilon_j}_j, v_{j+1})$. Then we have $v_0 = v_{m+1} = e$, and

\[ \tilde{\gamma} := g^{\varepsilon_0}_{i_0} \cdots g^{\varepsilon_m}_{i_m} \in \text{Aut}(\text{Nef}(X)) \]

is mapped to $e$ by $\Phi$. If $\pi_1(\Gamma, e)$ is generated by loops $\gamma_1, \ldots, \gamma_l$, then $\text{Aut}(X) = \text{Ker} \Phi$ is generated by $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_l$. 

Remark 7.8. Suppose that $X_{3,10}$ is generic. Applying the procedure in Remark 7.7 to the generators of $Aut(Nef(X_{3,10}))$ given in Theorem 6.9, we can obtain a finite set of generators of $Aut(X_{3,10})$. However, a naive application of the procedure would be inexecutable, because, when $p = 3$ and $\sigma = 10$, the order of $O(N_0,q_0)$ is
\[
2^{36} \cdot 3^{90} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 37 \cdot 41^2 \cdot 61 \cdot 73 \cdot 193 \cdot 547 \cdot 757 \cdot 1093 \cdot 1181,
\]
which is about $7.886 \times 10^{99}$.

For a non-zero vector $v \in S_X \otimes \mathbb{Q}$, we denote by $\langle v \rangle$ the linear subspace of $S_X \otimes \mathbb{Q}$ spanned by $v$, and put
\[
\bar{v} := (\langle v \rangle \cap S_X^0) / (\langle v \rangle \cap S_X),
\]
which is a linear subspace of $A_{S_X} \cong \mathbb{P}^{2\sigma}_p$. When $\bar{v} \neq 0$, we denote by
\[
[\bar{v}] \in \mathbb{P}(A_{S_X})
\]
the corresponding point of the projective space $\mathbb{P}(A_{S_X})$ over $\mathbb{F}_p$. We consider the action of $O(S_X)$ on $\mathbb{P}(A_{S_X})$.

Remark 7.9. By definition, the reflections $s_r$ with respect to $r \in R(S_X)$ act on $A_{S_X}$ trivially. Hence the restriction $\Phi$ of the homomorphism (7.4) to the subgroup $Aut(Nef(X))$ of $O(S_X)$ is also obtained by passing to the quotient $O(S_X)/\langle W(S_X) \times \{\pm 1\} \rangle \cong Aut(Nef(X))$. Thus the orbit of $[\bar{v}]$ under the action of $Aut(Nef(X))$ is equal to the orbit of $[\bar{v}]$ under the action of $O(S_X)$.

Corollary 7.10. Suppose that $X$ is generic. Let $v \in S_X$ be a vector such that $\bar{v} \subset A_{S_X}$ is not zero. Let $m$ be the cardinality of the orbit of $[\bar{v}] \in \mathbb{P}(A_{S_X})$ under the action of $O(S_X)$. Then the number of $Aut(X)$-orbits contained in the $O(S_X)$-orbit of $v$ in $S_X$ is at least $m$.

8. Existence of generic supersingular K3 surfaces

We prove Theorem 1.7. For the proof, we recall the construction by Ogus [24] of the scheme $\mathcal{M}$ parameterizing characteristic subspaces of the $2\sigma$-dimensional quadratic space $(N_0,q_0)$ over $\mathbb{F}_p$. This scheme $\mathcal{M}$ plays the role of the period domain for supersingular K3 surfaces. We continue to assume that $p$ is odd.

Let $\text{Grass}(\nu,N_0)$ denote the Grassmannian variety of $\nu$-dimensional subspaces of $N_0$, and let $\text{Isot}(\nu,q_0)$ be the subscheme of $\text{Grass}(\nu,N_0)$ parameterizing $\nu$-dimensional totally isotropic subspaces of $(N_0,q_0)$. We put
\[
\text{Gen} := \text{Isot}(\sigma,q_0),
\]
where $\text{Gen}$ is for “generatrix”. Note that $\text{Isot}(\nu,q_0)$ is defined over $\mathbb{F}_p$ for any $\nu$. Let $k$ be a field of characteristic $p$. For a subspace $L$ of $N_0 \otimes k$ with dimension $\nu$, we denote by $[L]$ the $k$-valued point of $\text{Grass}(\nu,N_0)$ corresponding to $L$. We then have the following:

(1) If $\nu < \sigma$, then $\text{Isot}(\nu,q_0)$ is geometrically connected.
(2) The scheme $\text{Gen} \otimes \mathbb{F}_{p^2}$ has two connected components $\text{Gen}_+$ and $\text{Gen}_-$, each of which is geometrically connected. Since $q_0$ is non-neutral, the action of $\text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ interchanges the two connected components.

(3) Let $K$ and $K'$ be two $\sigma$-dimensional totally isotropic subspaces of $(N_0, q_0) \otimes k$. Suppose that $\dim(K \cap K') = \sigma - 1$. Then the $k$-valued points $[K]$ and $[K']$ belong to distinct connected components of $\text{Gen}$.

(4) Suppose that $k$ is algebraically closed. Then, for each $k$-valued point $[L]$ of the scheme $\text{Isot}(\sigma - 1, q_0)$, there exist exactly two $\sigma$-dimensional totally isotropic subspaces of $(N_0, q_0) \otimes k$ that contain $L$.

(5) Let $P$ be the subscheme of $\text{Gen} \times \text{Gen}$ parameterizing pairs $(K, K')$ such that $\dim(K \cap K') = \sigma - 1$. Then the scheme $P \otimes \mathbb{F}_{p^2}$ has two connected components, each of which is isomorphic to $\text{Isot}(\sigma - 1, q_0)$ over $\mathbb{F}_{p^2}$. The action of $\text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ interchanges the two connected components.

Consider the graph

$$\text{id} \times \varphi : \text{Gen} \rightarrow \text{Gen} \times \text{Gen}$$

of the Frobenius morphism $\text{Gen} \rightarrow \text{Gen}$ given by $K \mapsto \varphi(K)$. The subscheme $\mathcal{M}$ of $\text{Gen}$ that parametrizes the characteristic subspaces of $(N_0, q_0)$ is defined by the fiber product

$$\mathcal{M} \hookrightarrow \text{Gen} \quad \downarrow \quad \square \quad \downarrow \text{id} \times \varphi \quad \quad \quad \quad \quad P \hookrightarrow \text{Gen} \times \text{Gen}.$$

Ogus [24] proved the following:

**Theorem 8.1.** The scheme $\mathcal{M}$ defined over $\mathbb{F}_p$ is smooth and projective of dimension $\sigma - 1$. The scheme $\mathcal{M} \otimes \mathbb{F}_{p^2}$ has two connected components $\mathcal{M}_+ = \mathcal{M} \cap \text{Gen}_+$ and $\mathcal{M}_- = \mathcal{M} \cap \text{Gen}_-$, each of which is geometrically connected. The action of $\text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ interchanges $\mathcal{M}_+$ and $\mathcal{M}_-$.

**Proof of Theorem 1.7.** Let $\kappa$ be an algebraic closure of the function field of the scheme $\mathcal{M}_+$ over $\mathbb{F}_{p^2}$, and let $[K_{\kappa}]$ be the geometric generic point of $\mathcal{M}_+$. By the surjectivity of the period mapping for supersingular $K3$ surfaces (Ogus [25, Theorem III′]), there exist a supersingular $K3$ surface $X$ of Artin invariant $\sigma$ defined over $\kappa$ and a marking $\eta : N \simeq S_X$ such that $K_{(X, \eta)} = K_{\kappa}$. We prove that this $X$ is generic, that is,

$$G_{\kappa} := \{ \gamma \in O(N_0, q_0) \mid K_{\kappa}^\gamma = K_{\kappa} \}$$

is equal to $\{ \pm 1 \}$. Note that the closure of the point $[K_{\kappa}]$ coincides with $\mathcal{M}_+$. Therefore we have the following: If a field $k$ contains $\mathbb{F}_{p^2}$, then the action of $G_{\kappa}$ leaves $K$ invariant for any $k$-valued point $[K]$ of $\mathcal{M}_+$. 
Suppose that $\sigma \geq 3$. Let $u$ be an arbitrary non-zero isotropic vector of $N_0$. We will prove that $u$ is an eigenvector of $G_\kappa$. Let

$$b_0 : N_0 \times N_0 \to \mathbb{F}_p$$

denote the symmetric bilinear form obtained from $q_0$. There exists a vector $v \in N_0$ such that $q_0(v) = 0$ and $b_0(u, v) = 1$, and hence $(N_0, q_0)$ has an orthogonal direct-sum decomposition

$$N_0 = U^\perp \oplus U,$$

where $U$ is the subspace spanned by $u$ and $v$. Repeating this procedure and noting that $q_0$ is non-neutral, we obtain a basis $a_1, \ldots, a_{2\sigma}$ of $N_0$ with $u = a_{2\sigma}$ such that $q_0(x_1a_1 + \cdots + x_{2\sigma}a_{2\sigma})$ is equal to the quadratic polynomial $f_-$ in (7.2). Let $\alpha$ and $\bar{\alpha} = \alpha^p$ be the roots in $\mathbb{F}_{p^2}$ of the irreducible polynomial $t^2 + ct + 1 \in \mathbb{F}_p[t]$. We consider the basis

(8.1) $b_i^{(-1)} := \alpha a_1 + a_2, \quad b_i^{(1)} := \bar{\alpha} a_1 + a_2, \quad \text{and} \quad b_i^{(-1)} := a_{2i-1}, \quad b_i^{(1)} := a_{2i} \quad (i = 2, \ldots, \sigma)$

of $N_0 \otimes \mathbb{F}_{p^2}$. Note that each $b_i^{(\pm 1)}$ is isotropic, and that

$$b_0(b_i^{(\alpha)}, b_j^{(\beta)}) = 0 \quad \text{if} \quad i \neq j, \quad b_0(b_i^{(1)}, b_i^{(-1)}) = \begin{cases} (1 - c^2)/2 & \text{if} \quad i = 1, \\ 1/2 & \text{if} \quad i \geq 2. \end{cases}$$

We put

$$\mathcal{E} := \{(1, -1)^\sigma\}.$$

For $e = (e_1, \ldots, e_\sigma) \in \mathcal{E}$, we denote by $K_e$ the linear subspace of $N_0 \otimes \mathbb{F}_{p^2}$ spanned by

$$b_1^{(e_1)} \ldots, b_\sigma^{(e_\sigma)}.$$

It is obvious that $K_e$ is isotropic. Moreover, since

$$\varphi(b_i^{(e)}) = b_i^{(-e)} \quad \text{and} \quad \varphi(b_i^{(e)}) = b_i^{(e)} \quad \text{if} \quad i \geq 2,$$

we have $\dim(K_e \cap \varphi(K_e)) = \sigma - 1$. Therefore $K_e$ is a characteristic subspace of $(N_0, q_0)$. Suppose that $e$ and $e' \in \mathcal{E}$ differ only at one component. Then we have $\dim(K_e \cap K_{e'}) = \sigma - 1$, and hence the $\mathbb{F}_{p^2}$-valued points $[K_e]$ and $[K_{e'}]$ of $\mathcal{M}$ belong to distinct connected components. We put

$$\mathcal{E}_+ := \{ e \in \mathcal{E} \mid \text{the number of } -1 \text{ in } e \text{ is even} \}, \quad 1 := (1, \ldots, 1) \in \mathcal{E}_+.$$

Interchanging $\alpha$ and $\bar{\alpha}$ if necessary, we can assume that $[K_1]$ is an $\mathbb{F}_{p^2}$-valued point of $\mathcal{M}_+$, and hence $[K_e]$ is an $\mathbb{F}_{p^2}$-valued point of $\mathcal{M}_+$ for any $e \in \mathcal{E}_+$. It follows that $K_e$ is invariant under the action of $G_\kappa$ for any $e \in \mathcal{E}_+$. Let $b_i^{(\alpha)}$ be an arbitrary element among the basis (8.1). Recall that we have assumed $\sigma \geq 3$. Therefore, for each element $b_j^{(\beta)}$ among the basis (8.1) that is distinct from $b_i^{(\alpha)}$, there exists $e(j, \beta) = (e_1, \ldots, e_\sigma) \in \mathcal{E}_+$ such that $e_1 = \alpha$ and $e_j \neq \beta$. Since

$$\bigcap_{(j, \beta) \neq (i, \alpha)} K_{e(j, \beta)} = (b_i^{(\alpha)})$$
is invariant under the action of $G_\kappa$, we see that $\lambda_i^{(\alpha)}$ is an eigenvector of $G_\kappa$. In particular, the isotropic vector $u = a_2 \sigma = b^{(1)}_\sigma$ given at the beginning is an eigenvector of $G_\kappa$.

Let

$$\lambda_i^{(\alpha)} : G_\kappa \to \mathbb{F}^*_p$$

be the character defined by $b_i^{(\alpha)}$. Suppose that $i, j \geq 2$ and $i \neq j$. Then $b_i^{(\alpha)} + b_j^{(\beta)}$ is an isotropic vector of $N_0$ for any choice of $\alpha, \beta \in \{\pm 1\}$, and hence is an eigenvector of $G_\kappa$. Therefore we have

$$(8.2) \quad \lambda_i^{(\alpha)} = \lambda_j^{(\beta)} \quad \text{if } i, j \geq 2 \text{ and } i \neq j.$$

Since the cardinality of $\{x^2 \mid x \in \mathbb{F}_p\}$ is $(p + 1)/2$, there exist $\xi, \eta \in \mathbb{F}_p$ such that

$$(4 - c^2) + \xi^2 + \eta^2 = 0.$$ \[9340714]

Then

$$b_1^{(1)} + b_1^{(-1)} + \xi(b_2^{(1)} + b_2^{(-1)}) + \eta(b_3^{(1)} + b_3^{(-1)})$$

is also an isotropic vector of $N_0$, and hence is an eigenvector of $G_\kappa$. Therefore we have

$$(8.3) \quad \lambda_1^{(1)} = \lambda_1^{(-1)} = \lambda_2^{(1)} = \lambda_2^{(-1)} \quad \text{or} \quad \lambda_1^{(1)} = \lambda_1^{(-1)} = \lambda_3^{(1)} = \lambda_3^{(-1)}.$$ \[9340714]

Combining (8.2) and (8.3), we see that all the characters $\lambda_i^{(\alpha)}$ are equal to each other. Thus $G_\kappa$ consists of only scalar multiplications.

Suppose that $\sigma = 2$. In this case, the scheme $\mathcal{M}$ coincides with $\text{Isot}(2, q_0)$, which is the scheme parametrizing lines on the smooth quadratic surface $Q_0 = \{q_0 = 0\}$ in the projective space $\mathbb{P} \times N_0 = \text{Grass}(1, N_0)$. Hence $\mathcal{M}_+$ and $\mathcal{M}_-$ correspond to the two rulings of $Q_0$. Let $g$ be an element of $G_\kappa$. Then $g$ leaves every line in the ruling of $Q_0$ corresponding to $\mathcal{M}_+$ invariant. Since $g$ is defined over $\mathbb{F}_p$ and $\text{Gal}(\mathbb{F}_p^\text{sep}/\mathbb{F}_p)$ interchanges $\mathcal{M}_+$ and $\mathcal{M}_-$, we see that $g$ also leaves every line in the other ruling of $Q_0$ invariant. Therefore $g$ fixes every point of $Q_0$, and hence every point of $\mathbb{P} \times N_0$. \[9340714\]

9. LATTICE EQUIVALENCE CLASSES VERSUS $\text{Aut}$-EQUIVALENCE CLASSES ON $X_{3,10}$

Suppose that $p > 2$ and $\sigma + \sigma' = 11$. We denote by $A_{p,\sigma'}$ the discriminant group $S_{p,\sigma'}^\vee/S_{p,\sigma'}$ of $S_{p,\sigma'}$, and use the notation in Section 7.

Let $\phi : X_{p,\sigma} \to \mathbb{P}^1$ be a genus one fibration, and let $\phi' : X_{p,\sigma'} \to \mathbb{P}^1$ be a genus one fibration whose lattice equivalence class $[\phi'] \in \mathbb{E}(X_{p,\sigma'})$ corresponds to $[\phi] \in \mathbb{E}(X_{p,\sigma})$ by Theorem 1.3. By Remark 4.6, we have an embedding $j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$ inducing $S_{p,\sigma'}^\vee(p) \cong S_{p,\sigma'}$ such that $j(f_\phi)$ is a positive scalar multiple of $f_{\phi'}$. Suppose that

$$\overline{f_{\phi'}} = j(f_\phi) = (f_{\phi'})_Q \cap S_{p,\sigma'}^\vee/(f_{\phi'})_Q \cap S_{p,\sigma'} \subset A_{p,\sigma'}$$

is not zero. Let $m$ be the cardinality of the orbit of $[\overline{f_{\phi'}}] \in \mathbb{P}(A_{p,\sigma'})$ by the action of $O(S_{p,\sigma'})$ (or equivalently, in virtue of Remark 7.9, by the action of $\text{Aut}(\text{Nef}(X_{p,\sigma'})))$. By Corollary 7.10, the
number of $\text{Aut}$-equivalence classes of genus one fibrations contained in the lattice equivalence class $[\phi']$ is at least $m$, provided that $X_{p,\sigma}$ is generic.

**Remark 9.1.** We can regard $S_{p,\sigma'}$ as a submodule of $S_{p,\sigma} \otimes \mathbb{Q}$ by $j$. Then $S_{p,\sigma'}$ is equal to $(1/p)S_{p,\sigma}$. Hence $(1/p)j(\phi)$ is contained in $S_{p,\sigma'}$.

As a consequence of the fact that $\text{Aut}(\text{Nef}(X_{3,10}))$ contains the subgroup $\text{PGU}(4, \mathbb{F}_9)'$ of order 13063680, we obtain the following:

**Proposition 9.2.** Suppose that $X_{3,10}$ is generic. Then there exists a genus one fibration on $X_{3,10}$ whose lattice equivalence class contains at least 6531840 $\text{Aut}$-equivalence classes.

**Proof.** Let $(w, x, y)$ be the affine coordinates of the Fermat quartic surface

$$X_{\mathbb{F}_q} = \{ w^4 + x^4 + y^4 + 1 = 0 \}$$

in characteristic 3, and let $i$ denote $\sqrt{-1} \in \mathbb{F}_9$. Consider the following ten lines on $X_{\mathbb{F}_q} \cong X_{3,1}:

$$\ell_1 := \{ w + (1+i) = x + (1+i)y = 0 \}, \quad \ell_2 := \{ w + (1+i) = x + (1-i)y = 0 \},$$

$$\ell_3 := \{ w + iy - i = x + iy + i = 0 \}, \quad \ell_4 := \{ w + iy + 1 = x + iy - 1 = 0 \},$$

$$\ell_5 := \{ w - y + 1 = x - y - 1 = 0 \}, \quad \ell_6 := \{ w - iy - 1 = x + iy - i = 0 \},$$

$$\ell_7 := \{ w + (1-i) = x + (1+i)y = 0 \}, \quad \ell_8 := \{ w - (1-i) = x + (1+i) = 0 \},$$

$$\ell_9 := \{ w + (1+i) = x + (1-i) = 0 \}, \quad \ell_{10} := \{ w + iy - 1 = x - iy - 1 = 0 \}. $$

They form a configuration of $(-2)$-curves whose dual graph is the affine Dynkin diagram of type $\tilde{A}_9$. Then the class $f_\phi := \sum_{k=1}^{10} [\ell_k]$ defines a genus one fibration $\phi : X_{3,1} \to \mathbb{P}^1$ in the lattice equivalence class No. 20 of Table 4.2. The line defined by $\{ w + y + 1 = x + iy - i = 0 \}$ provides us with a section of $\phi$ that intersects $\ell_{10}$.

Let $\phi' : X_{3,10} \to \mathbb{P}^1$ be a genus one fibration corresponding to $\phi$ by Theorem 1.3. Since the Néron-Severi lattice of $X_{\mathbb{F}_q}$ is generated by the classes of lines, we can calculate the action of $\text{PGU}(4, \mathbb{F}_9)$ on $S_{3,1}$ from the permutations of lines induced by $\text{PGU}(4, \mathbb{F}_9)$, and thus we can calculate the action of $\text{PGU}(4, \mathbb{F}_9)'$ on $S_{3,10}$. By computer, we calculate the action of $\text{PGU}(3, \mathbb{F}_4)'$ on the vector space $A_{3,10} \cong \mathbb{F}_3^{20}$. It turns out that the stabilizer subgroup of the non-zero vector

$$(1/3)j(\phi) \text{ mod } S_{3,10} \in A_{3,10}$$

is trivial. Hence the orbit of $[\overline{\phi'}] \in \mathbb{P}(A_{3,10}) \cong \mathbb{P}^{19}(\mathbb{F}_3)$ by the action of $\text{PGU}(4, \mathbb{F}_9)'$ contains at least $|\text{PGU}(4, \mathbb{F}_9)|/|\mathbb{F}_3^n|$ points. □

**References**


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