$K3$ surfaces with ten cusps

Ichiro Shimada and De-Qi Zhang

Dedicated to Professor Igor V. Dolgachev for his 60th birthday

Abstract. We show that normal $K3$ surfaces with ten cusps exist in and only in characteristic 3. We determine these $K3$ surfaces according to the degrees of the polarizations. Explicit examples are given.

1. Introduction

We work over an algebraically closed field k. A $K3$ surface always means an algebraic $K3$ surface.

An isolated singular point of an algebraic surface is called a *cusp* if it is a rational double point of type A_2 (Artin [**1, 2, 4**]).

In characteristic 0, the number of cusps on a normal $K3$ surface is at most nine. Barth showed in [**5**] that a complex normal $K3$ surface Y has nine cusps as its only singularities if and only if Y is the quotient of an abelian surface by a cyclic group of order 3. This is a generalization of the result of [**14**], in which Nikulin showed that a complex normal $K3$ surface Y has sixteen nodes as its only singularities if and only if Y is the quotient of an abelian surface by the involution. In [**6**], Barth classified normal $K3$ surfaces with nine cusps according to the degrees of the polarizations.

In positive characteristics, however, there exist normal $K3$ surfaces Y such that the singular locus $\operatorname{Sing} Y$ of Y consists of *ten* cusps. The purpose of this paper is to investigate such $K3$ surfaces.

A smooth $K3$ surface X is called *supersingular* (in the sense of Shioda [**25**]) if the Néron-Severi lattice $NS(X)$ of X is of rank 22. Supersingular $K3$ surfaces exist only in positive characteristics. Let X be a supersingular $K3$ surface in characteristic $p > 0$. Artin [**3**] showed that there exists a positive integer $\sigma(X) \leq 10$ such that $\operatorname{disc} NS(X) = -p^{2\sigma(X)}$ holds. This integer $\sigma(X)$ is called the *Artin invariant of X*.

1991 *Mathematics Subject Classification.* Primary 14J28; Secondary 14G50.
Key words and phrases. $K3$ surface, supersingular.

We denote by $U(m)$ the lattice of rank 2 whose intersection matrix is equal to

$$\begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}.$$

Our main results are Theorems 1.1 and 1.4 - 1.6.

THEOREM 1.1. *Let Y be a normal $K3$ surface such that* $\operatorname{Sing} Y$ *consists of ten cusps, and $\rho : X \to Y$ the minimal resolution of Y. Let R_ρ be the sublattice of $NS(X)$ generated by the classes of the (-2)-curves that are contracted by ρ. Then the following hold:*

(1) *The characteristic of the ground field k is* 3.

(2) *The orthogonal complement $R_\rho^\perp$ of R_ρ in $NS(X)$ is isomorphic to either $U(1)$ or $U(3)$.*

(3) *If $R_\rho^\perp \cong U(1)$, then $\sigma(X) \le 5$, while if $R_\rho^\perp \cong U(3)$, then $\sigma(X) \le 6$.*

Before we state the other main results, we fix the terminology below.

DEFINITION 1.2. Let L be a line bundle on a smooth $K3$ surface X. We say that L is *very ample modulo* (-2)*-curves* if the following hold:

- (i) The complete linear system $|L|$ has no fixed components, and hence has no base points by [**18**, Corollary 3.2]. In particular, $|L|$ defines a morphism $\Phi_{|L|} : X \to \mathbb{P}^N$, where $N = L^2/2 + 1$.
- (ii) The morphism $\Phi_{|L|}$ is birational onto the image $Y_{(X,L)} := \Phi_{|L|}(X)$.

A *polarized $K3$ surface* is a pair (X, L) of a $K3$ surface X and a line bundle L on X that is very ample modulo (-2)-curves. The *degree* of a polarized $K3$ surface (X, L) is defined to be L^2.

DEFINITION 1.3. Let (X, L) be a polarized $K3$ surface. We denote by

$$\rho_L : X \to Y_{(X,L)}$$

the birational morphism induced by $|L|$. By [**18**, Theorem 6.1], ρ_L is a contraction of an ADE-configuration of (-2)-curves on X. We denote by $R_{(X,L)}$ the sublattice of $NS(X)$ generated by the classes of the (-2)-curves that are contracted by ρ_L. We also denote by $\mathcal{R}_{(X,L)}$ the ADE-type of the configuration of these (-2)-curves.

Note that $\mathcal{R}_{(X,L)} = 10A_2$ is equivalent to saying that $\operatorname{Sing} Y_{(X,L)}$ consists of ten cusps. The degree of (X, L) can be completely determined:

THEOREM 1.4. *The following conditions on a positive integer d are equivalent:*

- (i) *$d = 2ab$, where a and b are integers ≥ 3 such that $a \ne b$.*
- (ii) *There exists a polarized supersingular $K3$ surface (X, L) of degree d such that $\mathcal{R}_{(X,L)} = 10A_2$ and $R_{(X,L)}^\perp \cong U(1)$.*
- (iii) *Every supersingular $K3$ surface X in characteristic 3 with $\sigma(X) \le 5$ admits a line bundle L such that (X, L) is a polarized $K3$ surface of degree d satisfying $\mathcal{R}_{(X,L)} = 10A_2$ and $R_{(X,L)}^\perp \cong U(1)$.*

THEOREM 1.5. *The following conditions on a positive integer d are equivalent:*

- (i) $d \equiv 0 \bmod 6$.
- (ii) *There exists a polarized supersingular $K3$ surface (X, L) of degree d such that $\mathcal{R}_{(X,L)} = 10A_2$ and $R_{(X,L)}^\perp \cong U(3)$.*

(iii) *Every supersingular* $K3$ *surface* X *in characteristic* 3 *with* $\sigma(X) \le 6$ *admits a line bundle* L *such that* (X, L) *is a polarized* $K3$ *surface of degree* d *satisfying* $\mathcal{R}_{(X,L)} = 10A_2$ *and* $R^{\perp}_{(X,L)} \cong U(3)$.

Supersingular $K3$ surfaces with ten cusps can be obtained as purely inseparable triple covers of the smooth quadric surface $Q = \mathbb{P}^1 \times \mathbb{P}^1$. From now on to the end of this paragraph, we assume that k is of characteristic 3. For integers a and b, we denote by $\mathcal{O}_Q(a, b)$ the invertible sheaf $\mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(b)$ of $Q = \mathbb{P}^1 \times \mathbb{P}^1$, and by $L_Q(a, b) \to Q = \mathbb{P}^1 \times \mathbb{P}^1$ the corresponding line bundle. Because we are in characteristic 3, the differential map

$$d : H^0(Q, \mathcal{O}_Q(3, 3)) \to H^0(Q, \Omega^1_Q(3, 3))$$

is well-defined by the isomorphism $L_Q(3, 3) \cong L_Q(1, 1)^{\otimes 3}$. For $G \in H^0(Q, \mathcal{O}_Q(3, 3))$, we denote by $Z(dG)$ the subscheme of Q defined by $dG = 0$. If $\dim Z(dG) = 0$, then

$$\operatorname{length} \mathcal{O}_{Z(dG)} = c_2(\Omega^1_Q(3, 3)) = 10$$

holds, where c_2 is the second Chern class. We put

$$\mathcal{U}_{3,3} := \{\, G \in H^0(Q, \mathcal{O}_Q(3, 3)) \;\mid\; Z(dG) \text{ is reduced and of dimension } 0 \,\},$$

which is a Zariski open dense subset of $H^0(Q, \mathcal{O}_Q(3, 3))$. For a non-zero $G \in H^0(Q, \mathcal{O}_Q(3, 3))$, we denote by

$$\pi_G : Y_G \to Q = \mathbb{P}^1 \times \mathbb{P}^1$$

the purely inseparable triple cover of Q defined by

$$W^3 = G,$$

where W is a fiber coordinate of the line bundle $L_Q(1, 1)$. It is easy to see that G is contained in $\mathcal{U}_{3,3}$ if and only if Y_G is a normal $K3$ surface such that $\operatorname{Sing} Y_G = \pi_G^{-1}(Z(dG))$ consists of ten cusps. In particular, if $G \in \mathcal{U}_{3,3}$, then the minimal resolution X_G of Y_G is a supersingular $K3$ surface with Artin invariant ≤ 6 by Theorem 1.1. Conversely, we have the following:

THEOREM 1.6. *Let* X *be a supersingular* $K3$ *surface in characteristic* 3 *with Artin invariant* ≤ 6*. Then there exists* $G \in \mathcal{U}_{3,3}$ *such that* X *is isomorphic to* X_G.

We put

$$\mathcal{V}_{1,1} := \{\, H^3 \in H^0(Q, \mathcal{O}_Q(3, 3)) \;\mid\; H \in H^0(Q, \mathcal{O}_Q(1, 1)) \,\},$$

which is an additive group acting on $\mathcal{U}_{3,3}$ by $G \mapsto G + H^3$ ($G \in \mathcal{U}_{3,3}, H^3 \in \mathcal{V}_{1,1}$). For $G, G' \in \mathcal{U}_{3,3}$, the triple covers Y_G and $Y_{G'}$ are isomorphic over Q if and only if $G = cG' + H^3$ holds for some $c \in k^{\times}$ and $H^3 \in \mathcal{V}_{1,1}$. Hence the space

$$\mathfrak{M} := (PGL(2, k) \times PGL(2, k)) \backslash \mathbb{P}_*(\mathcal{U}_{3,3}/\mathcal{V}_{1,1})$$

is a moduli space of supersingular $K3$ surfaces in characteristic 3 with Artin invariant ≤ 6. We remark that, since $\dim \mathcal{U}_{3,3} = 16$ and $\dim \mathcal{V}_{1,1} = 4$, we have

$$\dim \mathfrak{M} = 16 - 4 - 1 - (3 + 3) = 5,$$

as is predicted from the result of Artin [**3**]. In particular, the *unique* supersingular $K3$ surface of Artin invariant 1 has the following precise model:

EXAMPLE 1.7. We put

$$G_0 := (x^3 - x)(y^3 - y),$$

where x and y are affine coordinates of the two factors of $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Then $Z(dG_0)$ is equal to

$$\{ (\alpha, \beta) \mid \alpha, \beta \in \mathbb{F}_3 \} \cup \{(\infty, \infty)\}.$$

Therefore $G_0 \in \mathcal{U}_{3,3}$. It can be shown that the Artin invariant of the supersingular $K3$ surface X_{G_0} is 1. See Example 7.8.

Supersingular $K3$ surfaces in characteristic 2 with 21 nodes are investigated in [**21, 22, 23**]. In particular, it was shown there that every supersingular $K3$ surface in characteristic 2 is birational to a purely inseparable double cover of the projective plane with 21 nodes; that is, every supersingular $K3$ surface in characteristic 2 is obtained as a generic Zariski surface [**7**].

Quasi-elliptic $K3$ surfaces in characteristic 3 with a section and ten singular fibers of type A_2^* are constructed explicitly in [**13**]. The Artin invariants of these supersingular $K3$ surfaces are ≤ 5.

A family of smooth quartic surfaces in characteristic 3 containing an ADE-configuration of lines of type $10A_2$ is constructed in [**20**]. A general member of the family is of Artin invariant 6. See Example 4.3 for details.

This paper is organized as follows. In §2, we review the theory of discriminant forms of lattices due to Nikulin [**15**]. In §3, we quote from Artin [**3**], Rudakov-Shafarevich [**17**], Saint-Donat [**18**] and Nikulin [**16**] some known facts about Néron-Severi lattices and polarizations of supersingular $K3$ surfaces. In §4, we prove Theorem 1.1 using the theory of discriminant forms. In §5 and §6, we prove Theorems 1.5 and 1.4. We reduce the problem of existence of the polarizations on a supersingular $K3$ surface to a problem of existence of ternary codes with certain properties, and solve the latter by computer. In §7, we prove Theorem 1.6. The proof presented here seems to be quite lattice-intensive. We think there should be a more elementary proof. See Question 7.7.

In our preprint [**24**], we determine all possible Dynkin types R of rational double points of total Milnor number 20 on supersingular $K3$ surfaces in characteristic prime to $2\,\mathrm{disc}(R)$.

Acknowledgment. This work was done during the second author's visit to Hokkaido University who likes to express his thanks for the very warm hospitality.

2. Discriminant forms of lattices

For a finite abelian group A and a prime integer p, we denote by

$$A = A_{(p)} \times A_{(p')}$$

the decomposition of A into the p-part $A_{(p)}$ and the p-prime-part $A_{(p')}$ of A.

A lattice is, by definition, a free $\mathbb{Z}$-module of finite rank with a non-degenerate symmetric $\mathbb{Z}$-valued bilinear form. A lattice Λ is said to be *even* if $v^2 \in 2\mathbb{Z}$ holds for every $v \in \Lambda$. Let Λ be an even lattice. We denote by $\Lambda^\vee$ the *dual lattice* $\mathrm{Hom}(\Lambda, \mathbb{Z})$. We have a natural embedding $\Lambda \hookrightarrow \Lambda^\vee$ of finite cokernel, and a symmetric bilinear form $\Lambda^\vee \times \Lambda^\vee \to \mathbb{Q}$ that extends the $\mathbb{Z}$-valued symmetric bilinear form on Λ. We put

$$\mathrm{Disc}(\Lambda) := \Lambda^\vee / \Lambda,$$

and call it the *discriminant group of* Λ. We then define the *discriminant form*

$$\begin{aligned} q_\Lambda &: \operatorname{Disc}(\Lambda) \to \mathbb{Q}/2\mathbb{Z} \quad \text{and} \\ b_\Lambda &: \operatorname{Disc}(\Lambda) \times \operatorname{Disc}(\Lambda) \to \mathbb{Q}/\mathbb{Z} \end{aligned}$$

by

$$\begin{aligned} q_\Lambda(\bar{v}) &:= v^2 \bmod 2\mathbb{Z} \quad \text{and} \\ b_\Lambda(\bar{v}, \bar{w}) &:= vw \bmod \mathbb{Z} = (q_\Lambda(\bar{v}+\bar{w}) - q_\Lambda(\bar{v}) - q_\Lambda(\bar{w}))/2, \end{aligned}$$

where $v, w \in \Lambda^\vee$, and $\bar{v} := v \bmod \Lambda$, $\bar{w} := w \bmod \Lambda$. Let p be a prime integer dividing $|\operatorname{Disc}(\Lambda)| = |\operatorname{disc}\Lambda|$. Then $\operatorname{Disc}(\Lambda)_{(p)}$ and $\operatorname{Disc}(\Lambda)_{(p')}$ are orthogonal with respect to b_Λ. We put

$$\begin{aligned} q_{\Lambda(p)} &:= q_\Lambda | \operatorname{Disc}(\Lambda)_{(p)}, & q_{\Lambda(p')} &:= q_\Lambda | \operatorname{Disc}(\Lambda)_{(p')}, \\ b_{\Lambda(p)} &:= b_\Lambda | \operatorname{Disc}(\Lambda)_{(p)} \times \operatorname{Disc}(\Lambda)_{(p)}, & b_{\Lambda(p')} &:= b_\Lambda | \operatorname{Disc}(\Lambda)_{(p')} \times \operatorname{Disc}(\Lambda)_{(p')}. \end{aligned}$$

For a subgroup H of $\operatorname{Disc}(\Lambda)$, we denote by $H^\perp$ the orthogonal complement of H with respect to b_Λ. Note that $(H^\perp)_{(p)}$ is canonically isomorphic to

$$(H_{(p)})^\perp := \{\, x \in \operatorname{Disc}(\Lambda)_{(p)} \mid b_{\Lambda(p)}(x, y) = 0 \text{ for any } y \in H_{(p)} \,\}.$$

We will use the notation $H^\perp_{(p)}$ to denote $(H^\perp)_{(p)} = (H_{(p)})^\perp$. A subgroup $H \subset \operatorname{Disc}(\Lambda)$ is called *isotropic* if $q_\Lambda|H$ is constantly equal to 0. If H is isotropic, then H is contained in $H^\perp$. Note that we have

$$(H^\perp/H)_{(p)} = H^\perp_{(p)}/H_{(p)}.$$

An *overlattice* of Λ is, by definition, a submodule Λ' of $\Lambda^\vee$ containing Λ such that the $\mathbb{Q}$-valued symmetric bilinear form on $\Lambda^\vee$ takes values in $\mathbb{Z}$ on Λ'.

Proposition 2.1 (Nikulin [**15**]). *Let* $\operatorname{pr}_\Lambda : \Lambda^\vee \to \operatorname{Disc}(\Lambda)$ *be the natural projection. The correspondence*

$$H \mapsto \Lambda_H := \operatorname{pr}_\Lambda^{-1}(H)$$

gives a bijection from the set of isotropic subgroups of $\operatorname{Disc}(\Lambda)$ *to the set of even overlattices of* Λ*. For an isotropic subgroup* H*, the discriminant group of* Λ_H *is isomorphic to* $H^\perp/H$*.*

Remark 2.2. If Λ is of rank r, then $\operatorname{Disc}(\Lambda)$ is generated by at most r elements.

A vector v in an even *negative-definite* lattice Λ is called a *root* if $v^2 = -2$. We denote by $\operatorname{Roots}(\Lambda)$ the set of roots in Λ. It is known that $\operatorname{Roots}(\Lambda)$ forms a root system of type ADE ([**9, 12**]). An even negative-definite lattice Λ is called a *root lattice* if it is generated by $\operatorname{Roots}(\Lambda)$.

Let $\mathbb{Z}[10A_2]$ denote the root lattice of type $10A_2$. Then $\mathbb{Z}[10A_2]$ is generated by roots c_i, d_i $(i = 1, \dots, 10)$ satisfying

$$c_i^2 = d_i^2 = -2, \quad c_i d_i = 1, \quad \text{and} \quad \langle c_i, d_i \rangle \perp \langle c_j, d_j \rangle \ \text{ if } i \neq j.$$

We have

$$\operatorname{Roots}(\mathbb{Z}[10A_2]) = \{\pm c_i, \pm d_i, \pm(c_i + d_i) \quad (i = 1, \dots, 10)\},$$

and

$$\mathbb{Z}[10A_2]^\vee = \{\, \sum_{i=1}^{10} (s_i c_i + t_i d_i)/3 \ \mid \ s_i, t_i \in \mathbb{Z},\ s_i + t_i \equiv 0 \bmod 3 \ \ (i = 1, \dots, 10) \,\}.$$

We put

$$\gamma_i := (c_i + 2d_i)/3 \mod \mathbb{Z}[10A_2] \in \mathrm{Disc}(\mathbb{Z}[10A_2]).$$

Then we have

$$\mathrm{Disc}(\mathbb{Z}[10A_2]) = \mathbb{F}_3\gamma_1 \oplus \cdots \oplus \mathbb{F}_3\gamma_{10},$$

and

$$q_{\mathbb{Z}[10A_2]}(x_1\gamma_1 + \cdots + x_{10}\gamma_{10}) = -2(x_1^2 + \cdots + x_{10}^2)/3 \in \mathbb{Q}/2\mathbb{Z}. \tag{2.1}$$

For a vector

$$\mathbf{x} = (x_1, \ldots, x_{10}) = x_1\gamma_1 + \cdots + x_{10}\gamma_{10} \in \mathrm{Disc}(\mathbb{Z}[10A_2]) \cong \mathbb{F}_3^{10},$$

we define the *Hamming weight* $\mathrm{wt}(\mathbf{x})$ of $\mathbf{x}$ by

$$\mathrm{wt}(\mathbf{x}) := |\{\, i \mid x_i \neq 0 \,\}| \in \mathbb{Z}_{\geq 0}.$$

Then, for a vector $r \in \mathbb{Z}[10A_2]^\vee$, we have

$$r^2 \leq -(2/3)\,\mathrm{wt}(\bar{r}), \quad \text{where } \bar{r} := r \bmod \mathbb{Z}[10A_2] \in \mathrm{Disc}(\mathbb{Z}[10A_2]). \tag{2.2}$$

Moreover,

(2.3) for a vector $\mathbf{x} \in \mathrm{Disc}(\mathbb{Z}[10A_2])$, there exists a vector $r \in \mathbb{Z}[10A_2]^\vee$ such that $\bar{r} = \mathbf{x}$ and $r^2 = (-2/3)\,\mathrm{wt}(\mathbf{x})$ hold.

Let e and f be basis of the lattice $U(m)$ satisfying

$$e^2 = f^2 = 0, \qquad ef = m.$$

We put $e^\vee := f/m$ and $f^\vee := e/m$. Then $\mathrm{Disc}(U(m)) \cong (\mathbb{Z}/m\mathbb{Z})^2$ is generated by

$$\bar{e}^\vee := e^\vee \bmod U(m) \quad \text{and} \quad \bar{f}^\vee := f^\vee \bmod U(m),$$

and the discriminant form is given by

$$q_{U(m)}(y_1\bar{e}^\vee + y_2\bar{f}^\vee) = 2y_1y_2/m \in \mathbb{Q}/2\mathbb{Z}. \tag{2.4}$$

3. Néron-Severi lattices of supersingular $K3$ surfaces

A lattice Λ is called *hyperbolic* if the signature of Λ is $(1, \mathrm{rank}\,\Lambda - 1)$. Let p be a prime integer. A lattice Λ is called *p-elementary* if $\mathrm{Disc}(\Lambda)$ is a p-elementary abelian group; that is, $p\Lambda^\vee \subseteq \Lambda$ holds. An overlattice of a hyperbolic p-elementary lattice is again hyperbolic and p-elementary.

The following is due to Artin [**3**] and Rudakov-Shafarevich [**17**].

Theorem 3.1. *Let X be a supersingular $K3$ surface in characteristic $p > 0$. Then $NS(X)$ is an even hyperbolic p-elementary lattice.*

The following is due to Rudakov-Shafarevich [**17**, Section 1].

Theorem 3.2. *Suppose that p is odd. Let σ be a positive integer ≤ 10. Then the lattice N with the following properties is unique up to isomorphisms:*
(i) *N is even, hyperbolic of rank 22, and*
(ii) $\mathrm{Disc}(N) \cong \mathbb{F}_p^{2\sigma}$.

From now on to the end of this section, we assume that p is *odd*. We denote the lattice N in Theorem 3.2 by $N_{p,\sigma}$. Let X be a supersingular $K3$ surface in characteristic p with $\sigma(X) = \sigma$. By Theorems 3.1 and 3.2, there exists an isometry

$$\phi : N_{p,\sigma} \xrightarrow{\sim} NS(X).$$

More precisely, we have the following:

PROPOSITION 3.3. *Let h be a vector of $N_{p,\sigma}$ such that $h^2 \geq 4$, and let X be a supersingular $K3$ surface in characteristic p with $\sigma(X) = \sigma$.*

(1) *The following conditions are equivalent:*

(i) *There exist no vectors $u \in N_{p,\sigma}$ satisfying $hu = 1$ or 2 and $u^2 = 0$, and there exist no vectors $b \in N_{p,\sigma}$ satisfying $h = 2b$ and $b^2 = 2$.*

(ii) *There exists an isometry $\phi : N_{p,\sigma} \xrightarrow{\sim} NS(X)$ such that $\phi(h)$ is the class $[L]$ of a line bundle L that is very ample modulo (-2)-curves.*

(2) *Suppose that the conditions in (1) are fulfilled, and let L be a line bundle very ample modulo (-2)-curves such that $\phi(h) = [L]$ by some isometry ϕ. Then $Y_{(X,L)}$ has only rational double points as its singularities, and the ADE-type $\mathcal{R}_{(X,L)}$ of $\operatorname{Sing} Y_{(X,L)}$ is equal to that of the root system*

$$\operatorname{Roots}(h^{\perp}) := \{\ r \in N_{p,\sigma} \ \mid \ rh = 0,\ r^2 = -2\ \}.$$

For the proof, we use the following results due to Nikulin [**16**, Proposition 0.1] and Saint-Donat [**18**, Section 5].

PROPOSITION 3.4 (Nikulin [**16**]). *Let L be a nef line bundle on a $K3$ surface X with $L^2 > 0$. If $|L|$ has a fixed component, then $|L|$ is equal to $m|U| + \Gamma$, where Γ is the fixed part of $|L|$, $|U|$ is a (quasi-)elliptic pencil, and $U^2 = 0$, $U\Gamma = 1$, $\Gamma^2 = -2$, $m = \dim |L| = L^2/2 + 1$ hold. If $|L|$ has no fixed components, then a general member of $|L|$ is irreducible and $\dim |L| = L^2/2 + 1$.*

PROPOSITION 3.5 (Saint-Donat [**18**]). *Let $|L|$ be a complete linear system without fixed components on a $K3$ surface X such that $L^2 \geq 4$. Then the morphism $\Phi_{|L|}$ fails to be birational onto its image if and only if one of the following holds:*

(i) *There exists an irreducible curve U such that $U^2 = 0$ and $UL = 2$.*

(ii) *There exists an irreducible curve B such that $B^2 = 2$ and $L = \mathcal{O}_X(2B)$.*

PROOF OF PROPOSITION 3.3. The assertion (2) follows from [**18**, Theorem 6.1] and [**21**, Lemma 2.4]. We now prove (1).

Suppose that the condition (i) in (1) holds. By [**17**, Section 3, Proposition 3], there exists an isometry $\phi : N_{p,\sigma} \xrightarrow{\sim} NS(X)$ such that $\phi(h)$ is the class of a *nef* line bundle L. By Proposition 3.4, $|L|$ is fixed component free. By Proposition 3.5, $\Phi_{|L|}$ is birational onto its image. So (ii) is true.

Conversely, suppose that (ii) holds. We assume that there exists a vector $u \in N_{p,\sigma}$ satisfying $hu = 1$ or 2 and $u^2 = 0$, and derive a contradiction by the argument in [**27**, Proof of Proposition 1.7]. By the Riemann-Roch theorem, $\phi(u)$ is the class $[U]$ of an effective divisor U such that $\dim |U| \geq 1$. Let $D + \Delta$ be a general member of $|U|$, where Δ is the fixed part of $|U|$. We have $D \neq 0$ and $D^2 \geq 0$. If $DL = 0$, then $D^2 < 0$ would follow by Hodge index theorem, a contradiction. Since L is nef, $\Delta L \geq 0$. Therefore, we have $DL = 1$ or 2. Then the image of D by $\Phi_{|L|}$ is either a line or a plane conic. In any case, we have $\dim |D| = 0$, which is a contradiction.

Next we assume that there exists a vector $b \in N_{p,\sigma}$ such that $h = 2b$ and $b^2 = 2$. Let B be an effective divisor such that $\phi(b) = [B]$. Since $[B] = [L]/2$, B is nef. If there exists an irreducible member in $|B|$, then Proposition 3.5 implies that $\Phi_{|L|}$ is not birational onto its image. If there exist no irreducible members in $|B|$, then Proposition 3.4 implies that $|B|$ has a fixed component, and $|B|$ is written as $2|U| + \Gamma$, where $UB = 1$ and $U^2 = 0$. Then $UL = 2$ follows. Hence $\Phi_{|L|}$ is not birational onto its image, and we get a contradiction. So (i) is true. Thus the assertion (1) is proved. □

Remark 3.6. If there exists a vector b such that $h = 2b$ and $b^2 = 2$, then h is of degree 8.

4. Proof of Theorem 1.1

Theorem 1.1 follows from the structure theorem of Néron-Severi lattices of supersingular $K3$ surfaces (Theorems 3.1 and 3.2), and a purely lattice-theoretic Lemma 4.1 below. A sublattice $\Lambda' \subset \Lambda$ is called *primitive in* Λ if $(\Lambda' \otimes \mathbb{Q}) \cap \Lambda = \Lambda'$ holds.

Lemma 4.1. *Let N be an even hyperbolic p-elementary lattice of rank 22 such that* $\mathrm{Disc}(N)$ *is isomorphic to $\mathbb{F}_p^{2\sigma}$, where σ is a positive integer. Suppose that N contains a sublattice R isomorphic to $\mathbb{Z}[10A_2]$. Then $p = 3$, and the orthogonal complement $R^\perp$ of R in N is isomorphic to $U(1)$ or $U(3)$. If $R^\perp \cong U(1)$, then $\sigma \le 5$, while if $R^\perp \cong U(3)$, then $\sigma \le 6$.*

Proof. We put $S := R^\perp$, which is an even hyperbolic lattice of rank 2 primitive in N. Then N is an overlattice of the orthogonal direct sum $R \oplus S$. We put

$$H := N/(R \oplus S).$$

Clearly, we may assume that $H \neq (0)$.

Note that H is an isotropic subgroup of $\mathrm{Disc}(R \oplus S) = \mathrm{Disc}(R) \oplus \mathrm{Disc}(S)$ with respect to $q_{R\oplus S} = q_R \oplus q_S$, and $\mathrm{Disc}(N) \cong H^\perp/H$ is a p-elementary abelian group. Since S is primitive in N, we have

$$H \cap (0 \oplus \mathrm{Disc}(S)) = 0. \tag{4.1}$$

Let l be a prime integer different from 3 and p. Assume that $\mathrm{Disc}(S)_{(l)}$ is not 0. Since $\mathrm{Disc}(N)_{(l)} = 0$, we see that $H_{(l)}$ is not 0. Since $\mathrm{Disc}(\mathbb{Z}[10A_2])_{(l)} = 0$, we have $H_{(l)} \subset (0 \oplus \mathrm{Disc}(S)_{(l)})$, which contradicts (4.1). Hence we obtain

$$\mathrm{Disc}(S)_{(l)} = 0 \quad \text{for any prime } l \text{ distinct from 3 and } p. \tag{4.2}$$

Let $m_3 : \mathrm{Disc}(S)_{(3)} \to \mathrm{Disc}(S)_{(3)}$ be the homomorphism given by $m_3(x) := 3x$. Since every element of $\mathrm{Disc}(R)$ is annihilated by multiplication by 3, the image $H^S_{(3)} \subset \mathrm{Disc}(S)_{(3)}$ of $H_{(3)} \subset \mathrm{Disc}(R)_{(3)} \oplus \mathrm{Disc}(S)_{(3)}$ by the projection to the factor $\mathrm{Disc}(S)_{(3)}$ is contained in $\mathrm{Ker}\, m_3$ by (4.1):

$$H^S_{(3)} \subseteq \mathrm{Ker}\, m_3. \tag{4.3}$$

Therefore, $\mathrm{Im}\, m_3$ is contained in the orthogonal complement of $H^S_{(3)}$ with respect to $q_{S(3)}$. Hence we obtain

$$0 \oplus \mathrm{Im}\, m_3 \subset H^\perp_{(3)}. \tag{4.4}$$

We assume $p \neq 3$, and derive a contradiction. By (4.2), we have

$$\mathrm{Disc}(S) = \mathrm{Disc}(S)_{(3)} \times \mathrm{Disc}(S)_{(p)}. \tag{4.5}$$

Since $\mathrm{Disc}(R)_{(p)} = 0$, the property (4.1) implies $H_{(p)} = 0$. Therefore $\mathrm{Disc}(N) = \mathrm{Disc}(N)_{(p)}$ is isomorphic to $\mathrm{Disc}(S)_{(p)}$. Since $\dim_{\mathbb{F}_p} \mathrm{Disc}(N) = 2\sigma$ is positive and even, and S is of rank 2, we obtain

$$\mathrm{Disc}(S)_{(p)} \cong \mathbb{F}_p^2. \tag{4.6}$$

On the other hand, from $\mathrm{Disc}(N)_{(3)} = 0$, we obtain

$$H_{(3)} = H_{(3)}^{\perp}. \tag{4.7}$$

By (4.1), (4.4) and (4.7), we obtain $\mathrm{Im}\, m_3 = 0$; that is, $\mathrm{Disc}(S)_{(3)}$ is 3-elementary. From (4.7), we have $10 + \dim_{\mathbb{F}_3} \mathrm{Disc}(S)_{(3)} = 2 \dim_{\mathbb{F}_3} H_{(3)}$, and hence $\dim_{\mathbb{F}_3} \mathrm{Disc}(S)_{(3)}$ is even. Since S is of rank 2, we obtain

$$\mathrm{Disc}(S)_{(3)} \cong 0 \ \text{ or } \ \mathbb{F}_3^2. \tag{4.8}$$

Suppose that $\mathrm{Disc}(S)_{(3)} \cong 0$. Then $H_{(3)}$ can be regarded as an isotropic subgroup of $\mathrm{Disc}(R)$ with respect to q_R. Because $H_{(3)} = H_{(3)}^{\perp}$, the corresponding overlattice of R would be an even unimodular negative-definite lattice of rank 20. This contradicts the classification of unimodular lattices ([**19**, Chapter V]).

Suppose that $\mathrm{Disc}(S)_{(3)} \cong \mathbb{F}_3^2$. By (4.5) and (4.6), S is an even indefinite lattice of rank 2 such that $\mathrm{Disc}(S) \cong (\mathbb{Z}/3p\mathbb{Z})^2$. By the classification of indefinite lattices of rank 2 ([**10**, Chapter 15, Section 3]), we see that the intersection matrix of S with respect to an appropriate basis is

$$\begin{pmatrix} 0 & 3p \\ 3p & 0 \end{pmatrix}, \qquad \text{or} \qquad p = 2 \ \text{ and } \ \begin{pmatrix} 6 & 6 \\ 6 & 0 \end{pmatrix}.$$

In any case, the quadratic form $(\mathrm{Disc}(S)_{(3)}, q_{S(3)})$ is isomorphic to

$$\left(\mathbb{F}_3^2, \begin{bmatrix} 0 & 1/3 \\ 1/3 & 0 \end{bmatrix} \right) \ \cong \ (\mathrm{Disc}(U(3)), q_{U(3)}).$$

Therefore the isotropic subgroup $H_{(3)}$ of $\mathrm{Disc}(R) \oplus \mathrm{Disc}(S)_{(3)}$ satisfying $H_{(3)} = H_{(3)}^{\perp}$ would yield an even hyperbolic unimodular lattice of rank 22 as an overlattice of $R \oplus U(3)$, which again contradicts the classification of unimodular lattices.

Therefore $p = 3$ is proved.

By (4.2), we have $\mathrm{Disc}(S) = \mathrm{Disc}(S)_{(3)}$, and hence $H = H_{(3)}$ holds. Suppose that $(\xi, \eta) \in H^{\perp}$, where $\xi \in \mathrm{Disc}(R)$ and $\eta \in \mathrm{Disc}(S)$. Since $H^{\perp}/H$ is 3-elementary, we have $(3\xi, 3\eta) = (0, 3\eta) \in H$. By (4.1), we have $3\eta = 0$. Therefore the image $(H^{\perp})^S \subset \mathrm{Disc}(S)$ of $H^{\perp} \subset \mathrm{Disc}(R) \oplus \mathrm{Disc}(S)$ by the projection to the factor $\mathrm{Disc}(S)$ is contained in $\mathrm{Ker}\, m_3$:

$$(H^{\perp})^S \ \subset \ \mathrm{Ker}\, m_3. \tag{4.9}$$

Next we will show that S is isomorphic to $U(1)$ or $U(3)$. Since $H^{\perp}/H$ is 3-elementary, (4.1) and (4.4) implies that $m_3(\mathrm{Im}\, m_3) = 0$; that is, $9x = 0$ for any $x \in \mathrm{Disc}(S)$. Since

$$2\sigma = \dim_{\mathbb{F}_3}(H^{\perp}/H) = 10 + \log_3 |\mathrm{Disc}(S)| - 2 \log_3 |H|$$

is even and S is of rank 2, $\mathrm{Disc}(S)$ is isomorphic to 0, $\mathbb{F}_3^2$, $\mathbb{Z}/9\mathbb{Z}$ or $(\mathbb{Z}/9\mathbb{Z})^2$.

We first assume that $\mathrm{Disc}(S)$ is a cyclic group of order 9, and derive a contradiction. Let γ be a generator of $\mathrm{Disc}(S)$. We have $\mathrm{Im}\, m_3 = \mathrm{Ker}\, m_3 = \langle 3\gamma\rangle$. Let $H^R \subset \mathrm{Disc}(R)$ and $H^S \subset \mathrm{Disc}(S)$ be the images of $H \subset \mathrm{Disc}(R) \oplus \mathrm{Disc}(S)$ by the projections to the factors $\mathrm{Disc}(R)$ and $\mathrm{Disc}(S)$, respectively.

Claim 4.2. We have

$$H^\perp = (H^R)^\perp \oplus (H^S)^\perp,$$

where $(H^R)^\perp \subset \mathrm{Disc}(R)$ and $(H^S)^\perp \subset \mathrm{Disc}(S)$ are the orthogonal complements of H^R and H^S with respect to q_R and q_S, respectively. In particular, we have $(H^S)^\perp = (H^\perp)^S$.

Proof. It is obvious that $H^\perp$ contains $(H^R)^\perp \oplus (H^S)^\perp$. Suppose that $(\xi, \eta) \in H^\perp$, where $\xi \in \mathrm{Disc}(R)$ and $\eta \in \mathrm{Disc}(S)$. By (4.9), we have $\eta \in \mathrm{Ker}\, m_3 = \mathrm{Im}\, m_3$. By (4.4), we have $(0, \eta) \in H^\perp$ and hence $(\xi, 0) \in H^\perp$ hold. Because $(\xi, 0) \in (H^R)^\perp$ and $(0, \eta) \in (H^S)^\perp$, Claim 4.2 is proved. □

Because $H^\perp/H$ is 3-elementary, we have $(0, \gamma) \notin H^\perp$ by (4.1). Hence we obtain

$$(H^S)^\perp = (H^\perp)^S \neq \mathrm{Disc}(S). \tag{4.10}$$

By (4.3), H^S is either 0 or $\mathrm{Ker}\, m_3$. If $H^S = 0$, then $(H^S)^\perp = \mathrm{Disc}(S)$ and we get a contradiction to (4.10). Suppose that $H^S = \mathrm{Ker}\, m_3$. Then $(H^S)^\perp \supset \mathrm{Im}\, m_3$, and hence $(H^S)^\perp = \mathrm{Im}\, m_3$ by (4.10). In particular, we have

$$\log_3 |(H^S)^\perp| = 1. \tag{4.11}$$

Since $|(H^R)^\perp| = 3^{10}/|H^R|$ and $H \cong H^R$ by (4.1), we see that

$$2\sigma = \log_3 |H^\perp/H| = \log_3 |(H^R)^\perp| + \log_3 |(H^S)^\perp| - \log_3 |H| = 10 - 2\log_3 |H| + 1$$

is odd by (4.11), which is absurd. Therefore $\mathrm{Disc}(S) \not\cong \mathbb{Z}/9\mathbb{Z}$.

Because S is an even lattice, the classification of indefinite lattices of rank 2 ([**10**, Chapter 15, Section 3]) implies the following:

$$\begin{aligned} \mathrm{Disc}(S) = 0 \quad &\Longrightarrow \quad S \cong U(1), \\ \mathrm{Disc}(S) \cong \mathbb{F}_3^2 \quad &\Longrightarrow \quad S \cong U(3), \\ \mathrm{Disc}(S) \cong (\mathbb{Z}/9\mathbb{Z})^2 \quad &\Longrightarrow \quad S \cong U(9). \end{aligned}$$

Next we assume $S \cong U(9)$, and derive a contradiction. Note that $\mathrm{Ker}\, m_3$ is generated by

$$3\bar{e}^\vee = f/3 \mod S \quad \text{and} \quad 3\bar{f}^\vee = e/3 \mod S.$$

By (4.9), we have

$$H^\perp \quad \subset \quad \mathrm{Disc}(R) \oplus \mathrm{Ker}\, m_3. \tag{4.12}$$

Then H is also contained in $\mathrm{Disc}(R) \oplus \mathrm{Ker}\, m_3$. Suppose that H is generated by

$$g^{(\nu)} = \xi_1^{(\nu)}\gamma_1 + \cdots + \xi_{10}^{(\nu)}\gamma_{10} + \eta_1^{(\nu)}(3\bar{e}^\vee) + \eta_2^{(\nu)}(3\bar{f}^\vee) \qquad (\nu = 1, \ldots, r)$$

where $\xi_i^{(\nu)}, \eta_j^{(\nu)} \in \mathbb{F}_3$. We put

$$M := \left[\begin{array}{cccc|cc} \xi_1^{(1)} & \dots & \dots & \xi_{10}^{(1)} & \eta_2^{(1)} & \eta_1^{(1)} \\ & \dots & \dots & & & \\ & \dots & \dots & & & \\ \xi_1^{(r)} & \dots & \dots & \xi_{10}^{(r)} & \eta_2^{(r)} & \eta_1^{(r)} \end{array}\right].$$

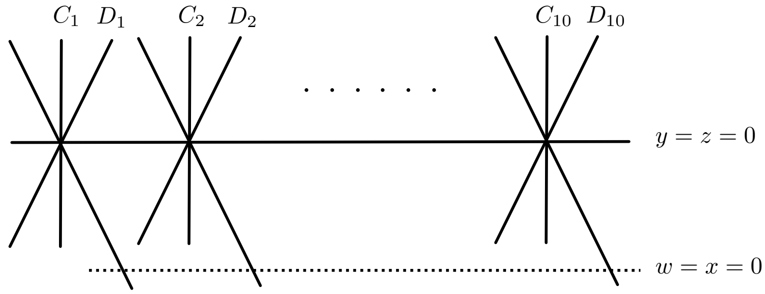

FIGURE 4.1. Lines on the quartic surface

From (2.1) and (2.4), an element

$$x_1\gamma_1 + \cdots + x_{10}\gamma_{10} + y_1\bar{e}^\vee + y_2\bar{f}^\vee \qquad (x_1, \ldots, x_{10} \in \mathbb{F}_3, \ \ y_1, y_2 \in \mathbb{Z}/9\mathbb{Z})$$

of $\mathrm{Disc}(R) \oplus \mathrm{Disc}(S)$ is contained in $H^\perp$ if and only if the vector $\mathbf{x} := [x_1, \ldots, x_{10}, y_1, y_2]$ satisfies the equation

$$M \cdot {}^T\mathbf{x} \equiv \mathbf{0} \mod 3. \tag{4.13}$$

We consider (4.13) as a system of linear equations over $\mathbb{F}_3$. The property (4.12) of $H^\perp$ implies that every solution of (4.13) in $\mathbb{F}_3$ must satisfy

$$y_1 = y_2 = 0. \tag{4.14}$$

Because of (4.1) and hence $H \cong H^R$, we can choose generators $g^{(1)}, \ldots, g^{(r)}$ of H in such a way that, after suitable permutations of 10 coordinates of $\mathrm{Disc}(R) = \mathbb{F}_3^{10}$ if necessary, the $r \times 12$ matrix M is of the form

$$M = \left[\begin{array}{c|c} \quad I_r \quad & * \end{array} \right],$$

where I_r ($r \leq 10$) is a diagonal matrix whose diagonal entries are 1. Now non-zero elements of the subgroup of $H \cong H^R$ of $H^\perp$ should be solutions of (4.13) in $\mathbb{F}_3$, but do not satisfy (4.14). Thus we get a contradiction.

Hence S is isomorphic to $U(1)$ or $U(3)$. If $S \cong U(1)$, then $2\sigma = 10 - 2\dim_{\mathbb{F}_3} H \leq 10$, while if $S \cong U(3)$, then $2\sigma = 10 + 2 - 2\dim_{\mathbb{F}_3} H \leq 12$. $\square$

EXAMPLE 4.3. Let $[w : x : y : z]$ be homogeneous coordinates of $\mathbb{P}^3$. For homogeneous polynomials $f(y, z)$, $g(y, z)$ and $h(y, z)$ of degrees 3, 3 and 4, we consider the quartic surface X defined in $\mathbb{P}^3$ by

$$w^3y + x^3z + wf(y, z) + xg(y, z) + h(y, z) = 0.$$

When X is smooth, X is a supersingular $K3$ surface, because X contains a configuration of lines drawn by thick lines in Figure 4.1. It was shown in [**20**, Section 6] that, when f, g and h are general, the Artin invariant of X is 6, and hence the orthogonal complement $R^\perp$ of the sublattice $R \subset NS(X)$ generated by the classes of the lines $C_1, D_1, \ldots, C_{10}, D_{10}$ is isomorphic to $U(3)$. When f, g are general and $h = 0$, the Artin invariant of X is 5 by [**26**, Section 4]. In this case, the line ℓ defined by $w = x = 0$ is contained in X. Since $\ell^2 = -2$ and $[\ell] \in R^\perp$, $R^\perp$ is isomorphic to $U(1)$.

5. Proof of Theorem 1.5

The discriminant group D of $\mathbb{Z}[10A_2] \oplus U(3)$ is equal to

$$\mathbb{F}_3\gamma_1 \oplus \cdots \oplus \mathbb{F}_3\gamma_{10} \oplus \mathbb{F}_3\bar{e}^\vee \oplus \mathbb{F}_3\bar{f}^\vee,$$

and the discriminant form q of $\mathbb{Z}[10A_2] \oplus U(3)$ is given by

$$q(x_1, \ldots, x_{10}, y_1, y_2) = -2(x_1^2 + \cdots + x_{10}^2)/3 + 2y_1y_2/3 \quad \in \quad \mathbb{Q}/2\mathbb{Z}.$$

We consider subgroups of D as ternary codes. Recall from §2 that the Hamming weight of a word $\mathbf{x} = (x_1, \ldots, x_{10}) \in \mathrm{Disc}(\mathbb{Z}[10A_2])$ is defined by

$$\mathrm{wt}(\mathbf{x}) := |\{\ i \ \mid \ x_i \neq 0\ \}|.$$

Then a ternary code $\mathcal{C} \subset D$ is isotropic with respect to q if and only if

$$\mathrm{wt}(\mathbf{x}) \equiv y_1y_2 \mod 3 \ \text{ for any } \ (\mathbf{x}, y_1, y_2) \in \mathcal{C} \tag{5.1}$$

holds. A ternary code $\mathcal{C} \subset D$ satisfying (5.1) is therefore called an *isotropic code.* For an isotropic code $\mathcal{C}$, we denote by $N_\mathcal{C}$ the overlattice of $\mathbb{Z}[10A_2] \oplus U(3)$ corresponding to $\mathcal{C}$ by Proposition 2.1. By Theorem 3.2, $N_\mathcal{C}$ is isomorphic to the lattice $N_{3,\sigma}$, where $\sigma = 6 - \dim \mathcal{C}$.

It is easy to see that the following conditions for an isotropic code $\mathcal{C}$ are equivalent:

(i) $\mathrm{wt}(\mathbf{x}) > 0$ for any non-zero word $(\mathbf{x}, y_1, y_2) \in \mathcal{C}$,
(ii) $U(3)$ is primitive in $N_\mathcal{C}$, and
(iii) $\mathbb{Z}[10A_2]^\perp = U(3)$ in $N_\mathcal{C}$.

We say that an isotropic code $\mathcal{C}$ is *admissible* if $\mathcal{C}$ satisfies the conditions above. Let $h = ae + bf$ be a vector of $U(3)$ with $a \geq 1$ and $b \geq 1$. We have $h^2 = 6ab$.

Lemma 5.1. *Let $\mathcal{C}$ be an admissible isotropic code.*

(1) *There exists a vector $u \in N_\mathcal{C}$ satisfying $hu = 1$ or 2 and $u^2 = 0$ if and only if the following hold:*

(α) *$a = b = 1$, and*
(β) *there exists $(\mathbf{x}, y_1, y_2) \in \mathcal{C}$ such that $\mathrm{wt}(\mathbf{x}) = 1$.*

(2) *The set of roots* $\mathrm{Roots}(h^\perp) := \{r \in N_\mathcal{C} \mid rh = 0,\ r^2 = -2\}$ *in $h^\perp$ is strictly larger than* $\mathrm{Roots}(\mathbb{Z}[10A_2]) = \{\pm c_i, \pm d_i, \pm(c_i + d_i)\}$ *if and only if one of the following holds:*

(a) *there exists $(\mathbf{x}, y_1, y_2) \in \mathcal{C}$ such that $\mathrm{wt}(\mathbf{x}) = 3$ and $y_1 = y_2 = 0$, or*
(b) *$a = b$, and there exists $(\mathbf{x}, y_1, y_2) \in \mathcal{C}$ such that $\mathrm{wt}(\mathbf{x}) = 2$, or*
(c) *$(a = 2b$ or $b = 2a)$ and there exists $(\mathbf{x}, y_1, y_2) \in \mathcal{C}$ such that $\mathrm{wt}(\mathbf{x}) = 1$.*

Proof. We prove (1) first. Suppose that a vector

$$u = r_u + \eta_1 e^\vee + \eta_2 f^\vee \quad (r_u \in \mathbb{Z}[10A_2]^\vee,\ \ \eta_1, \eta_2 \in \mathbb{Z}) \tag{5.2}$$

of $N_\mathcal{C}$ satisfies $hu = 1$ or 2 and $u^2 = 0$. Then we have

$$a\eta_1 + b\eta_2 = 1 \text{ or } 2, \tag{5.3}$$

$$r_u^2 + 2\eta_1\eta_2/3 = 0. \tag{5.4}$$

Note that $(\eta_1, \eta_2) \not\equiv (0, 0) \bmod 3$ by (5.3). Since $\mathcal{C}$ is admissible, we have $r_u \neq 0$, and hence $\eta_1\eta_2 > 0$ by (5.4). From (5.3), we obtain

$$a = b = 1, \qquad \eta_1 = \eta_2 = 1,$$

and hence, from (5.4), we have

$$r_u^2 = -2/3.$$

By (2.2), the word

$$\bar{u} = u \mod (\mathbb{Z}[10A_2] \oplus U(3)) = (\bar{r}_u, \bar{\eta}_1, \bar{\eta}_2) \qquad (\text{where } \bar{r}_u = r_u \bmod \mathbb{Z}[10A_2])$$

of $\mathcal{C}$ has the property $\mathrm{wt}(\bar{r}_u) = 1$.

Conversely, suppose that $a = b = 1$ and that there exists a word $(\bar{r}, y_1, y_2) \in \mathcal{C}$ such that $\mathrm{wt}(\bar{r}) = 1$. Replacing $(\bar{r}, y_1, y_2)$ by $(-\bar{r}, -y_1, -y_2)$ if necessary, we can assume that $y_1 = y_2 = 1$ by (5.1). Then, by (2.3), there exists a vector

$$u = r + e^\vee + f^\vee \qquad (r \in \mathbb{Z}[10A_2]^\vee)$$

in $N_\mathcal{C}$ satisfying $r^2 = -2/3$. This vector u satisfies $hu = 2$ and $u^2 = 0$. Thus the assertion (1) is proved.

We now prove (2). Suppose that a vector $u \in N_\mathcal{C}$ given by (5.2) satisfies $hu = 0$, $u^2 = -2$ and $u \notin \mathrm{Roots}(\mathbb{Z}[10A_2])$. Then we have

$$a\eta_1 + b\eta_2 = 0, \tag{5.5}$$

$$r_u^2 + 2\eta_1\eta_2/3 = -2. \tag{5.6}$$

Suppose that $\eta_1 = 0$ or $\eta_2 = 0$. Then (5.5) implies $\eta_1 = \eta_2 = 0$ and hence $\mathrm{wt}(\bar{r}_u) \equiv 0 \bmod 3$ holds because $\mathcal{C}$ is isotropic. By (2.2) and (5.6), we have $\mathrm{wt}(\bar{r}_u) \le 3$. If $\mathrm{wt}(\bar{r}_u) = 0$, then $u = r_u$ is contained in $\mathrm{Roots}(\mathbb{Z}[10A_2])$. Hence we have $\mathrm{wt}(\bar{r}_u) = 3$, and therefore the condition (a) is satisfied. Suppose that $\eta_1 \neq 0$ and $\eta_2 \neq 0$. By (5.5), we have $\eta_1\eta_2 < 0$. By (2.2) and (5.6), we see that the pair $(\eta_1\eta_2, \mathrm{wt}(\bar{r}_u))$ is either $(-1, 2)$ or $(-2, 1)$. In the former case, we have $a = b$ by (5.5) and hence (b) is satisfied. In the latter case, we have $a = 2b$ or $b = 2a$ by (5.5) and hence (c) is satisfied.

Conversely, suppose that (a) is fulfilled. Using (2.3), we have a lift

$$u = r + 0 + 0 \in N_\mathcal{C} \qquad (r \in \mathbb{Z}[10A_2]^\vee)$$

of the word $(\bar{r}, 0, 0) \in \mathcal{C}$ with $\mathrm{wt}(\bar{r}) = 3$ such that $r^2 = -2$. Then $u \in \mathrm{Roots}(h^\perp) \setminus \mathrm{Roots}(\mathbb{Z}[10A_2])$. Suppose that (b) is satisfied. A vector

$$u = r + e^\vee - f^\vee \quad \in \quad N_\mathcal{C}$$

with $\mathrm{wt}(\bar{r}) = 2$ and $r^2 = -4/3$ satisfies $u \in \mathrm{Roots}(h^\perp) \setminus \mathrm{Roots}(\mathbb{Z}[10A_2])$. Suppose that (c) is satisfied and assume that $a = 2b$. A vector

$$u = r + e^\vee - 2f^\vee \quad \in \quad N_\mathcal{C}$$

with $\mathrm{wt}(\bar{r}) = 1$ and $r^2 = -2/3$ satisfies $u \in \mathrm{Roots}(h^\perp) \setminus \mathrm{Roots}(\mathbb{Z}[10A_2])$. Thus the assertion (2) is proved. □

Proof of Theorem 1.5. The implication (iii) $\Longrightarrow$ (ii) is obvious. Since every vector h of $U(3)$ satisfies $h^2 \equiv 0 \bmod 6$, the implication (ii) $\Longrightarrow$ (i) is also obvious. Using computer, we can prove the following Claim 5.2. See Remark 5.3 and Table 5.1.

Claim 5.2. There exists an isotropic admissible code $\mathcal{C} \subset D$ of dimension 5 with the following property:

(5.7) every non-zero word $(\mathbf{x}, y_1, y_2) \in \mathcal{C}$ satisfies the following: (i) $\mathrm{wt}(\mathbf{x}) \ge 3$, and (ii) if $\mathrm{wt}(\mathbf{x}) = 3$, then $(y_1, y_2) \neq (0, 0)$.

We now prove (i) $\Longrightarrow$ (iii) Suppose that an integer $d = 6m$ $(m \in \mathbb{Z}_{>0})$ is given. Let X be a supersingular $K3$ surface in characteristic 3 with Artin invariant $\sigma \leq 6$. For the basis e, f of $U(3)$ at the end of Section 2, we put

$$h := e + mf.$$

Then $h^2 = d$. Let $\mathcal{C}(\sigma)$ be a linear subspace of the code $\mathcal{C}$ in Claim 5.2 with $\dim \mathcal{C}(\sigma) = 6 - \sigma$. Since $\mathcal{C}(\sigma)$ is isotropic, the corresponding overlattice $N_{\mathcal{C}(\sigma)}$ of $\mathbb{Z}[10A_2] \oplus U(3)$ is isomorphic to $N_{3,\sigma}$ by Theorem 3.2. Hence there exists an isometry

$$\phi : N_{\mathcal{C}(\sigma)} \xrightarrow{\sim} NS(X)$$

by Theorem 3.1. Since every word of $\mathcal{C}(\sigma)$ satisfies the conditions (i) and (ii) in (5.7), Lemma 5.1 implies that there exist no vectors u in $N_{\mathcal{C}(\sigma)}$ satisfying $hu = 1$ or 2 and $u^2 = 0$, and that the set of roots in the orthogonal complement $h^\perp$ of h in $N_{\mathcal{C}(\sigma)}$ coincides with $\mathrm{Roots}(\mathbb{Z}[10A_2])$. By Proposition 3.3 and Remark 3.6, we can choose the isometry $\phi : N_{\mathcal{C}(\sigma)} \xrightarrow{\sim} NS(X)$ in such a way that $\phi(h)$ is the class $[L]$ of a line bundle L very ample modulo (-2)-curves such that $\Phi_{|L|}$ induces a contraction $\rho_L : X \to Y_{(X,L)}$ of an ADE-configuration of (-2)-curves of type $10A_2$. Since $\mathcal{C}(\sigma)$ is admissible, we see that $R^\perp_{(X,L)} \subset NS(X)$ is isomorphic to $U(3)$. Thus X admits a polarization L of degree d with the hoped-for properties. □

Remark 5.3. Let G denote the group of linear automorphisms of $D \cong \mathbb{F}_3^{10} \oplus \mathbb{F}_3^2$ generated by

$$(x_1, \dots, x_{10}, y_1, y_2) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(10)}, y_{\tau(1)}, y_{\tau(2)}) \qquad (\sigma \in \mathfrak{S}_{10}, \tau \in \mathfrak{S}_2), \quad \text{and}$$
$$(x_1, \dots, x_{10}, y_1, y_2) \mapsto ((-1)^{\alpha_1} x_1, \dots, (-1)^{\alpha_{10}} x_{10}, (-1)^\beta y_1, (-1)^\beta y_2) \quad (\alpha_1, \dots, \alpha_{10} \in \mathbb{F}_2, \beta \in \mathbb{F}_2).$$

Note that, if $\mathcal{C} \subset D$ is an isotropic admissible code, then so is $g(\mathcal{C})$ for any $g \in G$. We define the weight enumerator of a ternary code $\mathcal{C}$ by

$$\mathrm{we}(\mathcal{C}) := \sum_{(\mathbf{x}, y_1, y_2) \in \mathcal{C}} z^{\mathrm{wt}(\mathbf{x})}.$$

Using computer, we have proved that there exist at least seven isomorphism classes of isotropic admissible codes of dimension 5 with the property (5.7). The representative codes $\mathcal{C}_1, \dots, \mathcal{C}_7$ of these classes are given in Table 5.1. Their weight-enumerators are given in Table 5.2.

Corollary 5.4. *Let X be a supersingular $K3$ surface in characteristic 3 with Artin invariant 1. Then there exist at least seven line bundles $L_1, \dots, L_7$ of degree 6 on X that are mutually non-isomorphic and that induce contractions of $10A_2$-configurations of (-2)-curves on X.*

See Example 7.8.

6. Proof of Theorem 1.4

The proof of Theorem 1.4 is similar to and simpler than that of Theorem 1.5.

The discriminant group D of $\mathbb{Z}[10A_2] \oplus U(1)$ is equal to

$$\mathbb{F}_3\gamma_1 \oplus \cdots \oplus \mathbb{F}_3\gamma_{10}.$$

$$\mathcal{C}_1 \quad : \quad \begin{bmatrix} 1&0&0&0&0&0&0&0&1&1&0&1\\ 0&1&0&0&0&0&0&1&0&1&2&0\\ 0&0&1&0&0&0&1&0&1&0&2&0\\ 0&0&0&1&0&0&1&1&0&0&0&1\\ 0&0&0&0&1&0&1&1&1&1&1&2 \end{bmatrix}$$

$$\mathcal{C}_2 \quad : \quad \begin{bmatrix} 1&0&0&0&0&0&0&0&1&1&0&1\\ 0&1&0&0&0&0&0&1&0&1&2&0\\ 0&0&1&0&0&0&1&0&1&0&2&0\\ 0&0&0&1&0&0&1&1&0&0&0&1\\ 0&0&0&0&1&1&1&2&2&1&0&0 \end{bmatrix}$$

$$\mathcal{C}_3 \quad : \quad \begin{bmatrix} 1&0&0&0&0&0&0&0&1&1&0&1\\ 0&1&0&0&0&0&1&1&0&0&0&1\\ 0&0&1&0&0&1&0&1&0&1&2&2\\ 0&0&0&1&0&1&1&0&1&0&2&2\\ 0&0&0&0&1&1&1&2&2&1&0&1 \end{bmatrix}$$

$$\mathcal{C}_4 \quad : \quad \begin{bmatrix} 1&0&0&0&0&0&0&0&1&1&0&1\\ 0&1&0&0&0&0&1&1&0&0&0&1\\ 0&0&1&0&0&1&0&1&0&1&2&2\\ 0&0&0&1&0&1&1&0&1&0&2&2\\ 0&0&0&0&1&1&2&2&2&2&2&0 \end{bmatrix}$$

$$\mathcal{C}_5 \quad : \quad \begin{bmatrix} 1&0&0&0&0&0&0&1&1&1&1&1\\ 0&1&0&0&0&0&1&0&1&1&2&2\\ 0&0&1&0&0&1&0&1&0&1&2&2\\ 0&0&0&1&0&1&1&0&0&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&0&0 \end{bmatrix}$$

$$\mathcal{C}_6 \quad : \quad \begin{bmatrix} 1&0&0&0&0&0&0&1&1&1&1&1\\ 0&1&0&0&0&0&1&0&1&1&2&2\\ 0&0&1&0&0&1&0&1&0&1&2&2\\ 0&0&0&1&0&1&1&0&0&1&1&1\\ 0&0&0&0&1&1&2&2&1&0&1&2 \end{bmatrix}$$

$$\mathcal{C}_7 \quad : \quad \begin{bmatrix} 1&0&0&0&0&0&1&1&1&1&1&2\\ 0&1&0&0&0&1&0&1&1&2&2&1\\ 0&0&1&0&0&1&1&0&2&1&2&1\\ 0&0&0&1&0&1&1&2&0&2&1&2\\ 0&0&0&0&1&1&2&1&2&0&1&2 \end{bmatrix}$$

TABLE 5.1. Bases of the codes $\mathcal{C}_1, \ldots, \mathcal{C}_7$

$$\begin{aligned}
\mathrm{we}(\mathcal{C}_1) &= 1+12\,z^3+18\,z^4+36\,z^5+108\,z^6+36\,z^7+18\,z^8+14\,z^9,\\
\mathrm{we}(\mathcal{C}_2) &= 1+8\,z^3+10\,z^4+24\,z^5+86\,z^6+40\,z^7+30\,z^8+40\,z^9+4\,z^{10},\\
\mathrm{we}(\mathcal{C}_3) &= 1+4\,z^3+8\,z^4+24\,z^5+94\,z^6+44\,z^7+30\,z^8+36\,z^9+2\,z^{10},\\
\mathrm{we}(\mathcal{C}_4) &= 1+6\,z^3+6\,z^4+18\,z^5+102\,z^6+42\,z^7+36\,z^8+26\,z^9+6\,z^{10},\\
\mathrm{we}(\mathcal{C}_5) &= 1+30\,z^4+60\,z^6+120\,z^7+20\,z^9+12\,z^{10},\\
\mathrm{we}(\mathcal{C}_6) &= 1+18\,z^4+18\,z^5+96\,z^6+36\,z^7+36\,z^8+38\,z^9,\\
\mathrm{we}(\mathcal{C}_7) &= 1+72\,z^5+60\,z^6+90\,z^8+20\,z^9.
\end{aligned}$$

TABLE 5.2. Weight-enumerators

A ternary code $\mathcal{C} \subset D$ is isotropic with respect to the discriminant form q of $\mathbb{Z}[10A_2] \oplus U(1)$ if and only if

$$\mathrm{wt}(\mathbf{x}) \equiv 0 \mod 3 \quad \text{for any} \quad \mathbf{x} \in \mathcal{C} \tag{6.1}$$

holds. For an isotropic code $\mathcal{C}$, we denote by $N_{\mathcal{C}}$ the overlattice of $\mathbb{Z}[10A_2] \oplus U(1)$ corresponding to $\mathcal{C}$. By Theorem 3.2, $N_{\mathcal{C}}$ is isomorphic to the lattice $N_{3,\sigma}$, where $\sigma = 5 - \dim \mathcal{C}$.

Let $h = ae + bf$ be a vector of $U(1)$ with $a \geq 1$ and $b \geq 1$. We have $h^2 = 2ab$.

LEMMA 6.1. *Let $\mathcal{C}$ be an isotropic code in $D \cong \mathbb{F}_3^{10}$.*

(1) *There exists a vector $u \in N_{\mathcal{C}}$ satisfying $hu = 1$ or 2 and $u^2 = 0$ if and only if $a \leq 2$ or $b \leq 2$.*

(2) *The set of roots* $\mathrm{Roots}(h^\perp) := \{r \in N_{\mathcal{C}} \mid rh = 0,\ r^2 = -2\}$ *in $h^\perp$ is strictly larger than* $\mathrm{Roots}(\mathbb{Z}[10A_2])$ *if and only if one of the following holds;*

(a) *there exists $\mathbf{x} \in \mathcal{C}$ such that $\mathrm{wt}(\mathbf{x}) = 3$, or*
(b) *$a = b$.*

PROOF. We prove (1) first. Suppose that a vector

$$u = r_u + \eta_1 f + \eta_2 e \quad (r_u \in \mathbb{Z}[10A_2]^\vee,\ \ \eta_1, \eta_2 \in \mathbb{Z}) \tag{6.2}$$

of $\mathbb{Z}[10A_2]^\vee \oplus U(1)^\vee = \mathbb{Z}[10A_2]^\vee \oplus U(1)$ satisfies $hu = 1$ or 2 and $u^2 = 0$. Then we have

$$a\eta_1 + b\eta_2 = 1 \text{ or } 2, \tag{6.3}$$

$$r_u^2 + 2\eta_1\eta_2 = 0. \tag{6.4}$$

By (6.4), we have $\eta_1\eta_2 \geq 0$. Using (6.3), we have $a \leq 2$ or $b \leq 2$. Conversely, if $a \leq 2$, then $u = f$ satisfies $hu = a = 1$ or 2 and $u^2 = 0$. Thus (1) is proved.

Next we prove (2). Suppose that a vector u given by (6.2) satisfies $hu = 0$, $u^2 = -2$ and $u \notin \mathrm{Roots}(\mathbb{Z}[10A_2])$. Then we have

$$a\eta_1 + b\eta_2 = 0, \tag{6.5}$$

$$r_u^2 + 2\eta_1\eta_2 = -2. \tag{6.6}$$

Because $\mathcal{C}$ is isotropic, $\mathrm{wt}(\bar{r}_u) \equiv 0 \mod 3$ holds. If $\eta_1 = 0$ or $\eta_2 = 0$, then (6.5) implies $\eta_1 = \eta_2 = 0$. By (2.2) and (6.6), we have $\mathrm{wt}(\bar{r}_u) \leq 3$. If $\mathrm{wt}(\bar{r}_u) = 0$, then $u = r_u$ is contained in $\mathrm{Roots}(\mathbb{Z}[10A_2])$. Hence we have $\mathrm{wt}(\bar{r}_u) = 3$, and therefore the condition (a) is satisfied. Suppose that $\eta_1 \neq 0$ and $\eta_2 \neq 0$. By (6.5), we have

$\eta_1\eta_2 < 0$. By (6.6), we have $r_u = 0$ and $\eta_1\eta_2 = -1$, and hence $a = b$ follows from (6.5).

Conversely, suppose that (a) is fulfilled. Using (2.3), we have a lift

$$u = r + 0 + 0 \in N_{\mathcal{C}} \qquad (r \in \mathbb{Z}[10A_2]^{\vee})$$

of the word $\bar{r} \in \mathcal{C}$ with $\mathrm{wt}(\bar{r}) = 3$ such that $r^2 = -2$. Then u is contained in $\mathrm{Roots}(h^{\perp}) \setminus \mathrm{Roots}(\mathbb{Z}[10A_2])$. Suppose that (b) is satisfied. The vector

$$u = e - f \quad \in \quad N_{\mathcal{C}}$$

satisfies $u \in \mathrm{Roots}(h^{\perp}) \setminus \mathrm{Roots}(\mathbb{Z}[10A_2])$. □

In order to prove Theorem 1.4, it is therefore enough to show the following:

Claim 6.2. There exists an isotropic code $\mathcal{C} \subset D \cong \mathbb{F}_3^{10}$ of dimension 4 such that $\mathrm{wt}(\mathbf{x}) \geq 6$ holds for any $\mathbf{x} \in \mathcal{C}$.

The code $\mathcal{C}$ generated by the row vectors of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 0 & 2 & 1 \end{bmatrix}$$

satisfies $\mathrm{wt}(\mathbf{x}) \geq 6$ for any $\mathbf{x} \in \mathcal{C}$. The weight-enumerator $\sum_{\mathbf{x}\in\mathcal{C}} z^{\mathrm{wt}(\mathbf{x})}$ of this code $\mathcal{C}$ is

$$1 + 60z^6 + 20z^9.$$

Remark 6.3. The code $\mathcal{C}$ above is obtained as a subcode of the extended ternary Golay code in $\mathbb{F}_3^{12}$. See [**12**, Chapter 5, Section 2].

7. Proof of Theorem 1.6

Let (X, L) be a polarized $K3$ surface of degree 6. Then $Y_{(X,L)}$ is a complete intersection of multi-degree $(2, 3)$ in $\mathbb{P}^4$ by [**18**, Theorem 6.1]. Let $\widetilde{Q}_{(X,L)}$ denote the unique quadric hypersurface in $\mathbb{P}^4$ containing $Y_{(X,L)}$.

Proposition 7.1. *Suppose that $\mathcal{R}_{(X,L)} = 10A_2$ and $R^{\perp}_{(X,L)} \cong U(3)$. Then $\widetilde{Q}_{(X,L)}$ is a cone over a non-singular quadric surface $Q = \mathbb{P}^1 \times \mathbb{P}^1$, and $Y_{(X,L)}$ does not pass through the vertex P of the cone $\widetilde{Q}_{(X,L)}$.*

Proof. By the assumption, $R^{\perp}_{(X,L)}$ is generated by the numerical equivalence classes $[E]$ and $[F]$ of divisors E and F satisfying

$$E^2 = F^2 = 0, \quad EF = 3, \quad [L] = [E] + [F]. \tag{7.1}$$

By the Riemann-Roch theorem, we can assume that E and F are effective. Suppose that $|E|$ has a fixed component. Let $M + \Gamma$ be a general member of $|E|$, where Γ is the fixed part of $|E|$. Because $\rho_L : X \to Y_{(X,L)}$ is birational, ρ_L induces a birational map from M to $\rho_L(M)$. Note that $\rho_L(M + \Gamma)$ is a cubic curve. If $\rho_L(\Gamma)$ is of dimension 1, then $\rho_L(M)$ is a line or a plane conic, and hence contradicts $\dim |M| > 0$. Therefore, ρ_L contracts every irreducible component of Γ to a point, and hence $[\Gamma] \in R_{(X,L)}$. From $[E] \in R^{\perp}_{(X,L)}$, we obtain $E\Gamma = 0$ and hence $M^2 = E^2 + \Gamma^2 < 0$. Thus we get a contradiction again. Hence $|E|$ has no fixed components. In particular, E is nef. Since ρ_L is birational and E is primitive in $R^{\perp}_{(X,L)}$ (being part of its basis), a general member E of $|E|$ is mapped by ρ_L birationally to a plane cubic curve in $\mathbb{P}^4$. Therefore a general member of $|E|$ is irreducible, and hence $|E|$ is a (quasi-)elliptic pencil by [**15**, Proposition 0.1]. Therefore the quadric hypersurface $\widetilde{Q}_{(X,L)}$ contains a one-dimensional family $\{\Pi_t^E\}$ of planes such that

$$|E| = \{\rho_L^*(\Pi_t^E \cap Y_{(X,L)})\}.$$

Hence $\widetilde{Q}_{(X,L)}$ is singular. Since $\widetilde{Q}_{(X,L)}$ contains two irreducible families $\{\Pi_t^E\}$ and $\{\Pi_t^F\}$ of planes corresponding to $|E|$ and $|F|$, we have $\dim \operatorname{Sing} \widetilde{Q}_{(X,L)} = 0$, and $\widetilde{Q}_{(X,L)}$ is a cone over a non-singular quadric surface $Q = \mathbb{P}^1 \times \mathbb{P}^1$. If $Y_{(X,L)}$ passed through the vertex P of the cone $\widetilde{Q}_{(X,L)}$, then the linear system $|E|$ would have a fixed component that is contracted to the point P. Hence P is not contained in $Y_{(X,L)}$. □

Note that a non-ordered pair of the numerical equivalence classes $[E]$ and $[F]$ in $R^{\perp}_{(X,L)}$ satisfying (7.1) is unique. The following has been shown in the proof above:

Corollary 7.2. *The divisors E and F are nef. The complete linear systems $|E|$ and $|F|$ are (quasi-)elliptic pencils.*

We denote by

$$\pi_P : Y_{(X,L)} \to Q = \mathbb{P}^1 \times \mathbb{P}^1$$

the projection from the vertex P of the cone $\widetilde{Q}_{(X,L)}$. Let x and y be affine coordinates of the two factors of $\mathbb{P}^1 \times \mathbb{P}^1$. The surface $Y_{(X,L)}$ is defined by an equation

$$\Psi \;:=\; W^3 \,+\, a(x,y)\,W^2 \,+\, b(x,y)\,W \,+\, c(x,y) \;=\; 0, \tag{7.2}$$

where W is a fiber coordinate of the affine line bundle $\widetilde{Q}_{(X,L)} \setminus \{P\} \cong L_Q(1,1)$ on $Q = \mathbb{P}^1 \times \mathbb{P}^1$, and a, b, c are polynomials of degrees 1, 2 and 3, respectively.

Let us consider the fibrations

$$\begin{aligned}\Phi_{|E|} &= \mathrm{pr}_1 \circ \pi_P \circ \rho_L \quad : \quad X \to \mathbb{P}^1, \qquad \text{and}\\ \Phi_{|F|} &= \mathrm{pr}_2 \circ \pi_P \circ \rho_L \quad : \quad X \to \mathbb{P}^1,\end{aligned}$$

where $\mathrm{pr}_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection onto the i-th factor. Because $Y_{(X,L)}$ has ten cusps, the classification of fibers of (quasi-)elliptic fibrations and the criterion [**17**, Section 4] for quasi-ellipticity imply the following:

Proposition 7.3. *The fibrations $\Phi_{|E|}$ and $\Phi_{|F|}$ are quasi-elliptic. Let Θ be a fiber of the quasi-elliptic fibration $\Phi_{|E|}$. Then Θ is either of type* II*, of type* IV *or of type* IV*. (*See Figure 7.1*). *Moreover, we have*

$$\begin{aligned}\Theta \text{ is of type II} &\iff \rho_L(\Theta) \text{ does not pass through any cusps of } Y_{(X,L)},\\ \Theta \text{ is of type IV} &\iff \rho_L(\Theta) \text{ passes through exactly one cusp of } Y_{(X,L)},\\ \Theta \text{ is of type IV}^* &\iff \rho_L(\Theta) \text{ is a line with multiplicity 3 passing through}\\ &\qquad \text{exactly three cusps of } Y_{(X,L)}.\end{aligned}$$

Same hold for fibers of $\Phi_{|F|}$.

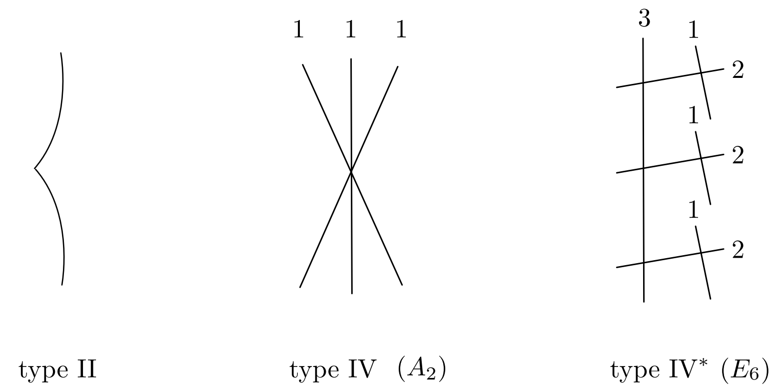

FIGURE 7.1. Fibers of quasi-elliptic fibrations

PROOF OF THEOREM 1.6. Let X be a supersingular $K3$ surface with $\sigma(X) \le 6$. We choose a subcode $\mathcal{C}$ of the isotropic admissible code $\mathcal{C}_7$ in Table 5.1 with

$$\dim \mathcal{C} = 6 - \sigma(X),$$

and consider the corresponding overlattice $N_{\mathcal{C}}$ of $\mathbb{Z}[10A_2] \oplus U(3)$. There exists an isometry

$$\phi : N_{\mathcal{C}} \xrightarrow{\sim} NS(X)$$

such that $\phi(e+f)$ is the class $[L]$ of a line bundle L that is very ample modulo (-2)-curves, where e, f form the canonical basis of $U(3)$; see the proof of Theorem 1.5. Then $Y_{(X,L)}$ is a complete intersection in $\mathbb{P}^4$ with multi-degree $(2,3)$ that has ten cusps as its only singularities. We will prove Theorem 1.6 by showing that, for this polarized supersingular $K3$ surface (X, L), the morphism π_P from $Y_{(X,L)}$ to $\mathbb{P}^1 \times \mathbb{P}^1$ is purely inseparable; that is, the polynomials a and b in (7.2) are zero.

We assume that π_P is separable, and derive a contradiction.

For $i = 1, \dots, 10$, let C_i and D_i be the (-2)-curves contracted by ρ_L satisfying

$$C_i^2 = D_i^2 = -2, \qquad C_i D_i = 1, \qquad \langle [C_i], [D_i] \rangle \perp \langle [C_j], [D_j] \rangle \quad (i \ne j),$$

and let E, F be divisors such that $\phi(e) = [E]$ and $\phi(f) = [F]$. Then E and F satisfy $[E], [F] \in R_{(X,L)}^{\perp}$ and (7.1). We put

$$\begin{aligned} \gamma_i &:= ([C_i] + 2[D_i])/3 \mod (R_{(X,L)} \oplus R_{(X,L)}^{\perp}), \\ \bar{f}^{\vee} &:= [E]/3 \mod (R_{(X,L)} \oplus R_{(X,L)}^{\perp}), \\ \bar{e}^{\vee} &:= [F]/3 \mod (R_{(X,L)} \oplus R_{(X,L)}^{\perp}). \end{aligned}$$

The code $\mathcal{C}_{(X,L)}$ defined by

$$\mathcal{C}_{(X,L)} := NS(X)/(R_{(X,L)} \oplus R_{(X,L)}^{\perp}) \quad \subset \quad \mathrm{Disc}(R_{(X,L)} \oplus R_{(X,L)}^{\perp}) \cong \mathbb{F}_3^{10} \oplus \mathbb{F}_3^2$$

is isomorphic to the subcode $\mathcal{C}$ of $\mathcal{C}_7$ chosen above. Let G be a divisor on X. Then $[G] \in NS(X)$ is written as

$$\frac{1}{3} \sum_{i=1}^{10} (s_i [C_i] + t_i [D_i]) + \frac{\alpha}{3} [E] + \frac{\beta}{3} [F],$$

where s_i, t_i, α, β are integers satisfying $s_i + t_i \equiv 0 \bmod 3$. We denote by

$$\langle G \rangle := [G] \mod (R_{(X,L)} \oplus R^{\perp}_{(X,L)})$$

the word of $\mathcal{C}_{(X,L)}$ corresponding to $[G]$, which is written as

$$(\mathbf{x}(G), \bar{\alpha}, \bar{\beta}) = \sum_{i=1}^{10} x_i \gamma_i + \bar{\alpha} \bar{f}^{\vee} + \bar{\beta} \bar{e}^{\vee},$$

where $\bar{\alpha} = \alpha \bmod 3$, $\bar{\beta} = \beta \bmod 3$, and

$$x_i = \begin{cases} 0 & \text{if } (s_i, t_i) \equiv (0,0) \mod 3, \\ 1 & \text{if } (s_i, t_i) \equiv (1,2) \mod 3, \\ 2 & \text{if } (s_i, t_i) \equiv (2,1) \mod 3. \end{cases}$$

We put

$$\begin{aligned} s(G) &:= \{ i \mid (s_i, t_i) \neq (0,0) \} = \{ i \mid C_i G \neq 0 \text{ or } D_i G \neq 0 \}, \\ s_1(\mathbf{x}(G)) &:= \{ i \mid x_i \neq 0 \} = \{ i \mid (s_i, t_i) \not\equiv (0,0) \mod 3 \}, \\ s_2(G) &:= \{ i \mid (s_i, t_i) \neq (0,0) \text{ and } (s_i, t_i) \equiv (0,0) \mod 3 \}. \end{aligned}$$

By definition, we have

$$s(G) = s_1(\mathbf{x}(G)) \sqcup s_2(G).$$

Lemma 7.4. *Suppose that G is a reduced irreducible curve on X. Then the following holds:*

$$|s_2(G)| \leq \frac{1}{3}(\alpha\beta - |s_1(\mathbf{x}(G))|) + 1. \tag{7.3}$$

In particular, we have $\alpha\beta - |s_1(\mathbf{x}(G))| \geq -3$.

Proof. Let s and t be integers such that $s + t \equiv 0 \mod 3$. If $(s,t) \neq (0,0)$, then

$$((sC_i + tD_i)/3)^2 \leq -2/3$$

holds. If $(s,t) \neq (0,0)$ and $(s,t) \equiv (0,0) \mod 3$, then

$$((sC_i + tD_i)/3)^2 \leq -2$$

holds. Therefore we have

$$G^2 \leq -\frac{2}{3}|s_1(\mathbf{x}(G))| - 2|s_2(G)| + \frac{2}{3}\alpha\beta.$$

On the other hand, we have $G^2 \geq -2$. Hence we get the inequality (7.3). □

Let us denote by $\bar{T}$ the Cartier divisor on $Y_{(X,L)}$ cut out by the equation

$$\frac{\partial \Psi}{\partial W} = -aW + b = 0, \tag{7.4}$$

and let T be the proper transform of $\bar{T}$ by ρ_L. By the assumption that π_P is separable, $\bar{T}$ is a divisor and π_P is étale outside $\bar{T}$. Hence the divisor $\bar{T}$ contains the ten cusps of $Y_{(X,L)}$. Therefore we have

$$s(T) = \{1, 2, \ldots, 10\}. \tag{7.5}$$

From the defining equation (7.4) of $\bar{T}$ on $Y_{(X,L)}$, we have

$$ET = FT = 6. \tag{7.6}$$

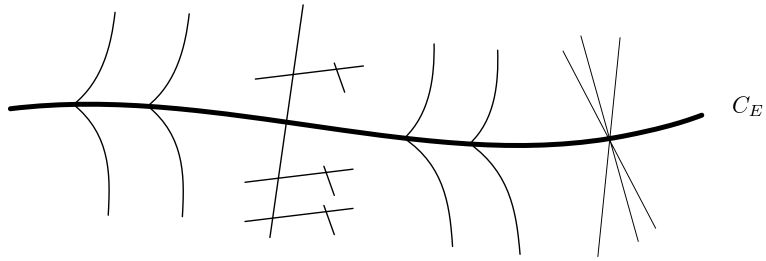

FIGURE 7.2. The curve C_E

We denote by C_E the closure of the locus

$$\{ x \in X \mid \text{the fiber of } \Phi_{|E|} \text{ passing through } x \text{ is of type II and is singular at } x \},$$

and equip C_E with the reduced structure. A general member E of $|E|$ intersects C_E at one point with multiplicity 3 ([**8**]). See Figure 7.2. We define C_F in the same way. Both of C_E and C_F are irreducible, and we have

$$C_E E = C_F F = 3. \tag{7.7}$$

Because $\mathrm{pr}_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is smooth, if $\mathrm{pr}_1 \circ \pi_P$ is not smooth at a non-singular point of $Y_{(X,L)}$, then π_P is not smooth at that point. Therefore the divisor T contains C_E as a reduced irreducible component. Same holds for C_F.

CLAIM 7.5. The two curves C_E and C_F are distinct.

PROOF. Suppose that $C_E = C_F$ holds. Let x be a general point of $C_E = C_F$. Since the fibers E_x of $\Phi_{|E|}$ and F_x of $\Phi_{|F|}$ passing through x are both singular at x, we have $E_x F_x \geq 4$, which contradicts $EF = 3$. □

Let

$$T = C_E + C_F + T_1 + \cdots + T_t$$

be the decomposition of T into reduced irreducible components. We put

$$\begin{aligned} [C_E] &= \sum (s_{E,i}[C_i] + t_{E,i}[D_i])/3 + (\alpha_E[E] + \beta_E[F])/3, \\ [C_F] &= \sum (s_{F,i}[C_i] + t_{F,i}[D_i])/3 + (\alpha_F[E] + \beta_F[F])/3, \\ [T_\nu] &= \sum (s_{\nu,i}[C_i] + t_{\nu,i}[D_i])/3 + (\alpha_\nu[E] + \beta_\nu[F])/3 \quad (\nu = 1, \ldots, t). \end{aligned}$$

Since E and F are nef, we have

$$\alpha_E \geq 0, \;\; \beta_E \geq 0, \;\; \alpha_F \geq 0, \;\; \beta_F \geq 0, \;\; \alpha_\nu \geq 0, \;\; \beta_\nu \geq 0 \quad (\nu = 1, \ldots, t). \tag{7.8}$$

Since π_P is finite, $\pi_P \circ \rho_L$ maps each irreducible component of T to a curve on $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore we have

$$\alpha_\nu > 0 \;\text{ or }\; \beta_\nu > 0 \qquad (\nu = 1, \ldots, t). \tag{7.9}$$

By (7.7), we have

$$\beta_E = 3, \quad \alpha_F = 3. \tag{7.10}$$

Then, from (7.6), we have

$$\alpha_E + \sum_{\nu=1}^{t} \alpha_\nu = 3, \quad \text{and} \quad \beta_F + \sum_{\nu=1}^{t} \beta_\nu = 3. \tag{7.11}$$

Consider the words

$$\langle C_E\rangle = (\mathbf{x}_E, \bar\alpha_E, 0), \quad \langle C_F\rangle = (\mathbf{x}_F, 0, \bar\beta_F), \quad \langle T_\nu\rangle = (\mathbf{x}_\nu, \bar\alpha_\nu, \bar\beta_\nu) \qquad (\nu = 1, \dots, t)$$

in the code $\mathcal{C}_{(X,L)}$. From Lemma 7.4, we have

$$-\operatorname{wt}(\mathbf{x}_\nu) + \alpha_\nu\beta_\nu \geq -3 \qquad (\nu = 1, \dots, t). \tag{7.12}$$

Claim 7.6. $\mathbf{x}_E = \mathbf{x}_F = \mathbf{0}$.

Proof. Let Θ be a fiber of $\Phi_{|E|}$ such that $\rho_L(\Theta)$ passes through a cusp $q_i := \rho_L(C_i) = \rho_L(D_i)$ of $Y_{(X,L)}$. Then Θ is of type IV or IV*. Suppose that Θ is of type IV. Then Θ consists of three irreducible components of multiplicity one, two of which are C_i and D_i, that intersect at one point. The curve C_E passes through the intersection point. Since $\Theta C_E = 3$, we have $C_EC_i = C_ED_i = 1$, and therefore $s_{i,E} = t_{i,E} = -3$ holds. Suppose that Θ is of type IV*. Then C_E passes through a point of the multiplicity 3 component of Θ, and does not intersect other irreducible components. This fact can be proved by considering the pull-back of the quasi-elliptic fibration $\Phi_{|E|}$ by the base change $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 2 branching at the point $\Phi_{|E|}(\Theta)$, which makes the fiber Θ into type IV. Then it follows that $s_{i,E} = t_{i,E} = 0$. In any case, we have $i \notin s_1(\mathbf{x}_E)$. Since this holds for any cusp q_i of $Y_{(X,L)}$, we have $s_1(\mathbf{x}_E) = \emptyset$. □

Since $\bar T$ is a Cartier divisor of $Y_{(X,L)}$, the total transform $\rho_L^*(\bar T)$ is contained in $R^\perp_{(X,L)} = \mathbb{Z}[E] \oplus \mathbb{Z}[F]$, and hence $\langle T\rangle = 0$. Therefore we obtain

$$\mathbf{x}_1 + \cdots + \mathbf{x}_t = \mathbf{0}. \tag{7.13}$$

By (7.5), we have

$$s(C_E) \cup s(C_F) \cup s(T_1) \cup \cdots \cup s(T_t) = \{1, 2, \dots, 10\}.$$

Using Lemma 7.4, we obtain

$$t + 2 + \frac{1}{3}\Big(\alpha_E\beta_E + \alpha_F\beta_F + \sum_{\nu=1}^{t}(\alpha_\nu\beta_\nu - |s_1(\mathbf{x}_\nu)|)\Big) \geq 10 - |s_1(\mathbf{x}_1) \cup \cdots \cup s_1(\mathbf{x}_t)|. \tag{7.14}$$

Because $\mathcal{C}_{(X,L)}$ is isomorphic to a subcode of $\mathcal{C}_7$, we have shown that there exist integers α_E, β_F, α_ν, β_ν $(\nu = 1, \dots, t)$ and words

$$(\mathbf{0}, \bar\alpha_E, 0), \quad (\mathbf{0}, 0, \bar\beta_F), \quad (\mathbf{x}_\nu, \bar\alpha_\nu, \bar\beta_\nu) \qquad (\nu = 1, \dots, t)$$

in the code $\mathcal{C}_7$ satisfying (7.8)-(7.14). Using computer, however, we can show that such integers and words do not exist. Thus we get a contradiction. □

Instead of the code $\mathcal{C}_7$, we can use the codes $\mathcal{C}_3, \dots, \mathcal{C}_6$ in Table 5.1. However, we cannot use $\mathcal{C}_2$ or $\mathcal{C}_1$. Indeed, in $\mathcal{C}_1$, for example, we have the following integers and words:

$$\begin{aligned}
&[0,0,0,0,0,0,0,0,0,0,3,0],\\
&[0,0,0,0,0,0,0,0,0,0,0,3],\\
&[1,0,0,0,0,0,0,0,1,1,0,1],\\
&[0,2,0,0,0,0,0,2,0,2,1,0],\\
&[0,0,2,0,0,0,2,0,2,0,1,0],\\
&[0,0,0,1,0,0,1,1,0,0,0,1],\\
&[2,1,1,2,0,0,0,0,0,0,1,1].
\end{aligned}$$

Nevertheless, we can ask the following:

QUESTION 7.7. Is $\pi_P : Y_{(X,L)} \to \mathbb{P}^1 \times \mathbb{P}^1$ inseparable for any polarized supersingular $K3$ surface (X, L) of degree 6 with $\mathcal{R}_{(X,L)} = 10A_2$ and $R^{\perp}_{(X,L)} \cong U(3)$?

EXAMPLE 7.8. Consider the purely inseparable triple cover of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$W^3 = (x^3 - x)(y^3 - y),$$

and the corresponding polarized supersingular $K3$ surface (X, L). We will show that the Artin invariant of X is 1, and that the 5-dimensional ternary code $\mathcal{C}_{(X,L)}$ is isomorphic to $\mathcal{C}_1$. For $\alpha \in \mathbb{F}_3$, let l_α and m_α be the lines on $\mathbb{P}^1 \times \mathbb{P}^1$ defined by $x = \alpha$ and $y = \alpha$, respectively. The strict transforms of l_α and m_α by $\pi_P \circ \rho_L$ are written as $3\tilde{l}_\alpha$ and $3\tilde{m}_\alpha$, respectively. Numbering the twenty (-2)-curves $C_1, D_1, \ldots, C_{10}, D_{10}$ in an appropriate way, we can write the numerical equivalence classes $[\tilde{l}_\alpha]$, $[\tilde{m}_\alpha]$ as follows:

$$\begin{aligned}
[\tilde{l}_0] &= A_1 + A_2 + A_3 + [E]/3,\\
[\tilde{m}_0] &= A'_1 + A'_4 + A'_7 + [F]/3,\\
[\tilde{l}_1] &= A_4 + A_5 + A_6 + [E]/3,\\
[\tilde{m}_1] &= A'_2 + A'_5 + A'_8 + [F]/3,\\
[\tilde{l}_2] &= A_7 + A_8 + A_9 + [E]/3,\\
[\tilde{m}_2] &= A'_3 + A'_6 + A'_9 + [F]/3,
\end{aligned}$$

where

$$A_i = -([C_i] + 2[D_i])/3, \qquad A'_i = -(2[C_i] + [D_i])/3.$$

The discriminant of the sublattice of $NS(X)$ generated by the classes $[E], [F]$, the classes of the twenty exceptional curves, and the 6 classes above is equal to -9. Hence these classes span $NS(X)$, and the Artin invariant of X is 1. The 6 words $\langle \tilde{l}_\alpha \rangle$, $\langle \tilde{m}_\alpha \rangle$ generate a 5-dimensional ternary code isomorphic to $\mathcal{C}_1$.

QUESTION 7.9. Find the defining equations of purely inseparable triple covers of $Q = \mathbb{P}^1 \times \mathbb{P}^1$ corresponding to the other ternary codes $\mathcal{C}_2, \ldots, \mathcal{C}_7$ of dimension 5 in Table 5.1. (See Corollary 5.4.)

In [**11**], Dolgachev and Kondo gave various defining equations of the supersingular $K3$ surface in characteristic 2 with Artin invariant 1, and determined the full automorphism group of this $K3$ surface. We expect that various defining equations of the supersingular $K3$ surface in characteristic 3 with Artin invariant 1 would be also helpful in the study of the automorphism group of this surface.

References

[1] M. Artin, *Some numerical criteria for contractability of curves on algebraic surfaces*, Amer. J. Math. **84** (1962), 485–496.

[2] ———, *On isolated rational singularities of surfaces*, Amer. J. Math. **88** (1966), 129–136.

[3] ———, *Supersingular $K3$ surfaces*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 543–567 (1975).

[4] ———, *Coverings of the rational double points in characteristic p*, Complex analysis and algebraic geometry, (a collection of papers dedicated to K. Kodaira. Edited by W. L. Baily, Jr., and T. Shioda), Iwanami Shoten, Tokyo, 1977, pp. 11–22.

[5] W. Barth, *$K3$ surfaces with nine cusps*, Geom. Dedicata, **72** (1998), 171–178.

[6] ———, *On the classification of $K3$ surfaces with nine cusps*, Complex analysis and algebraic geometry, de Gruyter, Berlin, 2000, pp. 41–59.

[7] P. Blass and J. Lang, *Zariski surfaces and differential equations in characteristic $p > 0$*, Marcel Dekker Inc., New York, 1987.

[8] E. Bombieri and D. Mumford, *Enriques' classification of surfaces in char. p. III*, Invent. Math. **35** (1979), 197–232.

[9] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: Systèmes de racines*, Hermann, Paris, 1968.

[10] J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups*, third ed., Springer-Verlag, New York, 1999.

[11] I. Dolgachev and S. Kondo, *A supersingular $K3$ surface in characteristic 2 and the Leech lattice*, Int. Math. Res. Not. 2003, no. 1, 1–23 (2001).

[12] W. Ebeling, *Lattices and codes*, A course partially based on lectures by F. Hirzebruch. Second revised edition. Friedr. Vieweg & Sohn, Braunschweig, 2002.

[13] H. Ito, *The Mordell-Weil groups of unirational quasi-elliptic surfaces in characteristic* 3, Math. Z. **211** (1992), no. 1, 1–39.

[14] V. V. Nikulin, *Kummer surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), no. 2, 278–293, 471.

[15] ———, *Integer symmetric bilinear forms and some of their geometric applications*, Math. USSR-Izv. **14** (1979), no. 1, 103–167.

[16] ———, *Weil linear systems on singular $K3$ surfaces*, Algebraic geometry and analytic geometry (Tokyo, 1990), Springer, Tokyo, 1991, pp. 138–164.

[17] A. N. Rudakov and I. R. Šafarevič, *Surfaces of type $K3$ over fields of finite characteristic*, Current problems in mathematics, Vol. 18, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1981, pp. 115–207: Igor R. Shafarevich, Collected mathematical papers, Springer-Verlag, Berlin, 1989, pp. 657–714.

[18] B. Saint-Donat, *Projective models of $K-3$ surfaces*, Amer. J. Math. **96** (1974), 602–639.

[19] J.-P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics, No. 7, Springer-Verlag, New York, 1973.

[20] I. Shimada, *On supercuspidal families of curves on a surface in positive characteristic*, Math. Ann. **292** (1992), no. 4, 645–669.

[21] ———, *Rational double points on supersingular $K3$ surfaces*, Math. Comp. **73** (2004), no. 248, 1989–2017 (electronic).

[22] ———, *Supersingular $K3$ surfaces in characteristic* 2 *as double covers of a projective plane*, Asian J. Math. **8** (2004), no. 3, 531–586.

[23] ———, *Moduli curves of supersingular $K3$ surfaces in characteristic* 2 *with Artin invariant* 2, Proc. Edinburgh Math. Soc. **49** (2006), 435–503.

[24] I. Shimada and De-Qi Zhang, *Dynkin diagrams of rank* 20 *on supersingular $K3$ surfaces*, Preprint, http://www.math.sci.hokudai.ac.jp/˜shimada/preprints.html.

[25] T. Shioda, *Supersingular $K3$ surfaces*, Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), Lecture Notes in Math., Vol. 732, Springer, Berlin, 1979, pp. 564–591.

[26] ———, *Some results on unirationality of algebraic surfaces*, Math. Ann. **230** (1977), no. 2, 153–168.

[27] T. Urabe, *Dynkin graphs and combinations of singularities on plane sextic curves*, Singularities (Iowa City, IA, 1986), Amer. Math. Soc., Providence, RI, 1989, pp. 295–316.

Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-0810, JAPAN

E-mail address: shimada@math.sci.hokudai.ac.jp

Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, SINGAPORE

E-mail address: matzdq@math.nus.edu.sg