数体上定義された特異 $K3$ 曲面

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1. Introduction

First, we fix some notions and notation.

By a *lattice*, we mean a finitely generated free $\mathbb{Z}$-module Λ equipped with a non-degenerate symmetric bilinear form

$$(\ , \ ) \ : \ \Lambda \times \Lambda \ \to \ \mathbb{Z}.$$

The discriminant $\operatorname{disc}(\Lambda) \in \mathbb{Z}$ of a lattice Λ is the determinant of a symmetric matrix expressing the bilinear form. A lattice Λ is said to be *even* if $(v, v) \in 2\mathbb{Z}$ holds for every $v \in \Lambda$. Let Λ and Λ' be lattices. A homomorphism $\Lambda \to \Lambda'$ of $\mathbb{Z}$-modules is called an *isometry* if it preserves the symmetric bilinear forms. By definition, an isometry is injective. Let $\Lambda \hookrightarrow \Lambda'$ be an isometry. We denote by

$$(\Lambda \hookrightarrow \Lambda')^{\perp}$$

the *orthogonal complement* of Λ in Λ'. A sublattice $\Lambda \subset \Lambda'$ is called *primitive* if Λ'/Λ is torsion-free. For a lattice Λ, we denote by $\Lambda[-1]$ the lattice obtained from Λ by multiplying the symmetric bilinear form by -1.

Let F be a number field. We denote by $\mathbb{Z}_F$ the integer ring of F, and by

$$\pi_F : \operatorname{Spec} \mathbb{Z}_F \to \operatorname{Spec} \mathbb{Z}$$

the natural projection.

Let k be a field of characteristic 0. We denote by $\operatorname{Emb}(k, \mathbb{C})$ the set of embeddings $\sigma : k \hookrightarrow \mathbb{C}$ of k into $\mathbb{C}$. For a variety X over k and an embedding $\sigma \in \operatorname{Emb}(k, \mathbb{C})$, we define a complex variety X^{σ} by the following diagram of the fiber product:

$$\begin{array}{ccc} X^{\sigma} & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ \operatorname{Spec} \mathbb{C} & \xrightarrow{\sigma^*} & \operatorname{Spec} k. \end{array}$$

Two complex varieties X and X' are said to be *conjugate* if there exists $\sigma \in \operatorname{Emb}(\mathbb{C}, \mathbb{C})$ such that X^{σ} is isomorphic to X' over $\mathbb{C}$. It is obvious that the relation of being conjugate is an equivalence relation. If is easy to see that a complex algebraic surface conjugate to a $K3$ surface is also a $K3$ surface.

For a $K3$ surface X defined over a field k, we denote by $\mathrm{NS}(X)$ the Néron-Severi lattice of $X \otimes \bar{k}$, where $\bar{k}$ is the algebraic closure of k; that is, $\mathrm{NS}(X)$ is the lattice of numerical equivalence classes of divisors on $X \otimes \bar{k}$ with the intersection pairing.

Definition 1.1. A $K3$ surface X defined over a field of characteristic 0 is said to be *singular* if $\mathrm{rank}(\mathrm{NS}(X)) = 20$.

Definition 1.2. A $K3$ surface X defined over a field of characteristic $p > 0$ is said to be *supersingular* if $\mathrm{rank}(\mathrm{NS}(X)) = 22$.

By the Hodge index theorem, if X is singular or supersingular, then

$$d(X) := \mathrm{disc}(\mathrm{NS}(X))$$

is a negative integer.

Shioda and Inose [27] showed that every singular $K3$ surface is defined over a number field. Let X be a singular $K3$ surface defined over a number field F. We consider a smooth proper family

$$\mathcal{X} \to U$$

of $K3$ surfaces over a non-empty Zariski open subset U of $\mathrm{Spec}\,\mathbb{Z}_F$ such that the generic fiber X_η is isomorphic to X. For a closed point $\mathfrak{p}$ of U, we denote by $X_\mathfrak{p}$ the reduction of $\mathcal{X}$ at $\mathfrak{p}$. For a prime integer p, we put

$$\mathcal{S}_p(\mathcal{X}) := \{\ \mathfrak{p} \in \pi_F^{-1}(p) \cap U \ \mid\ X_\mathfrak{p} \text{ is supersingular } \}.$$

We investigate the following lattices of rank 2:

- the transcendental lattice

$$T(X^\sigma) := (\mathrm{NS}(X) \hookrightarrow \mathrm{H}^2(X^\sigma, \mathbb{Z}))^\perp$$

  for each $\sigma \in \mathrm{Emb}(F, \mathbb{C})$, where $\mathrm{H}^2(X^\sigma, \mathbb{Z}))$ is the Betti cohomology group of the complex surface X^σ with the cup-product, and
- the supersingular reduction lattice

$$L(\mathcal{X}, \mathfrak{p}) := (\mathrm{NS}(X) \hookrightarrow \mathrm{NS}(X_\mathfrak{p}))^\perp$$

  for each $\mathfrak{p} \in \mathcal{S}_p(\mathcal{X})$, where $\mathrm{NS}(X) \hookrightarrow \mathrm{NS}(X_\mathfrak{p})$ is the specialization isometry. (See [6, Exp. X], [13, §4] or [14, §20.3] for the definition of the specialization isometry.)

As an application of our main results, we present new examples of non-homeomorphic conjugate complex varieties, and arithmetic Zariski pairs of maximizing sextics.

Remark 1.3. The supersingular reduction lattices and their relation with transcendental lattices were first considered by Shioda in [28] for certain elliptic $K3$ surfaces.

2. The genus theory of lattices and discriminant forms

We recall the notions of genera and discriminant forms of lattices. See [7] and [19] for details.

Definition 2.1. Two lattices

$$\lambda : \Lambda \times \Lambda \to \mathbb{Z} \quad \text{and} \quad \lambda' : \Lambda' \times \Lambda' \to \mathbb{Z}$$

are said to be *in the same genus* if

$$\begin{array}{rcccl} \lambda \otimes \mathbb{Z}_p & : & (\Lambda \otimes \mathbb{Z}_p) \times (\Lambda \otimes \mathbb{Z}_p) & \to & \mathbb{Z}_p \quad \text{and} \\ \lambda' \otimes \mathbb{Z}_p & : & (\Lambda' \otimes \mathbb{Z}_p) \times (\Lambda' \otimes \mathbb{Z}_p) & \to & \mathbb{Z}_p \end{array}$$

are isomorphic for any p including $p = \infty$, where $\mathbb{Z}_\infty = \mathbb{R}$.

If Λ and Λ' are in the same genus and Λ is even, then Λ' is also even.

Definition 2.2. Let Λ be an even lattice. We put

$$\Lambda^\vee := \mathrm{Hom}(\Lambda, \mathbb{Z}).$$

Then Λ is canonically embedded into $\Lambda^\vee$ as a subgroup of index equal to $|\operatorname{disc}(\Lambda)|$, and we have a natural symmetric bilinear form

$$\Lambda^\vee \times \Lambda^\vee \to \mathbb{Q}$$

that extends the symmetric bilinear form on Λ. The finite abelian group $D_\Lambda := \Lambda^\vee/\Lambda$ together with the natural quadratic form

$$q_\Lambda : D_\Lambda \to \mathbb{Q}/2\mathbb{Z}$$

is called the *discriminant form of* Λ.

The following are due to Nikulin [19].

Theorem 2.3. *Two even lattices of the same rank are in the same genus if and only if they have the same signature and their discriminant forms are isomorphic.*

Theorem 2.4. *Let L be an even lattice, and let $M \subset L$ be a primitive sublattice. We put $N := (M \hookrightarrow L)^\perp$. Suppose that* $\operatorname{disc}(M)$ *and* $\operatorname{disc}(L)$ *are prime to each other. Then there exists an isomorphism*

$$(D_N, q_N) \;\cong\; (D_L, q_L) \oplus (D_M, -q_M)$$

of finite quadratic forms. In particular, we have $\operatorname{disc}(N) = \operatorname{disc}(L)\operatorname{disc}(M)$.

3. Transcendental lattices

Let X be a singular $K3$ surface defined over a number field F. For an embedding $\sigma : F \hookrightarrow \mathbb{C}$, the transcendental lattice $T(X^\sigma)$ of the complex singular $K3$ surface $X^\sigma := X \otimes_{F,\sigma} \mathbb{C}$ is an even positive-definite lattice of rank 2 with discriminant equal to $-d(X)$, where $d(X) = \mathrm{disc}(\mathrm{NS}(X))$.

Proposition 3.1. *For $\sigma, \sigma' \in \mathrm{Emb}(F, \mathbb{C})$, the lattices $T(X^\sigma)$ and $T(X^{\sigma'})$ are in the same genus.*

Proof. Because the Néron-Severi lattice is defined algebraically, we have

$$\mathrm{NS}(X) \;\cong\; \mathrm{NS}(X^\sigma) \;\cong\; \mathrm{NS}(X^{\sigma'}).$$

Since $\mathrm{H}^2(X^\sigma, \mathbb{Z})$ is unimodular, it follows from Theorem 2.4 that

$$(D_{T(X^\sigma)}, q_{T(X^\sigma)}) \;\cong\; (D_{\mathrm{NS}(X^\sigma)}, -q_{\mathrm{NS}(X^\sigma)}).$$

The same holds for $T(X^{\sigma'})$. Hence $T(X^\sigma)$ and $T(X^{\sigma'})$ have the isomorphic discriminant forms. □

For a negative integer d, we put

$$\mathcal{M}_d := \left\{ \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \;\middle|\; \begin{array}{l} a, b, c \in \mathbb{Z},\;\; a > 0,\;\; c > 0, \\ b^2 - 4ac = d \end{array} \right\},$$

on which $GL_2(\mathbb{Z})$ acts by $M \mapsto {}^t g M g$ ($M \in \mathcal{M}_d$, $g \in GL_2(\mathbb{Z})$). We then denote by

$$\mathcal{L}_d := \mathcal{M}_d / \, GL_2(\mathbb{Z}) \qquad (\text{resp. } \widetilde{\mathcal{L}}_d := \mathcal{M}_d / \, SL_2(\mathbb{Z}) \;)$$

the set of isomorphism classes of even, positive-definite lattices (resp. even, positive-definite *oriented* lattices) of rank 2 with discriminant $-d$.

Let S be a complex singular $K3$ surface. By the Hodge decomposition

$$T(S) \otimes \mathbb{C} = \mathrm{H}^{2,0}(S) \oplus \mathrm{H}^{0,2}(S),$$

we can define a canonical orientation on $T(S)$.

Definition 3.2. For a complex singular $K3$ surface S, we denote by $\widetilde{T}(S)$ the *oriented* transcendental lattice of S, and by $[\widetilde{T}(S)] \in \widetilde{\mathcal{L}}_{d(S)}$ the isomorphism class of the oriented transcendental lattice.

The following is due to Shioda-Inose [27].

Theorem 3.3. *The map $S \mapsto [\widetilde{T}(S)]$ induces a bijection from the set of isomorphism classes of complex singular $K3$ surfaces to the set of isomorphism classes of even, positive-definite oriented lattices of rank 2.*

In [25] and [21], the author and M. Schütt have proved the following existence theorem. (See Remark 4.15.)

Theorem 3.4. *Let $\mathcal{G} \subset \mathcal{L}_d$ be a genus of even positive-definite lattices of rank 2, and let $\widetilde{\mathcal{G}} \subset \widetilde{\mathcal{L}}_d$ be the pull-back of $\mathcal{G}$ by the natural projection $\widetilde{\mathcal{L}}_d \to \mathcal{L}_d$. Then there exists a singular $K3$ surface X defined over a number field F with $d(X) = d$ such that the set*

$$\{ \, [\widetilde{T}(X^\sigma)] \; | \; \sigma \in \mathrm{Emb}(F, \mathbb{C}) \, \} \; \subset \; \widetilde{\mathcal{L}}_d$$

coincides with the oriented genus $\widetilde{\mathcal{G}}$.

Corollary 3.5. *Complex singular $K3$ surfaces S and S' are conjugate if and only if $T(S)$ and $T(S')$ are in the same genus.*

Proof. The "only if" part is proved in the same way as the proof of Proposition 3.1. Suppose that $T(S)$ and $T(S')$ are in the same genus. Let $\widetilde{\mathcal{G}}_S \subset \widetilde{\mathcal{L}}_{d(S)}$ be the oriented genus containing $[\widetilde{T}(S)] \in \widetilde{\mathcal{L}}_{d(S)}$, and let X be the singular $K3$ surface defined over a number field F such that

$$\{ \, [\widetilde{T}(X^\sigma)] \; | \; \sigma \in \mathrm{Emb}(F, \mathbb{C}) \, \} \; = \; \widetilde{\mathcal{G}}_S. \tag{3.1}$$

By the assumption, we have $[\widetilde{T}(S')] \in \widetilde{\mathcal{G}}_S$. By the injectivity of the theorem of Shioda-Inose, there exist $\tau \in \mathrm{Emb}(F, \mathbb{C})$ and $\tau' \in \mathrm{Emb}(F, \mathbb{C})$ such that $X^\tau \cong S$ and $X^{\tau'} \cong S'$. There exists $\sigma : \mathbb{C} \hookrightarrow \mathbb{C}$ such that $\sigma \circ \tau = \tau'$. □

Corollary 3.6. *Let S be a complex singular $K3$ surface, and let $\widetilde{\mathcal{G}}_S \subset \widetilde{\mathcal{L}}_{d(S)}$ be the oriented genus containing $[\widetilde{T}(S)] \in \widetilde{\mathcal{L}}_{d(S)}$. If S is defined over a number field L, then $[L : \mathbb{Q}] \geq |\widetilde{\mathcal{G}}_S|$.*

Proof. Let X be a $K3$ surface defined over a number field F such that (3.1) holds. Then $X^{\sigma_0} \cong S$ for some $\sigma_0 \in \mathrm{Emb}(F, \mathbb{C})$. Let Y be a $K3$ surface defined over L such that $Y^{\tau_0} \cong S$ for some $\tau_0 \in \mathrm{Emb}(L, \mathbb{C})$. Then there exists a number field $M \subset \mathbb{C}$ containing both of $\sigma_0(F)$ and $\tau_0(L)$ such that

$$X \otimes M \cong Y \otimes M \quad \text{ over } M.$$

Therefore, for each $\sigma \in \mathrm{Emb}(F, \mathbb{C})$, there exists $\tau \in \mathrm{Emb}(L, \mathbb{C})$ such that $X^\sigma \cong Y^\tau$ over $\mathbb{C}$. Since there exist exactly $|\widetilde{\mathcal{G}}_S|$ isomorphism classes of complex $K3$ surfaces among X^σ, we have $|\,\mathrm{Emb}(L, \mathbb{C})| \geq |\widetilde{\mathcal{G}}_S|$. □

Corollary 3.7. *Let S and S' be complex singular $K3$ surfaces. If* $\mathrm{NS}(S)$ *and* $\mathrm{NS}(S')$ *are in the same genus, then* $\mathrm{NS}(S)$ *and* $\mathrm{NS}(S')$ *are isomorphic.*

Proof. If $\mathrm{NS}(S)$ and $\mathrm{NS}(S')$ are in the same genus, then $T(S)$ and $T(S')$ are in the same genus, and hence S and S' are conjugate. □

4. Supersingular reduction lattices

Definition 4.1. Let Y be a supersingular $K3$ surface in characteristic p. Artin [4] and Rudakov-Shafarevich [20] showed that there exists a positive integer $\sigma(Y) \leq 10$ such that $d(Y) := \mathrm{disc}(\mathrm{NS}(Y))$ is written as $-p^{2\sigma(Y)}$. This integer $\sigma(Y)$ is called the *Artin invariant* of Y.

We describe the Néron-Severi lattice of a supersingular $K3$ surface in *odd* characteristic $p > 0$. In [20], Rudakov-Shafarevich showed the following:

Theorem 4.2. *Let p be an odd prime, and let σ be a positive integer ≤ 10. Then there exists a lattice $\Lambda_{p,\sigma}$ of rank 22 with the following properties, and it is unique up to isomorphism:* (i) *even,* (ii) *of signature* $(1, 21)$, *and* (iii) *the discriminant group is isomorphic to* $(\mathbb{Z}/p\mathbb{Z})^{2\sigma}$.

Definition 4.3. We call $\Lambda_{p,\sigma}$ the *Rudakov-Shafarevich lattice.*

Let $\chi_p : \mathbb{F}_p^\times \to \{\pm 1\}$ be the Legendre character.

Remark 4.4. The discriminant form of $\Lambda_{p,\sigma}$ is calculated in [24]. For an odd prime p, let v_p be an *even* integer such that $\chi_p(v_p) = -1$. Let $\langle \gamma \rangle$ be the cyclic group of order p generated by γ. We define quadratic forms

$$q_1 : \langle \gamma \rangle \to \mathbb{Q}/2\mathbb{Z} \quad \text{and} \quad q_v : \langle \gamma \rangle \to \mathbb{Q}/2\mathbb{Z}$$

by $q_1(\gamma) := (p+1)/p$ and $q_v(\gamma) := v_p/p$. Then the discriminant form $(D_{p,\sigma}, q_{p,\sigma})$ of $\Lambda_{p,\sigma}$ for an odd prime p is isomorphic to

$$\begin{cases} (\langle \gamma \rangle, q_1)^{\oplus 2\sigma} & \text{if } \sigma(p-1) \equiv 2 \bmod 4, \\ (\langle \gamma \rangle, q_1)^{\oplus (2\sigma - 1)} \oplus (\langle \gamma \rangle, q_v) & \text{if } \sigma(p-1) \equiv 0 \bmod 4. \end{cases}$$

Artin [4] and Rudakov-Shafarevich [20] showed the following:

Theorem 4.5. *Let Y be a supersingular $K3$ surface in odd characteristic p with the Artin invariant σ. Then $\mathrm{NS}(Y)$ is isomorphic to $\Lambda_{p,\sigma}$.*

We fix a smooth proper family $\mathcal{X} \to U$ of $K3$ surfaces over an open subset $U \subset \operatorname{Spec} \mathbb{Z}_F$ such that the generic fiber X_η is singular, and investigate the set

$$\mathcal{S}_p(\mathcal{X}) := \{\, \mathfrak{p} \in \pi_F^{-1}(p) \cap U \mid X_\mathfrak{p} \text{ is supersingular} \,\}.$$

In [24] and [25], we have obtained the following:

Theorem 4.6. *Suppose that p does not divide $2d(X_\eta) = 2\operatorname{disc}(\mathrm{NS}(X_\eta))$.*

(1) *If $\mathfrak{p} \in \mathcal{S}_p(\mathcal{X})$, then the Artin invariant of $X_\mathfrak{p}$ is 1.*

(2) *There exists a finite set N of prime integers containing the prime divisors of $2d(X_\eta)$ such that*

$$p \notin N \quad \Rightarrow \quad \mathcal{S}_p(\mathcal{X}) = \begin{cases} \emptyset & \text{if } \chi_p(d(X_\eta)) = 1, \\ \pi_F^{-1}(p) & \text{if } \chi_p(d(X_\eta)) = -1. \end{cases}$$

Recall that the supersingular reduction lattice $L(\mathcal{X}, \mathfrak{p})$ of $\mathcal{X}$ at $\mathfrak{p} \in \mathcal{S}_p(\mathcal{X})$ is defined to be $(\mathrm{NS}(X_\eta) \hookrightarrow \mathrm{NS}(X_\mathfrak{p}))^\perp$. If $p \not| 2d(X_\eta)$ and $\mathfrak{p} \in \mathcal{S}_p(\mathcal{X})$, then the Artin invariant of $X_\mathfrak{p}$ is 1, and hence

$$\mathrm{NS}(X_\mathfrak{p}) \cong \Lambda_{p,1}.$$

Using the standard technique of [15, Exp. XI], we have obtained the following in [25]:

Proposition 4.7. *Suppose that $p \not| 2d(X_\eta)$, and let $\mathfrak{p}$ be a point of $\mathcal{S}_p(\mathcal{X})$. Then the image of the specialization isometry $\mathrm{NS}(X_\eta) \hookrightarrow \mathrm{NS}(X_\mathfrak{p})$ is primitive.*

Combining Proposition 4.7 and Theorem 2.4, we obtain the following:

Corollary 4.8. *Suppose that $p \not| 2d(X_\eta)$, and let $\mathfrak{p}$ be a point of $\mathcal{S}_p(\mathcal{X})$. Then $L(\mathcal{X}, \mathfrak{p})$ is an even, negative-definite lattice of rank 2 with discriminant $-p^2 d(X_\eta)$, and its discriminant form is isomorphic to*

$$(D_{\mathrm{NS}}, -q_{\mathrm{NS}}) \oplus (D_{p,1}, q_{p,1}) \quad \cong \quad (D_T, q_T) \oplus (D_{p,1}, q_{p,1}),$$

where $\mathrm{NS} = \mathrm{NS}(X_\eta)$, $T = T(X_\eta^\sigma)$ for any $\sigma \in \mathrm{Emb}(F, \mathbb{C})$, and $(D_{p,1}, q_{p,1})$ is the discriminant form of the Rudakov-Shafarevich lattice $\Lambda_{p,1}$.

Definition 4.9. For any $[T] \in \mathcal{L}_d$ and a prime integer $p \not| 2d$, we denote by

$$\mathcal{G}(p, T) \ \subset \ \mathcal{L}_{p^2 d}[-1] := \{-M \mid M \in \mathcal{M}_{p^2 d}\} /\ GL(2, \mathbb{Z})$$

the genus consisting of even, negative-definite lattices of rank 2 whose discriminant form is isomorphic to $(D_T, q_T) \oplus (D_{p,1}, q_{p,1})$.

In fact, the genus $\mathcal{G}(p, T)$ depends only on the genus containing $[T]$. By Theorem 2.3, we have the following:

Corollary 4.10. *Suppose that $p \not| 2d(X_\eta)$. Then $L(\mathcal{X}, \mathfrak{p})$ is contained in the genus $\mathcal{G}(p, T(X_\eta^\sigma))$ for any $\mathfrak{p} \in \mathcal{S}_p(\mathcal{X})$.*

In view of Theorem 3.4, it is natural to raise the following:

Problem 4.11. For a given $[T] \in \mathcal{L}_d$, does there exist a smooth proper family $\mathcal{X} \to U$ of $K3$ surfaces over an open subset $U \subset \operatorname{Spec} \mathbb{Z}_F$ with the following properties?

- (i) $(D_{\mathrm{NS}(X_\eta)}, q_{\mathrm{NS}(X_\eta)}) \cong (D_T, -q_T)$, and
- (ii) except for a finite number of primes, if $\chi_p(d) = -1$, then the set of isomorphism classes $[L(\mathcal{X}, \mathfrak{p})]$, where $\mathfrak{p}$ runs through $\mathcal{S}_p(\mathcal{X}) = \pi_F^{-1}(p)$, coincides with the genus $\mathcal{G}(p, T)$.

In [25], we have proved a partial affirmative answer to this problem.

Definition 4.12. A negative integer d is called a *fundamental discriminant* if it is the discriminant of an imaginary quadratic field.

Definition 4.13. An even lattice of rank 2 is said to be *primitive* if it is expressed by a matrix

$$\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \quad \text{with} \quad \gcd(a, b, c) = 1.$$

Theorem 4.14. *Let d be a negative integer, and let T be an even positive-definite lattice of rank 2 with discriminant $-d$. Assume the following:*

- *d is odd,*
- *d is a fundamental discriminant, and*

- *T is primitive.*

Then there exists a smooth proper family of $K3$ surfaces $\mathcal{X} \to U$ over an open subset $U \subset \operatorname{Spec} \mathbb{Z}_F$, where F is a number field, such that

(i) $(D_{\mathrm{NS}(X_\eta)}, q_{\mathrm{NS}(X_\eta)}) \cong (D_T, -q_T)$, *and*
(ii) *except for a finite number of primes, if $\chi_p(d) = -1$, then the set*

$$\{ \, [L(\mathcal{X}, \mathfrak{p})] \;\mid\; \mathfrak{p} \in \mathcal{S}_p(\mathcal{X}) = \pi_F^{-1}(p) \, \}$$

of isomorphism classes of supersingular reduction lattices at the points of $\mathcal{S}_p(\mathcal{X}) = \pi_F^{-1}(p)$ coincides with the genus $\mathcal{G}(p, T)$.

Remark 4.15. The author proved Theorem 3.4 in [25] under the assumption that d be a fundamental discriminant, and that T be primitive. Then Schütt [21] removed these assumptions.

5. The theory of Shioda, Mitani and Inose

We give a sketch of the proof of Theorems 3.4 and 4.14.

Suppose that a matrix

$$\widetilde{T} = \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \quad \text{with} \quad a, b, c \in \mathbb{Z}, \;\; a > 0, \;\; c > 0, \;\; d := b^2 - 4ac < 0,$$

is given. Let $\sqrt{d} \in \mathbb{C}$ be in the upper-half plane. We consider elliptic curves

$$E' := \mathbb{C}/(\mathbb{Z} + \tau'\mathbb{Z}) \quad \text{and} \quad E := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}),$$

where $\tau' = \dfrac{-b + \sqrt{d}}{2a}$ and $\tau = \dfrac{b + \sqrt{d}}{2}$. Shioda and Mitani [29] showed the following:

Theorem 5.1. *The oriented transcendental lattice $\widetilde{T}(A)$ of the abelian surface*

$$A := E' \times E.$$

is expressed by the given matrix $\widetilde{T}$.

We consider the *Kummer diagram*

$$\mathrm{Km}(A) \;\longleftarrow\; \widetilde{A} \;\longrightarrow\; A,$$

where $\widetilde{A} \to A$ is the blowing-up of A at the 2-torsion points, and $\mathrm{Km}(A) \leftarrow \widetilde{A}$ is the quotient by the lift of the inversion of A. Shioda and Inose [27] showed that, on the Kummer surface $\mathrm{Km}(A)$, there exist reduced effective divisors C and Θ such that

(i) C and Θ are disjoint,
(ii) C is an ADE-configuration of (-2)-curves $C_1, \dots, C_8$ of type $\mathbb{E}_8$,
(iii) Θ is an ADE-configuration of (-2)-curves $\Theta_1, \dots, \Theta_8$ of type $8\mathbb{A}_1$, and
(iv) there exists a class $[\mathcal{L}] \in \mathrm{NS}(\mathrm{Km}(A))$ such that $2[\mathcal{L}] = [\Theta]$.

We consider the *Shioda-Inose diagram*

$$Y \longleftarrow \widetilde{Y} \longrightarrow \mathrm{Km}(A),$$

where $\widetilde{Y} \to \mathrm{Km}(A)$ is the double covering branched exactly along Θ, and $Y \leftarrow \widetilde{Y}$ is the contraction of the (-1)-curves on $\widetilde{Y}$ (that is, the inverse images of $\Theta_1, \dots, \Theta_8$). Shioda and Inose [27] proved the following:

Theorem 5.2. *The surface Y is a singular $K3$ surface, and the diagram*

$$Y \longleftarrow \widetilde{Y} \longrightarrow \mathrm{Km}(A) \longleftarrow \widetilde{A} \longrightarrow A$$

induces an isomorphism

$$\widetilde{T}(Y) \cong \widetilde{T}(A) \ (\cong \widetilde{T} \ )$$

of the oriented transcendental lattices.

Suppose that we have a Shioda-Inose-Kummer diagram

$$\mathcal{Y} \longleftarrow \widetilde{\mathcal{Y}} \longrightarrow \mathrm{Km}(\mathcal{A}) \longleftarrow \widetilde{\mathcal{A}} \longrightarrow \mathcal{A} = \mathcal{E}' \times \mathcal{E}$$

over an open subset U of $\operatorname{Spec} \mathbb{Z}_F$, where F is a number field. We denote by

$$Y_\eta \longleftarrow \widetilde{Y}_\eta \longrightarrow \mathrm{Km}(A_\eta) \longleftarrow \widetilde{A}_\eta \longrightarrow A_\eta = E'_\eta \times E_\eta$$

the generic fiber of the diagram. For a closed point $\mathfrak{p} \in U$, we denote by

$$Y_\mathfrak{p} \longleftarrow \widetilde{Y}_\mathfrak{p} \longrightarrow \mathrm{Km}(A_\mathfrak{p}) \longleftarrow \widetilde{A}_\mathfrak{p} \longrightarrow A_\mathfrak{p} = E'_\mathfrak{p} \times E_\mathfrak{p}$$

the fiber over $\mathfrak{p}$ of the diagram.

Analyzing the arguments of Shioda and Inose, we obtain the following theorems.

Theorem 5.3. *The above diagram over η induces an isomorphism $\widetilde{T}(Y_\eta^\sigma) \cong \widetilde{T}(A_\eta^\sigma)$ for any $\sigma \in \mathrm{Emb}(F, \mathbb{C})$.*

Definition 5.4. For elliptic curves E_1, E_2 defined over a field k, we denote by $\mathrm{Hom}(E_1, E_2)$ the $\mathbb{Z}$-module of homomorphisms

$$\phi : E_1 \otimes \bar{k} \to E_2 \otimes \bar{k},$$

and we regard $\mathrm{Hom}(E_1, E_2)$ as a lattice by

$$(\phi, \phi) := 2 \deg \phi.$$

Theorem 5.5. *Except for a finite number of closed points $\mathfrak{p}$ of U, we have*

$$Y_\mathfrak{p} \textit{ is supersingular } \iff E'_\mathfrak{p} \textit{ and } E_\mathfrak{p} \textit{ are supersingular},$$

and if this is the case, then the above diagram over $\mathfrak{p}$ induces an isomorphism

$$L(\mathcal{Y}, \mathfrak{p}) \ \cong \ (\mathrm{Hom}(E'_\eta, E_\eta) \hookrightarrow \mathrm{Hom}(E'_\mathfrak{p}, E_\mathfrak{p}))^\perp[-1],$$

where $\mathrm{Hom}(E'_\eta, E_\eta) \hookrightarrow \mathrm{Hom}(E'_\mathfrak{p}, E_\mathfrak{p})$ is the specialization isometry.

Thus Theorems 3.4 and 4.14 are reduced to the statements about elliptic curves. The lattices $\widetilde{T}(A^{\sigma}_{\eta}) = \widetilde{T}({E'_{\eta}}^{\sigma} \times {E_{\eta}}^{\sigma})$ for $\sigma \in \mathrm{Emb}(F, \mathbb{C})$) are calculated by the classical theory of complex multiplications in the class field theory ([18], [30]). The lattices

$$(\mathrm{Hom}(E'_{\eta}, E_{\eta}) \hookrightarrow \mathrm{Hom}({E'}_{\mathfrak{p}}, E_{\mathfrak{p}}))^{\perp}$$

are calculated by Deuring's theory [10] of endmorphism rings of supersingular elliptic curves. We use Dorman's description [11] of optimal embeddings of the integer ring of an imaginary quadratic fields into the Deuring order.

6. An application to topology

It is obvious from the definition that conjugate complex varieties are homeomorphic in Zariski topology. On the other hand, for the complex topology, we have the following classical example by Serre [22].

Example 6.1. There exist conjugate complex smooth projective varieties X and X^{σ} such that their topological fundamental groups are *not* isomorphic. In particular, X and X^{σ} are not homotopically equivalent.

We also have Grothendieck's *dessins d'enfant* ([16], [17]).

Example 6.2. Let $f : C \to \mathbb{P}^1$ be a finite covering defined over $\overline{\mathbb{Q}} \subset \mathbb{C}$ branching only at $0, 1, \infty \in \mathbb{P}^1$. For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, consider the conjugate covering

$$f^{\sigma} : C^{\sigma} \to \mathbb{P}^1.$$

Then f and f^{σ} are topologically distinct in general. Belyi's theorem asserts that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of topological types of the coverings of $\mathbb{P}^1$ branching only at $0, 1, \infty$ is faithful.

See Abelson [1], Artal, Carmona and Cogolludo [3], Easton and Vakil [12], Bauer, Catanese and Grunewald [5] and Charles [8] for other examples. Using Corollary 3.5, we also have obtained simple and explicit examples of non-homeomorphic conjugate complex varieties in [23] and [26]. (Note that, except for [22] and [1], all these papers have appeared quite recently.)

We present our construction of examples in [23] and [26]. Let V be an oriented topological manifold of real dimension 4. We put

$$\mathrm{H}_2(V) := \mathrm{H}_2(V, \mathbb{Z})/\mathrm{torsion},$$

on which we have the intersection pairing

$$\iota_V \;:\; \mathrm{H}_2(V) \times \mathrm{H}_2(V) \;\to\; \mathbb{Z}.$$

We then put

$$J_{\infty}(V) \;:=\; \bigcap_{K} \mathrm{Im}(\mathrm{H}_2(V \setminus K) \to \mathrm{H}_2(V)),$$

where K runs through the set of compact subsets of V, and set

$$\widetilde{B}_V := \mathrm{H}_2(V)/J_\infty(V) \quad \text{and} \quad B_V := (\widetilde{B}_V)/\text{torsion}.$$

Since any topological cycle is compact, the intersection pairing ι_V induces a symmetric bilinear form

$$\beta_V \,:\, B_V \times B_V \,\to\, \mathbb{Z}.$$

It is obvious that the isomorphism class of (B_V, β_V) is a topological invariant of V.

Theorem 6.3. *Let S be a complex smooth projective surface, and let $C_1, \ldots, C_n$ be irreducible curves on S. We put*

$$V := S \setminus \bigcup C_i.$$

Suppose that the classes $[C_1], \ldots, [C_n]$ span $\mathrm{NS}(S) \otimes \mathbb{Q}$. Then (B_V, β_V) is isomorphic to the transcendental lattice

$$T(S) := (\mathrm{NS}(S) \hookrightarrow \mathrm{H}^2(S, \mathbb{Z}))^\perp/\text{torsion}.$$

Using Corollary 3.5 and Theorem 6.3, we obtain the following examples of non-homeomorphic conjugate complex varieties.

Example 6.4. Let T_1 and T_2 be even positive-definite lattices of rank 2 that are in the same genus but not isomorphic. We have a singular $K3$ surface X defined over a number field F and two embeddings $\sigma_1, \sigma_2 \in \mathrm{Emb}(F, \mathbb{C})$ such that

$$T(X^{\sigma_1}) \cong T_1 \quad \text{and} \quad T(X^{\sigma_2}) \cong T_2.$$

Let $C_1, \ldots, C_n$ be irreducible curves on X whose classes span $\mathrm{NS}(X) \otimes \mathbb{Q}$. Enlarging F, we can assume that the Zariski open subset $V := X \setminus \bigcup C_i$ of X is also defined over F. Then the conjugate open varieties V^{σ_1} and V^{σ_2} are not homeomorphic.

Definition 6.5. A pair $[C, C']$ of complex projective plane curves is said to be an *arithmetic Zariski pair* if the following hold:

(i) Suppose that $C = \{\Phi = 0\}$, where Φ is a homogeneous polynomial in three variables. Then there exists $\sigma \in \mathrm{Emb}(\mathbb{C}, \mathbb{C})$ such that $C' \subset \mathbb{P}^2$ is projectively isomorphic to the plane curve $C^\sigma := \{\Phi^\sigma = 0\}$.

(ii) There exist tubular neighborhoods $\mathcal{T} \subset \mathbb{P}^2$ of C and $\mathcal{T}' \subset \mathbb{P}^2$ of C' such that $(\mathcal{T}, C)$ and $(\mathcal{T}', C')$ are diffeomorphic, while $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are *not* homeomorphic.

Remark 6.6. The first example of an arithmetic Zariski pair was discovered by Artal, Carmona, Cogolludo [3] in degree 12. They used the invariant of *braid monodromies* in order to distinguish $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ topologically.

Definition 6.7. A complex plane curve $C \subset \mathbb{P}^2$ of degree 6 is called a *maximizing sextic* if C has only simple singularities and the total Milnor number of C attains the possible maximum 19.

If C is a maximizing sextic, the minimal resolution $X_C \to Y_C$ of the double cover $Y_C \to \mathbb{P}^2$ branching exactly along C is a singular $K3$ surface. We denote by $T[C]$ the transcendental lattice of X_C.

Remark 6.8. Using Urabe's idea [31], Yang [32] has made the complete list of all possible ADE-configurations of singular points of sextic curves with only simple singularities. Recently, Degtyarev [9] has described the connected components of the equisingular family of sextic curves with only simple singularities of a given ADE-configuration.

Example 6.9. In the following example, we employ a calculation of Artal, Carmona and Cogolludo in [2]. We consider the following cubic extension of $\mathbb{Q}$:

$$K := \mathbb{Q}[t]/(\varphi), \quad \text{where} \quad \varphi = 17t^3 - 18t^2 - 228t + 556.$$

The roots of $\varphi = 0$ are $\alpha, \bar{\alpha}, \beta$, where

$$\alpha = 2.590\cdots + 1.108\cdots\sqrt{-1}, \qquad \beta = -4.121\cdots.$$

There are three corresponding embeddings

$$\sigma_\alpha : K \hookrightarrow \mathbb{C}, \quad \sigma_{\bar{\alpha}} : K \hookrightarrow \mathbb{C} \quad \text{and} \quad \sigma_\beta : K \hookrightarrow \mathbb{C}.$$

There exists a homogeneous polynomial $\Phi(x_0, x_1, x_2)$ of degree 6 with coefficients in K such that the plane curve $C = \{\Phi = 0\}$ has three simple singular points of type $A_{16} + A_2 + A_1$ as its only singularities. Consider the conjugate plane curves

$$C_\alpha = \{\Phi^{\sigma_\alpha} = 0\}, \;\; C_{\bar{\alpha}} = \{\Phi^{\sigma_{\bar{\alpha}}} = 0\} \quad \text{and} \quad C_\beta = \{\Phi^{\sigma_\beta} = 0\}.$$

Artal, Carmona and Cogolludo showed that, if $C' \subset \mathbb{P}^2$ is a complex projective plane curve possessing $A_{16} + A_2 + A_1$ as its only singularities, then C' is projectively isomorphic to C_α, $C_{\bar{\alpha}}$ or C_β.

On the other hand, by the surjectivity of the period map for complex $K3$ surfaces, we can prove that there are exactly three singular $K3$ surfaces (up to isomorphism) that is a double cover of $\mathbb{P}^2$ with a sextic branch curve possessing $A_{16} + A_2 + A_1$ as its only singularities. Their oriented transcendental lattices are

$$[10, \pm 4, 22] := \begin{bmatrix} 10 & \pm 4 \\ \pm 4 & 22 \end{bmatrix} \quad \text{and} \quad [6, 0, 34] := \begin{bmatrix} 6 & 0 \\ 0 & 34 \end{bmatrix},$$

which are in the same genus. The non-oriented lattices $[10, 4, 22]$ and $[10, -4, 22]$ are isomorphic, while the non-oriented lattices $[10, 4, 22]$ and $[6, 0, 34]$ are not isomorphic. Therefore we have

$$T[C_\alpha] \cong [10, 4, 22] \text{ or } [10, -4, 22] \quad \text{and} \quad T[C_\beta] \cong [6, 0, 34].$$

(The homeomorphism $(\mathbb{P}^2, C_\alpha) \cong (\mathbb{P}^2, C_{\bar{\alpha}})$ induced by the complex conjugate corresponds to the orientation reversing of the transcendental lattices.) Let $V \subset Y_C$ be the pull-back of $\mathbb{P}^2 \setminus C$ by $Y_C \to \mathbb{P}^2$, which is a smooth open surface defined over K.

Then the conjugate varieties V^{σ_α} and V^{σ_β} are not homeomorphic. Hence the pair $[C_\alpha, C_\beta]$ is an arithmetic Zariski pair.

By the same method, we have found examples of arithmetic Zariski pair of maximizing sextics listed in the table below.

No.	$\mathrm{Sing}(C) = \mathrm{Sing}(C')$	$T[C]$ and $T[C']$ (non-oriented)	
1	$E_8 + A_{10} + A_1$	$[6, 2, 8]$,	$[2, 0, 22]$
2	$E_8 + A_6 + A_4 + A_1$	$[8, 2, 18]$,	$[2, 0, 70]$
3	$E_6 + D_5 + A_6 + A_2$	$[12, 0, 42]$,	$[6, 0, 84]$
4	$E_6 + A_{10} + A_3$	$[12, 0, 22]$,	$[4, 0, 66]$
5	$E_6 + A_{10} + A_2 + A_1$	$[18, 6, 24]$,	$[6, 0, 66]$
6	$E_6 + A_7 + A_4 + A_2$	$[24, 0, 30]$,	$[6, 0, 120]$
7	$E_6 + A_6 + A_4 + A_2 + A_1$	$[30, 0, 42]$,	$[18, 6, 72]$
8	$D_8 + A_{10} + A_1$	$[6, 2, 8]$,	$[2, 0, 22]$
9	$D_8 + A_6 + A_4 + A_1$	$[8, 2, 18]$,	$[2, 0, 70]$
10	$D_7 + A_{12}$	$[6, 2, 18]$,	$[2, 0, 52]$
11	$D_7 + A_8 + A_4$	$[18, 0, 20]$,	$[2, 0, 180]$
12	$D_5 + A_{10} + A_4$	$[20, 0, 22]$,	$[12, 4, 38]$
13	$D_5 + A_6 + A_5 + A_2 + A_1$	$[12, 0, 42]$,	$[6, 0, 84]$
14	$D_5 + A_6 + 2A_4$	$[20, 0, 70]$,	$[10, 0, 140]$
15	$A_{18} + A_1$	$[8, 2, 10]$,	$[2, 0, 38]$
16	$A_{16} + A_3$	$[4, 0, 34]$,	$[2, 0, 68]$
17	$A_{16} + A_2 + A_1$	$[10, 4, 22]$,	$[6, 0, 34]$
18	$A_{13} + A_4 + 2A_1$	$[8, 2, 18]$,	$[2, 0, 70]$
19	$A_{12} + A_6 + A_1$	$[8, 2, 46]$,	$[2, 0, 182]$
20	$A_{12} + A_5 + 2A_1$	$[12, 6, 16]$,	$[4, 2, 40]$
21	$A_{12} + A_4 + A_2 + A_1$	$[24, 6, 34]$,	$[6, 0, 130]$
22	$A_{10} + A_9$	$[10, 0, 22]$,	$[2, 0, 110]$
23	$A_{10} + A_9$	$[8, 3, 8]$,	$[2, 1, 28]$
24	$A_{10} + A_8 + A_1$	$[18, 0, 22]$,	$[10, 2, 40]$
25	$A_{10} + A_7 + A_2$	$[22, 0, 24]$,	$[6, 0, 88]$
26	$A_{10} + A_7 + 2A_1$	$[10, 2, 18]$,	$[2, 0, 88]$
27	$A_{10} + A_6 + A_2 + A_1$	$[22, 0, 42]$,	$[16, 2, 58]$
28	$A_{10} + A_5 + A_3 + A_1$	$[12, 0, 22]$,	$[4, 0, 66]$
29	$A_{10} + 2A_4 + A_1$	$[30, 10, 40]$,	$[10, 0, 110]$
30	$A_{10} + A_4 + 2A_2 + A_1$	$[30, 0, 66]$,	$[6, 0, 330]$
31	$A_8 + A_6 + A_4 + A_1$	$[22, 4, 58]$,	$[18, 0, 70]$
32	$A_7 + A_6 + A_4 + A_2$	$[24, 0, 70]$,	$[6, 0, 280]$
33	$A_7 + A_6 + A_4 + 2A_1$	$[18, 4, 32]$,	$[2, 0, 280]$
34	$A_7 + A_5 + A_4 + A_2 + A_1$	$[24, 0, 30]$,	$[6, 0, 120]$

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