Moduli of supersingular K3 surfaces in characteristic 2

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§2. Stratification by codes
§3. Geometry of splitting curves and codes
§4. The case of Artin invariant 2
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We work over an algebraically closed field $k$ of characteristic 2.
§1. Construction of the Moduli Space

Let $X$ be a supersingular $K3$ surface.

Let $\mathcal{L}$ be a line bundle on $X$ with $\mathcal{L}^2 = 2$. We say that $\mathcal{L}$ is a polarization of type (♯) if the following conditions are satisfied:

- the complete linear system $|\mathcal{L}|$ has no fixed components, and
- the set of curves contracted by the morphism $\Phi_{|\mathcal{L}|} : X \to \mathbb{P}^2$

defined by $|\mathcal{L}|$ consists of 21 disjoint $(-2)$-curves.

If $(X, \mathcal{L})$ is a polarized supersingular $K3$ surface of type (♯), then $\Phi_{|\mathcal{L}|} : X \to \mathbb{P}^2$ is purely inseparable.

Every supersingular $K3$ surface has a polarization of type (♯).

We will construct the moduli space $\mathcal{M}$ of polarized supersingular $K3$ surfaces of type (♯).
Let $G = G(X_0, X_1, X_2)$ be a non-zero homogeneous polynomial of degree 6.

We can define

$$dG \in \Gamma(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1(6)),$$

because we are in characteristic 2 and we have $\mathcal{O}_{\mathbb{P}^2}(6) \cong \mathcal{O}_{\mathbb{P}^2}(3)^{\otimes 2}$.

We put

$$Z(dG) := \{dG = 0\} = \left\{ \frac{\partial G}{\partial X_0} = \frac{\partial G}{\partial X_1} = \frac{\partial G}{\partial X_2} = 0 \right\} \subset \mathbb{P}^2.$$

If $\dim Z(dG) = 0$, then

$$\text{length } \mathcal{O}_{Z(dG)} = c_2(\Omega_{\mathbb{P}^2}^1(6)) = 21.$$
We then put
\[ \mathcal{L}_G := (\pi_G \circ \rho_G)^* \mathcal{O}_{\mathbb{P}^2}(1). \]

\((X, \mathcal{L})\) is a polarized supersingular \(K3\) surface of type (♯)
\[ \Downarrow \]
there exists \(G \in \mathcal{U}\) such that \((X, \mathcal{L}) \cong (X_G, \mathcal{L}_G)\)

We put
\[ \mathcal{V} := H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)). \]
Because we have \(d(G + H^2) = dG\) for \(H \in \mathcal{V}\), the additive group \(\mathcal{V}\) acts on the space \(\mathcal{U}\) by
\[ (G, H) \in \mathcal{U} \times \mathcal{V} \mapsto G + H^2 \in \mathcal{U}. \]
Let \(G\) and \(G'\) be homogeneous polynomials in \(\mathcal{U}\).
Then the following conditions are equivalent:

(i) \(Y_G\) and \(Y_{G'}\) are isomorphic over \(\mathbb{P}^2\),
(ii) \(Z(dG) = Z(dG')\), and
(iii) there exist \(c \in k^\times\) and \(H \in \mathcal{V}\) such that \(G' = cG + H^2\).
Therefore the moduli space $\mathcal{M}$ of polarized supersingular K3 surfaces of type ($\sharp$) is constructed by

$$\mathcal{M} = \text{PGL}(3, k) \backslash \mathbb{P}^*(\mathcal{U}/\mathcal{V}).$$

We put

$$\mathcal{P} := \{P_1, \ldots, P_{21}\},$$
on which the full symmetric group $S_{21}$ acts from left.

We denote by $\mathcal{G}$ the space of all injective maps

$$\gamma : \mathcal{P} \hookrightarrow \mathbb{P}^2$$
such that there exists $G \in \mathcal{U}$ satisfying $\gamma(\mathcal{P}) = Z(dG)$.

Then we can construct $\mathcal{M}$ by

$$\mathcal{M} = \text{PGL}(3, k) \backslash \mathcal{G}/S_{21}.$$

Example by Dolgachev-Kondo:

$$G_{\text{DK}} := X_0X_1X_2(X_0^3 + X_1^3 + X_2^3),$$

$$Z(dG_{\text{DK}}) = \mathbb{P}^2(\mathbb{P}^4).$$

The Artin invariant of the supersingular K3 surface $X_{G_{\text{DK}}}$ is 1.

$[G_{\text{DK}}] \in \mathcal{M}$: the Dolgachev-Kondo point.
§2. Stratification by Isomorphism Classes of Codes

Let $G$ be a polynomial in $\mathcal{U}$.

$\text{NS}(X_G)$: the Néron-Severi lattice of $X_G$,

$\text{disc } \text{NS}(X_G) = -2^{2\sigma(X_G)}$,

($\sigma(X_G)$ is the Artin invariant of $X_G$).

Let $\gamma: \mathcal{P} \hookrightarrow \mathbb{P}^2$ be an injective map such that

$\gamma(\mathcal{P}) = Z(dG) = \pi_G(\text{Sing } Y_G)$,

that is, $\gamma$ is a numbering of the singular points of $Y_G$.

$E_i \subset X_G$: the $(-2)$-curve that is contracted to $\gamma(P_i)$.

Then $\text{NS}(X_G)$ contains a sublattice

$$ S_0 = \langle [E_1], \ldots, [E_{21}], [\mathcal{L}_G] \rangle = \begin{bmatrix} -2 & \quad & \quad \\ \quad & -2 & \quad \\ \quad & \quad & -2 \end{bmatrix}. $$

$$ S_0^\vee = \text{Hom}(S_0, \mathbb{Z}) = \langle [E_1]/2, \ldots, [E_{21}]/2, [\mathcal{L}_G]/2 \rangle \supset \text{NS}(X_G). $$
We put
\[ \tilde{C}_G := \text{NS}(X_G)/S_0 \subset S_0^\vee/S_0 = \mathbb{F}_2^{\oplus 21} \oplus \mathbb{F}_2, \]
\[ C_G := \text{pr}(\tilde{C}_G) \subset \mathbb{F}_2^{\oplus 21} \cong 2^\mathcal{P} (\text{the power set of } \mathcal{P}). \]
Here the identification \( \mathbb{F}_2^{\oplus 21} \cong 2^\mathcal{P} \) is given by
\[ v \mapsto \{ P_i \in \mathcal{P} \mid \text{the } i\text{-th coordinate of } v \text{ is } 1 \}. \]
We have
\[ \dim \tilde{C}_G = \dim C_G = 11 - \sigma(X_G). \]

We say that a reduced irreducible curve \( C \subset \mathbb{P}^2 \) \textit{splits in } \( X_G \) if the proper transform of \( C \) in \( X_G \) is non-reduced, that is, of the form \( 2F_C \), where \( F_C \subset X_G \) is a reduced curve in \( X_G \).

We say that a reduced curve \( C \subset \mathbb{P}^2 \) \textit{splits in } \( X_G \) if every irreducible component of \( C \) splits in \( X_G \).
\( C \subset \mathbb{P}^2 \) : a curve of degree \( d \) splitting in \( X_G \),

\( m_i(C) \) : the multiplicity of \( C \) at \( \gamma(P_i) \in Z(dG) \).

\[ [F_C] = \frac{1}{2} \left( d \cdot [\mathcal{L}_G] - \sum_{i=1}^{21} m_i(C)[E_i] \right) \in NS(X_G), \]

\( \tilde{w}(C) := [F_C] \mod S_0 \in \tilde{\mathcal{C}}_G = NS(X_G)/S_0, \)

\( w(C) := \text{pr}(\tilde{w}(C)) \)

\[ = \{ P_i \in \mathcal{P} \mid m_i(C) \text{ is odd} \} \in \mathcal{C}_G. \]

A general member \( Q \) of the linear system

\[ |\mathcal{I}_{Z(dG)}(5)| = \left\langle \frac{\partial G}{\partial X_0}, \frac{\partial G}{\partial X_1}, \frac{\partial G}{\partial X_2} \right\rangle \]

splits in \( X_G \).

In particular,

\( w(Q) = \mathcal{P} = (1, 1, \ldots, 1) \in \mathcal{C}_G. \)

What kind of codes can appear as \( \mathcal{C}_G \) for some \( G \in \mathcal{U} \)?
$NS(X_G)$ has the following properties;

- type II (that is, $v^2 \in \mathbb{Z}$ for any $v \in NS(X_G)\vee$),
- there are no $u \in NS(X_G)$ such that $u \cdot [L_G] = 1$ and $u^2 = 0$ (that is, $|L_G|$ is fixed component free), and
- if $u \in NS(X_G)$ satisfies $u \cdot [L_G] = 0$ and $u^2 = -2$, then $u = [E_i]$ or $-[E_i]$ for some $i$ (that is, Sing $Y_G$ consists of 21 ordinary nodes).

$C_G$ has the following properties;

- $\mathcal{P} = (1,1,\ldots,1) \in C_G$, and
- $|w| \in \{0,5,8,9,12,13,16,21\}$ for any $w \in C_G$.

The isomorphism classes $[\mathcal{C}]$ of codes $\mathcal{C} \subset \mathbb{F}_2^{\oplus 21} = 2^\mathcal{P}$ satisfying these conditions are classified:

- $\sigma = 11 - \dim \mathcal{C}$,
- $r(\sigma) = \text{the number of the isomorphism classes}$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r(\sigma)$</td>
<td>1</td>
<td>3</td>
<td>13</td>
<td>41</td>
<td>58</td>
<td>43</td>
<td>21</td>
<td>8</td>
<td>3</td>
<td>1</td>
<td>193</td>
</tr>
</tbody>
</table>
the isomorphism class of $(X_G, \mathcal{L}_G) \in \mathcal{M}_[C]$ 
\[ \iff C_G \in [C] \]

\[ \mathcal{M} = PGL(3, k) \backslash \mathbb{P}^*(U/V) = \bigsqcup \text{the isom. classes} \mathcal{M}_[C]. \]

Each $\mathcal{M}_[C]$ is non-empty.

\[ \dim \mathcal{M}_[C] = \sigma - 1 = 10 - \dim C. \]

Case of $\sigma = 1$.

There exists only one isomorphism class $[\mathcal{C}_{\text{DK}}]$ with dimension 10.

\[ \mathcal{P} \cong \mathbb{P}^2(F_4), \]

\[ \mathcal{C}_{\text{DK}} := \langle L(F_4) \mid L : F_4\text{-rational lines} \rangle \subset \mathcal{P}. \]

The weight enumerator of $\mathcal{C}_{\text{DK}}$ is

\[ 1 + 21z^5 + 210z^8 + 280z^9 + 280z^{12} + 210z^{13} + 21z^{16} + z^{21}. \]

The 0-dimensional stratum $\mathcal{M}_{\text{DK}}$ consists of a single point $[(X_{\text{DK}}, \mathcal{L}_{\text{DK}})]$, where $X_{\text{DK}}$ is the resolution of

\[ W^2 = X_0X_1X_2(X_0^3 + X_1^3 + X_2^3). \]
§3. Geometry of Splitting Curves and Codes

$G \in \mathcal{U}$. We fix a bijection

$$\gamma : \mathcal{P} \sim Z(dG) = \pi_G(\text{Sing } Y_G).$$

Let $L \subset \mathbb{P}^2$ be a line.

$L$ splits in $(X_G, \mathcal{L}_G)$,

$$\iff |L \cap Z(dG)| \geq 3,$$

$$\iff |L \cap Z(dG)| = 5.$$

Let $Q \subset \mathbb{P}^2$ be a non-singular conic curve.

$Q$ splits in $(X_G, \mathcal{L}_G)$,

$$\iff |Q \cap Z(dG)| \geq 6,$$

$$\iff |Q \cap Z(dG)| = 8.$$

The word $w(L) = \gamma^{-1}(L \cap Z(dG))$ of a splitting line $L$ is of weight 5.

The word $w(Q) = \gamma^{-1}(Q \cap Z(dG))$ of a splitting non-singular conic curve $Q$ is of weight 8.
A pencil $E$ of cubic curves in $\mathbb{P}^2$ is called a regular pencil if the following hold:

- the base locus $\text{Bs}(E)$ consists of distinct 9 points, and
- every singular member has only one ordinary node.

We say that a regular pencil $E$ splits in $(X_G, \mathcal{L}_G)$ if every member of $E$ splits in $(X_G, \mathcal{L}_G)$.

Let $E$ be a regular pencil of cubic curves spanned by $E_0$ and $E_\infty$. Let $H_0 = 0$ and $H_\infty = 0$ be the defining equations of $E_0$ and $E_\infty$, respectively. Then $E$ splits in $(X_G, \mathcal{L}_G)$ if and only if

$$Z(dG) = Z(d(H_0 H_\infty)),$$

or equivalently

$Y_G$ and $Y_{H_0 H_\infty}$ are isomorphic over $\mathbb{P}^2$,

or equivalently

$$\exists c \in k^\times, \exists H \in \mathcal{N}, \ H_0 H_\infty = cG + H^2.$$ 

If $E$ splits in $(X_G, \mathcal{L}_G)$, then $\text{Bs}(E)$ is contained in $Z(dG)$, and

$$w(E_t) = \gamma^{-1}(\text{Bs}(E))$$

holds for every member $E_t$ of $E$. In particular, the word $w(E_t)$ is of weight 9.
Let $A$ be a word of $\mathcal{C}_G$.

(i) We say that $A$ is a \textit{linear word} if $|A| = 5$.

(ii) Suppose $|A| = 8$. If $A$ is \textit{not} a sum of two linear words, then we say that $A$ is a \textit{quadratic word}.

(iii) Suppose $|A| = 9$. If $A$ is neither a sum of three linear words nor a sum of a linear and a quadratic words, then we say that $A$ is a \textit{cubic word}.

By $C \mapsto w(C)$, we obtain the following bijections:

\[
\{ \text{lines splitting in } (X_G, \mathcal{L}_G) \} \quad \cong \quad \{ \text{linear words in } \mathcal{C}_G \},
\]

\[
\{ \text{non-singular conic curves splitting in } (X_G, \mathcal{L}_G) \} \quad \cong \quad \{ \text{quadratic words in } \mathcal{C}_G \}.
\]

By $\mathcal{E} \mapsto w(E_t) = \gamma^{-1}(\text{Bs}(\mathcal{E}))$, we obtain the bijection

\[
\{ \text{regular pencils of cubic curves splitting in } (X_G, \mathcal{L}_G) \} \quad \cong \quad \{ \text{cubic words in } \mathcal{C}_G \}.
\]
§4. The Case of Artin Invariant 2

We start from a code $\mathcal{C} \subset 2^\mathcal{P}$ such that
\begin{itemize}
\item $\mathcal{P} = (1, 1, \ldots, 1) \in \mathcal{C}$, and
\item $|w| \in \{0, 5, 8, 9, 12, 13, 16, 21\}$ for any $w \in \mathcal{C}$,
\end{itemize}
and construct the stratum $\mathcal{M}_{[\mathcal{C}]}$.

For simplicity, we assume that $\mathcal{C}$ is generated by $\mathcal{P}$ and words of weight 5 and 8.

We denote by $\mathcal{G}_\mathcal{C}$ the space of all injective maps
$$\gamma : \mathcal{P} \hookrightarrow \mathbb{P}^2$$
with the following properties:
\begin{enumerate}
\item $\gamma(\mathcal{P}) = Z(dG)$ for some $G \in \mathcal{U}$ (that is, $\gamma \in \mathcal{G}$),
\item for a subset $A \subset \mathcal{P}$ of weight 5, $\gamma(A)$ is collinear if and only if $A \in \mathcal{C}$,
\item for a subset $A \subset \mathcal{P}$ of weight 8, $\gamma(A)$ is on a non-singular conic curve if and only if $A \in \mathcal{C}$ and $A$ is not a sum of words of weight 5 in $\mathcal{C}$.
\end{enumerate}
\[ \mathcal{M} = PGL(3, k) \backslash \mathcal{G} / S_{21} \supset \]
\[ \mathcal{M}_{[C]} = PGL(3, k) \backslash \mathcal{G}_C / \text{Aut}(C). \]

Suppose that the isomorphism class of \((X_G, \mathcal{L}_G)\) is a point of \(\mathcal{M}_{[C]}\).
Let \(\gamma \in \mathcal{G}_C\) be the injective map such that \(\gamma(P) = Z(dG)\).

Then
\[ \text{Aut}(X_G, \mathcal{L}_G) = \{ g \in PGL(3, k) \mid g(Z(dG)) = Z(dG) \} \]
is the stabilizer subgroup
\[ \text{Stab}(\langle \gamma \rangle) \subset \text{Aut}(C) \]
of the projective equivalence class \(\langle \gamma \rangle \in PGL(3, k) \backslash \mathcal{G}_C\).

We carry out this construction of \(\mathcal{M}_{[C]}\) for the three isomorphism classes \([C_A], [C_B], [C_C]\) of codes with dimension 9, that is, the Artin invariant 2.
Generators of the code $C_A$

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}
\]

Generators of the code $C_B$

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

Generators of the code $C_C$

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
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0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
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0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0
\end{bmatrix}
\]
The weight enumerators of these codes are as follows:

\[ C_A : 1 + z^{21} + 13(z^5 + z^{16}) + 106(z^8 + z^{13}) + 136(z^9 + z^{12}), \]
\[ C_B : 1 + z^{21} + 9(z^5 + z^{16}) + 102(z^8 + z^{13}) + 144(z^9 + z^{12}), \]
\[ C_C : 1 + z^{21} + 5(z^5 + z^{16}) + 130(z^8 + z^{13}) + 120(z^9 + z^{12}). \]

The numbers of linear, quadratic and cubic words in these codes, and the order of the automorphism group are given in the following table:

|     | linear | quadratic | cubic | \(| \text{Aut}(C) | \) |
|-----|--------|-----------|-------|-----------------
| \( C_A \) | 13     | 28        | 0     | 1152            |
| \( C_B \) | 9      | 66        | 0     | 432             |
| \( C_C \) | 5      | 120       | 0     | 23040           |

These codes are generated by \( \mathcal{P} \) and linear and quadratic words.
For $T = A, B$ and $C$, the following hold.
($\omega$ is the third root of unity, and $\bar{\omega} = \omega + 1$.)

The space $PGL(3, k) / G_T$ has exactly two connected components, both of which are isomorphic to

$$\text{Spec } k[\lambda, 1/(\lambda^4 + \lambda)] = \mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}.$$  

Let $N_T \subset \text{Aut}(\mathcal{C}_T)$ be the subgroup of index 2 that preserves the connected components, and let $\Gamma_T$ be the image of $N_T$ in

$$\text{Aut}(\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}).$$

The moduli curve

$$\mathcal{M}_T = (\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}) / \Gamma_T$$

is isomorphic to a punctured affine line

$$\text{Spec } k[J_T, 1/J_T] = \mathbb{A}^1 \setminus \{0\}.$$  

The punctured origin $J_T = 0$ corresponds to the Dolgachev-Kondo point.

The action of $\Gamma_T$ on $\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}$ is free. Hence the order of $\text{Stab}(\langle \gamma \rangle) \subset \text{Aut}(\mathcal{C}_T)$ is constant on $PGL(3, k) / G_T$.

We have an exact sequence

$$1 \to \text{Aut}(X, \mathcal{L}) \to N_T \to \Gamma_T \to 1.$$
The case $A$:

$$\Gamma_A = \left\{ \lambda, \lambda + 1, \frac{1}{\lambda}, \frac{1}{\lambda + 1}, \frac{\lambda}{\lambda + 1}, \frac{\lambda + 1}{\lambda} \right\} \cong S_3,$$

$$J_A = \frac{(\lambda^2 + \lambda + 1)^3}{\lambda^2 (\lambda + 1)^2},$$

$$GA[\lambda] := X_0 X_1 X_2 (X_0 + X_1 + X_2) \cdot (X_0^2 + X_1^2 + (\lambda^2 + \lambda) X_2^2 + X_0 X_1 + X_1 X_2 + X_2 X_0).$$

The family

$$W^2 = GA[\lambda]$$

is the universal family of polarized supersingular $K3$ surfaces over the $\lambda$-line.

For $\alpha \in k \setminus \{0, 1, \omega, \bar{\omega}\}$, $\text{Aut}(X_{GA[\alpha]}, \mathcal{L}_{GA[\alpha]})$ is equal to the group

$$\left\{ \begin{bmatrix} A & a \\ b & 1 \end{bmatrix} \in PGL(3, k) \mid A \in GL(2, \mathbb{F}_2), \quad a, b \in \{0, 1, \alpha, \alpha + 1\} \right\}.$$
\( \Gamma_B \) is isomorphic to the alternating group \( A_4 \).

\[ J_B = (\lambda + \omega)^{12}/(\lambda^3(\lambda + 1)^3(\lambda + \bar{\omega})^3). \]

\[ GB[\lambda] = X_0X_1X_2 (X_0 + X_1 + X_2) \cdot \\
(\bar{\omega}\lambda + \omega)X_0^2 + \bar{\omega}X_1^2 + \omega\lambda X_2^2 + \\
(\lambda + 1)X_0X_1 + (\bar{\omega}\lambda + \omega)X_1X_2 + (\lambda + 1)X_2X_0. \]

\( \Gamma_C \) is the group of affine transformations of an affine line over \( \mathbb{F}_4 \).

\[ J_C = (\lambda^4 + \lambda)^3. \]

\[ GC[\lambda] = X_0X_1X_2 (X_0^3 + X_1^3 + X_2^3) + (\lambda^4 + \lambda)X_0^3X_1^3. \]

The orders of the groups above are given as follows.

| \( T \) | \( | \text{Aut}(C_T)\| = 2 \times |\Gamma_T| \times |\text{Aut}(X, \mathcal{L})| \) |
|-------|----------------------------------|
| \( A \) | 1152 = 2 \times 6 \times 96 |
| \( B \) | 432 = 2 \times 12 \times 18 |
| \( C \) | 23040 = 2 \times 12 \times 960 |
§5. Cremona transformations

Let \( \Sigma = \{p_1, \ldots, p_6\} \subset Z(dG) \) be a subset with \(|\Sigma| = 6\) satisfying the following:

- no three points of \( \Sigma \) are collinear, and
- for each \( i \), the non-singular conic curve \( Q_i \) containing \( \Sigma \setminus \{p_i\} \) satisfies \( Q_i \cap Z(dG) = \Sigma \setminus \{p_i\} \).

Let \( \beta : S \rightarrow \mathbb{P}^2 \) be the blowing up at the points in \( \Sigma \), and let \( \beta' : S \rightarrow \mathbb{P}^2 \) be the blowing down of the strict transforms \( Q'_i \) of the conic curves \( Q_i \).

The birational map

\[ c := \beta' \circ \beta^{-1} : \mathbb{P}^2 \rightarrow \mathbb{P}^2 \]

is called the Cremona transformation with the center \( \Sigma \).

There exists \( G' \in \mathcal{U} \) such that

\[ c(Z(dG) \setminus \Sigma) \cup \{\beta'(Q'_i) \mid i = 1, \ldots, 6\} = Z(dG'). \]

Obviously, \( X_G \) and \( X_{G'} \) are isomorphic.
But \((X_G, \mathcal{L}_G)\) and \((X_{G'}, \mathcal{L}_{G'})\) may fail to be isomorphic.
A curve $D \subset \mathcal{M}_T \times \mathcal{M}_{T'}$ is called an isomorphism correspondence if, for any pair

$$([X, \mathcal{L}], [X', \mathcal{L}']) \in D,$$

the $K3$ surfaces $X$ and $X'$ are isomorphic as non-polarized surfaces.

Using Cremona transformations, we obtain an example of non-trivial isomorphism correspondences.

Let $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ be polarized supersingular $K3$ surfaces of type (‡) with Artin invariant 2, and let $J_T$ and $J_{T'}$ be their $J$-invariants.

If $T = T' = A$ and

$$1 + J_A J'_A + J_A^2 J'^2_A + J_A^2 J'^3_A + J_A^3 J'^2_A = 0,$$

then $X$ and $X'$ are isomorphic.

If $T = A$ and $T' = B$ and

$$J_B + J_A J_B + J_A J_B^2 + J_A^2 J_B + J_A^4 = 0,$$

then $X$ and $X'$ are isomorphic.
The isomorphism correspondence

\[ 1 + J_A J'_A + J_A^2 J'_A^2 + J_A^2 J'_A^3 + J_A^3 J'_A^2 = 0 \]

intersects with the diagonal \( \Delta_A \subset \mathcal{M}_A \times \mathcal{M}_A \) at two points \((J_A, J'_A) = (\omega, \omega)\) and \((\bar{\omega}, \bar{\omega})\).

At these points, the automorphism group \( \text{Aut}(X) \) of the supersingular \( K3 \) surface \textit{jumps}.

Do all isomorphism correspondences come from Cremona transformations?