Singularity of discriminant varieties in characteristic 2 and 3

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We work over an algebraically closed field $k$. 
§1. An Example

Let $E \subset \mathbb{P}^2$ be a smooth cubic plane curve. We fix a flex point $O \in E$, and consider the elliptic curve $(E, O)$.

Let $(\mathbb{P}^2)^\vee$ be the dual projective plane, and let $E^\vee \subset (\mathbb{P}^2)^\vee$ be the dual curve of $E$. We denote by

$$\phi : E \to E^\vee$$

the morphism that maps a point $P \in E$ to the tangent line $T_P(E) \in E^\vee$ to $E$ at $P$.

Suppose that $\text{char}(k) \neq 2$.

Then $E^\vee$ is of degree 6, and $\phi$ is birational. The singular points $\text{Sing}(E^\vee)$ of $E^\vee$ are in one-to-one correspondence with the flex points of $E$ via $\phi$. On the other hand, the flex points of $E$ are in one-to-one correspondence with the 3-torsion subgroup $E[3]$ of $(E, O)$. 
We have
\[ E[3] \cong \begin{cases} 
\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} & \text{if char}(k) \neq 3, \\
\mathbb{Z}/3\mathbb{Z} & \text{if char}(k) = 3 \text{ and } E \text{ is not supersingular}, \\
0 & \text{if char}(k) = 3 \text{ and } E \text{ is supersingular}.
\end{cases} \]

Then we have
\[ \text{Sing}(E^\vee) \text{ consists of} \]
\[ \begin{cases} 
9 \text{ points of type } A_2 & \text{if char}(k) \neq 3, \\
3 \text{ points of type } E_6 & \text{if char}(k) = 3 \text{ and } E \text{ is not s-singular}, \\
1 \text{ point of type } T_3 & \text{if char}(k) = 3 \text{ and } E \text{ is s-singular}.
\end{cases} \]

<table>
<thead>
<tr>
<th>type</th>
<th>defining equation</th>
<th>normalization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_2$</td>
<td>$x^2 + y^3 = 0$</td>
<td>$t \mapsto (t^3, t^2)$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$x^4 + y^3 + x^2y^2 = 0$ or $x^4 + y^3 = 0$</td>
<td>$t \mapsto (t^4, t^3 + t^5)$ or $t \mapsto (t^4, t^3)$</td>
</tr>
<tr>
<td>$T_3$</td>
<td>$x^{10} + y^3 + x^6y^2 = 0$</td>
<td>$t \mapsto (t^{10}, t^3 + t^{11})$</td>
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Remark. When char($k$) $\neq 3$, then the two types of the $E_6$-singular point are isomorphic.
Suppose that $\text{char}(k) = 2$.

Then $E^\vee$ is a smooth cubic curve, and $\phi : E \to E^\vee$ is a purely inseparable finite morphism of degree 2.

If $E$ is defined by

$$x^3 + y^3 + z^3 + a\ xyz = 0,$$

then $E^\vee$ is defined by

$$\xi^3 + \eta^3 + \zeta^3 + a^2\ \xi\eta\zeta = 0,$$

where $[\xi : \eta : \zeta]$ are the homogeneous coordinates dual to $[x : y : z]$ (C. T. C. Wall).
§2. Introduction

The aim of this talk is to investigate the singularity of the discriminant variety of a smooth projective variety $X \subset \mathbb{P}^m$ in arbitrary characteristics.

It turns out that the nature of the singularity differs according to the following cases:

- $\text{char}(k) > 3$ or $\text{char}(k) = 0$ (the classical case),
- $\text{char}(k) = 3$,
- $\text{char}(k) = 2$ and $\dim X$ is even,
- $\text{char}(k) = 2$ and $\dim X$ is odd (I could not analyze the singularity in this case).
§3. Definition of the discriminant variety

We need some preparation.

Let $V$ be a variety, and let $E$ and $F$ be vector bundles on $V$ with rank $e$ and $f$, respectively. For a bundle homomorphism $\sigma : E \to F$, we define the degeneracy subscheme of $\sigma$ to be the closed subscheme of $V$ defined locally on $V$ by all $r$-minors of the $f \times e$-matrix expressing $\sigma$, where $r := \min(e, f)$.

Let $V$ and $W$ be smooth varieties, and let $\phi : V \to W$ be a morphism.

The critical subscheme of $\phi$ is the degeneracy subscheme of the homomorphism $d\phi : T(V) \to \phi^* T(W)$.

Suppose that $\dim V \leq \dim W$. We say that $\phi$ is a closed immersion formally at $P \in V$ if $d_P \phi : T_P(V) \to T_{\phi(P)}(W)$ is injective, or equivalently, the induced homomorphism $(\mathcal{O}_{W,\phi(P)})^\wedge \to (\mathcal{O}_{V,P})^\wedge$ is surjective.

When $\dim V \leq \dim W$, a point $P \in V$ is in the support of the critical subscheme of $\phi$ if and only if $\phi$ is not a closed immersion formally at $P$. 
Let $X \subset \mathbb{P}^m$ be a smooth projective variety with $\dim X = n > 0$. We put
\[ \mathcal{L} := \mathcal{O}_X(1). \]
We assume that $X$ is not contained in any hyperplane of $\mathbb{P}^m$. Then the dual projective space
\[ \mathbb{P} := (\mathbb{P}^m)\vee \]
is regarded as a linear system $|M|$ of divisors on $X$, where $M$ is a linear subspace of $H^0(X, \mathcal{L})$.

Let $\mathcal{D} \subset X \times \mathbb{P}$ be the universal family of the hyperplane sections of $X$, which is smooth of dimension $n + m - 1$. The support of $\mathcal{D}$ is equal to
\[ \{ (p, H) \in X \times \mathbb{P} \mid p \in H \cap X \}. \]
Let $\mathcal{C} \subset \mathcal{D}$ be the critical subscheme of the second projection $\mathcal{D} \to \mathbb{P}$. It turns out that $\mathcal{C}$ is smooth of dimension $m - 1$. The support of $\mathcal{C}$ is equal to
\[ \{ (p, H) \in \mathcal{D} \mid H \cap X \text{ is singular at } p \}. \]
Let $\mathcal{E} \subset \mathcal{C}$ be the critical subscheme of the second projection $\pi_2 : \mathcal{C} \to \mathbb{P}$. The support of $\mathcal{E}$ is equal to
\[ \{ (p, H) \in \mathcal{C} \mid \text{the Hessian of } H \cap X \text{ at } p \text{ is degenerate} \}. \]

The image of $\pi_2 : \mathcal{C} \to \mathbb{P}$ is called the discriminant variety of $X \subset \mathbb{P}^m$. 


We will study the singularity of the discriminant variety by investigating the morphism $\pi_2 : \mathcal{C} \to \mathbb{P}$ at a point of the critical subscheme $\mathcal{E}$

Let $P = (p, H) \in X \times \mathbb{P}$ be a point of $\mathcal{E}$, so that $H \cap X$ has a degenerate singularity at $p$. Let $\Lambda \subset \mathbb{P}$ be a general plane passing through the point $\pi_2(P) = H \in \mathbb{P}$.

We denote by $C_\Lambda \subset \mathcal{C}$ the pull-back of $\Lambda$ by $\pi_2$, and by $\pi_\Lambda : C_\Lambda \to \Lambda$ the restriction of $\pi_2$ to $C_\Lambda$.

- What type of singular point does the plane curve $\Lambda \cap \pi_2(\mathcal{C})$ have at $H$?
- Does there exist any normal form for the morphism $\pi_\Lambda : C_\Lambda \to \Lambda$ at $P$?
§4. The scheme $\mathcal{E}$

For $P = (p, H) \in \mathcal{C}$, we have the Hessian

$$H_P : T_p(X) \times T_p(X) \to k$$

of the hypersurface singularity $p \in H \cap X \subset X$. If $H \cap X$ is defined locally by $f = 0$ in $X$, then $H_P$ is expressed by the symmetric matrix

$$M_P := \left( \frac{\partial^2 f}{\partial x_i \partial x_j} (p) \right).$$

Over $\mathcal{C}$, we can define the universal Hessian

$$\mathcal{H} : \pi_1^* T(X) \otimes \pi_1^* T(X) \to \tilde{\mathcal{L}} := \pi_1^* \mathcal{L} \otimes \pi_2^* \mathcal{O}_\mathbb{P}(1),$$

where $\pi_1 : \mathcal{C} \to X$ and $\pi_2 : \mathcal{C} \to \mathbb{P}$ are the projections.

The critical subscheme $\mathcal{E}$ of $\pi_2 : \mathcal{C} \to \mathbb{P}$ coincides with the degeneracy subscheme of the homomorphism $\pi_1^* T(X) \to \pi_1^* T(X)^\vee \otimes \tilde{\mathcal{L}}$ induced from $\mathcal{H}$.

From this proposition, we see that $\mathcal{E} \subset \mathcal{C}$ is either empty or of codimension $\leq 1$. In positive characteristics, we sometimes have $\mathcal{E} = \mathcal{C}$. 
Example.

Suppose that char$(k) = 2$. Then the Hessian $H_P$ is not only symmetric but also anti-symmetric, because we have

$$M_P = tM_P = -tM_P$$

and

$$\frac{\partial^2 \phi}{\partial x^2_i}(p) = 0.$$ 

On the other hand, the rank of an anti-symmetric bilinear form is always even. Hence we obtain the following:

If char$(k) = 2$ and dim $X$ is odd, then $C = E$.

Example.

Let $X \subset \mathbb{P}^{n+1}$ be the Fermat hypersurface of degree $q+1$, where $q$ is a power of the characteristic of the base field $k$. Then, at every point $(p, H)$ of $C$, the singularity of $H \cap X$ at $p$ is always degenerate. In particular, we have $C = E$.

The discriminant variety of a hypersurface is the dual hypersurface. The dual hypersurface $X^\vee$ of the Fermat hypersurface $X$ of degree $q+1$ is isomorphic to the Fermat hypersurface of degree $q+1$, and the natural morphism $X \to X^\vee$ is purely inseparable of degree $q^n$. 

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§5. The quotient morphism by an integrable tangent subbundle

In order to describe the situation in characteristic 2 and 3, we need the notion of the quotient morphism by an integrable tangent subbundle.

In this section, we assume that $k$ is of characteristic $p > 0$. Let $V$ be a smooth variety.

A subbundle $\mathcal{N}$ of $T(V)$ is called integrable if $\mathcal{N}$ is closed under the $p$-th power operation and the bracket product of Lie.

The following is due to Seshadri:

Let $\mathcal{N}$ be an integrable subbundle of $T(V)$. Then there exists a unique morphism $q : V \to V^\mathcal{N}$ with the following properties;

(i) $q$ induces a homeomorphism on the underlying topological spaces,
(ii) $q$ is a radical covering of height 1, and
(iii) the kernel of $dq : T(V) \to q^* T(V^\mathcal{N})$ is equal to $\mathcal{N}$.

Moreover, the variety $V^\mathcal{N}$ is smooth, and the morphism $q$ is finite of degree $p^r$, where $r = \text{rank } \mathcal{N}$. 
For an integrable subbundle $\mathcal{N}$ of $T(V)$, the morphism $q : V \to V^\mathcal{N}$ is called the \textit{quotient morphism} by $\mathcal{N}$.

The construction of $q : V \to V^\mathcal{N}$.

Let $V$ be covered by affine schemes $U_i := \text{Spec} \, A_i$. We put

$$A_i^\mathcal{N} := \{ f \in A_i \mid Df = 0 \text{ for all } D \in \Gamma(U_i, \mathcal{N}) \}.$$

Then the natural morphisms $\text{Spec} \, A_i \to \text{Spec} \, A_i^\mathcal{N}$ patch together to form $q : V \to V^\mathcal{N}$.

Let $\phi : V \to W$ be a morphism from a smooth variety $V$ to a smooth variety $W$. Suppose that the kernel $\mathcal{K}$ of $d\phi : T(V) \to \phi^*T(W)$ is a subbundle of $T(V)$, which is always the case if we restrict $\phi$ to a Zariski open dense subset of $V$. Then $\mathcal{K}$ is integrable, and $\phi$ factors through the quotient morphism by $\mathcal{K}$. 
The case where char$(k) = 2$ and dim $X$ is odd.

Suppose that char$(k) = 2$ and dim $X$ is odd, so that $\mathcal{C} = \mathcal{E}$ holds. Let $\mathcal{K}$ be the kernel of the homomorphism $\pi_1^* T(X) \to \pi_1^* T(X) \vee \otimes \widetilde{\mathcal{L}}$ induced from the universal Hessian $\mathcal{H}$, which is of rank $\geq 1$ at the generic point of every irreducible component of $\mathcal{C}$. Then the subsheaf $\mathcal{K} \subset \pi_1^* T(X) \subset \pi_1^* T(X) \oplus \pi_2^* T(\mathbb{P}) = T(X \times \mathbb{P})|_\mathcal{C}$ is in fact contained in $T(\mathcal{C}) \subset T(X \times \mathbb{P})|_\mathcal{C}$.

Let $U \subset \mathcal{C}$ be a Zariski open dense subset of $\mathcal{C}$ over which $\mathcal{K}$ is a subbundle of $T(\mathcal{C})$. Then the restriction of $\pi_2$ to $U$ factors through the quotient morphism by $\mathcal{K}$. In particular, the projection $\mathcal{C} \to \mathbb{P}$ is inseparable onto its image.
§6. The case where char($k$) $\neq 2$

Suppose that the characteristic of $k$ is not 2.

Let $(p, H)$ be a point of $\mathcal{E}$, so that the divisor $H \cap X$ has a degenerate singularity at $p$.

We say that the singularity of $H \cap X$ at $p$ is of type $A_2$ if there exists a formal parameter system $(x_1, \ldots, x_n)$ of $X$ at $p$ such that $H \cap X$ is given as the zero of the function of the form

$$x_1^2 + \cdots + x_{n-1}^2 + x_n^3 + (\text{higher degree terms}).$$

We then put

$$\mathcal{E}^{A_2} := \left\{ (p, H) \in \mathcal{E} \mid \text{the singularity of } H \cap X \text{ at } p \text{ is of type } A_2 \right\}.$$

We also put

$$\mathcal{E}^{\text{sm}} := \left\{ (p, H) \in \mathcal{E} \mid \mathcal{E} \text{ is smooth of dimension } m - 2 \text{ at } (p, H) \right\}.$$

We see that $\mathcal{E}$ is irreducible and the loci $\mathcal{E}^{A_2}$ and $\mathcal{E}^{\text{sm}}$ are dense in $\mathcal{E}$ if the linear system $|M|$ is sufficiently ample; e.g., if the evaluation homomorphism

$$v_p^{[3]} : M \to \mathcal{L}_p/m_p^4\mathcal{L}_p$$

is surjective at every point $p$ of $X$, where $m_p \subset \mathcal{O}_{X,p}$ is the maximal ideal.
The case where $\text{char}(k) > 3$ or $\text{char}(k) = 0$.

In this case, we have the following:

Let $P = (p, H)$ be a point of $\mathcal{E}$. The following two conditions are equivalent:

- $P \in \mathcal{E}^{A_2}$, 
- $P \in \mathcal{E}^{\text{sm}}$, and the projection $\mathcal{E} \rightarrow \mathbb{P}$ is a closed immersion formally at $P$.

Moreover, if these conditions are satisfied, then the curve $C_\Lambda = \pi_2^{-1}(\Lambda)$ is smooth at $P$, and $\pi_\Lambda : C_\Lambda \rightarrow \Lambda$ has a critical point of $A_2$-type at $P$; that is,

\begin{align*}
\pi_\Lambda^* u &= a t^2 + b t^3 + (\text{terms of degree } \geq 4) \quad \text{and} \\
\pi_\Lambda^* v &= c t^2 + d t^3 + (\text{terms of degree } \geq 4)
\end{align*}

with $ad - bc \neq 0$ hold for a formal parameter system $(u, v)$ of $\Lambda$ at $\pi(P) = H$ and a formal parameter $t$ of $C_\Lambda$ at $P$.

By suitable choice of formal parameters, we have

\begin{align*}
\pi_\Lambda^* u &= t^3, \\
\pi_\Lambda^* v &= t^2,
\end{align*}

and the plane curve $\pi_2(C) \cap \Lambda \subset \Lambda$ is defined by $u^2 - v^3 = 0$ locally at $H \in \Lambda$. 
The case where \( \text{char}(k) = 3 \).

In this case, \( P \in \mathcal{E}^{A_2} \) does not necessarily imply \( P \in \mathcal{E}^{\text{sm}} \). Our main results are as follows.

(I) Let \( \varpi : \mathcal{E}^{\text{sm}} \to \mathbb{P} \) be the projection. Then the kernel \( \mathcal{K} \) of \( d\varpi : T(\mathcal{E}^{\text{sm}}) \to \varpi^*T(\mathbb{P}) \) is a subbundle of \( T(\mathcal{E}^{\text{sm}}) \) with rank 1. Hence \( \varpi \) factors as

\[
\begin{align*}
\mathcal{E}^{\text{sm}} & \xrightarrow{q} (\mathcal{E}^{\text{sm}})^{\mathcal{K}} \xrightarrow{\tau} \mathbb{P},
\end{align*}
\]

where \( \mathcal{E}^{\text{sm}} \to (\mathcal{E}^{\text{sm}})^{\mathcal{K}} \) is the quotient morphism by \( \mathcal{K} \).

(II) Suppose that \( P \) is a point of \( \mathcal{E}^{\text{sm}} \cap \mathcal{E}^{A_2} \). Then \( \tau : (\mathcal{E}^{\text{sm}})^{\mathcal{K}} \to \mathbb{P} \) is a closed immersion formally at \( q(P) \). Moreover the curve \( C_\Lambda \) is smooth at \( P \), and \( \pi_\Lambda : C_\Lambda \to \Lambda \) has a critical point of \( E_6 \)-type at \( P \); i. e.,

\[
\begin{align*}
\pi_\Lambda^* u &= a t^3 + b t^4 + (\text{terms of degree } \geq 5) \quad \text{and} \\
\pi_\Lambda^* v &= c t^3 + d t^4 + (\text{terms of degree } \geq 5)
\end{align*}
\]

with \( ad - bc \neq 0 \) hold.

By suitable choice of formal parameters, we have either

\( (\pi_\Lambda^* u = t^3, \pi_\Lambda^* v = t^4) \) or \( (\pi_\Lambda^* u = t^3 + t^5, \pi_\Lambda^* v = t^4) \).

The plane curve \( \pi_2(C) \cap \Lambda \subset \Lambda \) is defined at \( H \in \Lambda \) by either

\[
x^4 + y^3 = 0 \quad \text{or} \quad x^4 + y^3 + x^2 y^2 = 0.
\]
In the case of a projective plane curve (i.e., the case where \((n, m) = (1, 2)\)), the locus \(E^{sm}\) is always empty. In this case, we have the following:

(III) Suppose that \((n, m) = (1, 2)\), and that the projection \(C \rightarrow \mathbb{P}\) is separable onto its image. (This assumption excludes the case of, for example, the Fermat curve of degree \(3^\nu + 1\).)

Then \(\dim E = 0\). Let \(P = (p, H)\) be a point of \(E\). Then the length of \(O_{E,P}\) is divisible by 3. If \(P \in E^{A_2}\) (that is, \(H\) is an ordinary flex tangent line to \(X\) at \(p\)), then, with appropriate choice of formal parameters, the formal completion of \(\pi_2 : C \rightarrow \mathbb{P}\) at \(P\) is given by

\[
T_l : t \mapsto (t^{3l+1}, t^3 + t^{3l+2}),
\]

where \(l := \text{length } O_{E,P}/3\). 
§7. The case where $\text{char}(k) = 2$ and $\dim X$ is even.

For simplicity, we assume that $|M|$ is so ample that the evaluation homomorphism

$$v_p^{[4]} : M \to \mathcal{L}_p/m_p^5\mathcal{L}_p$$

is surjective at every point $p$ of $X$.

Then $\mathcal{E}$ is an irreducible divisor of $\mathcal{C}$, and is written as $2\mathcal{R}$, where $\mathcal{R}$ is a reduced divisor of $\mathcal{C}$.

We denote by $\mathcal{R}^{\text{sm}}$ the smooth locus of $\mathcal{R}$, and by $\varpi : \mathcal{R}^{\text{sm}} \to \mathbb{P}$ the projection.

Then we have the following:

(I) The kernel $\mathcal{K}$ of $d\varpi : T(\mathcal{R}^{\text{sm}}) \to \varpi^*T(\mathbb{P})$ is a sub-bundle of $T(\mathcal{R}^{\text{sm}})$ with rank 2.

In particular, the projection $\varpi$ factors through a finite inseparable morphism of degree 4.
(II) Let $P = (p, H)$ be a general point of $\mathcal{R}$.
Let $L \subset \mathbb{P}$ be a general linear subspace of dimension 3 containing $\Lambda$. We put $S_L := \pi_2^{-1}(L) \subset \mathcal{C}$.
Then $S_L$ is smooth of dimension 2 at $P$, and $C_\Lambda$ is a curve on $S_L$ that has an ordinary cusp at $P$.
Let $\nu : \tilde{C}_\Lambda \to C_\Lambda$ be the normalization of $C_\Lambda$ at $P$, and let $z$ be a formal parameter of $\tilde{C}_\Lambda$ at the inverse image $P' \in \tilde{C}_\Lambda$ of $P$. Then the formal completion at $P'$ of $\pi_\Lambda \circ \nu : \tilde{C}_\Lambda \to \Lambda$ is written as
\[(\pi_\Lambda \circ \nu)^* u = a z^4 + (\text{terms of degree } \geq 6) \quad \text{and} \quad (\pi_\Lambda \circ \nu)^* v = b z^4 + (\text{terms of degree } \geq 6)\]
for some $a, b \in k$, where $(u, v)$ is a formal parameter system of $\Lambda$ at $H$.
Hence the plane curve singularity of $\pi_2(\mathcal{C}) \cap \Lambda$ at $H$ is \textit{not} a rational double point any more.