Fundamental groups of complements of dual varieties in Grassmannian

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§1. Introduction

This work is motivated by the conjecture in the paper

[ADKY]
Topology, 43(6): 1285-1318, 2004,

on the fundamental group

\[ \pi_1(\mathbb{P}^2 \setminus B), \]

where \( B \) is the branch curve of a general projection \( S \to \mathbb{P}^2 \)
from a smooth projective surface \( S \subset \mathbb{P}^N \).

By the previous work of Moishezon-Teicher-Robb and by their own new examples, they conjectured in [ADKY] that \( \pi_1(\mathbb{P}^2 \setminus B) \)
is “small”.
Let $G^2(\mathbb{P}^N)$ be the Grassmannian variety of linear subspaces in $\mathbb{P}^N$ with codimension 2. We put

$$U_0(S, \mathbb{P}^N) := \{ L \in G^2(\mathbb{P}^N) \mid L \cap S \text{ is smooth of dimension 0} \},$$

which is a Zariski open subset of the Grassmannian $G^2(\mathbb{P}^N)$.

It is easy to see that there exists a natural inclusion

$$\mathbb{P}^2 \setminus B \hookrightarrow U_0(S, \mathbb{P}^N),$$

which induces a surjective homomorphism

$$\pi_1(\mathbb{P}^2 \setminus B) \longrightarrow \pi_1(U_0(S, \mathbb{P}^N)).$$

Hence, if the conjecture is true, the fundamental group

$$\pi_1(U_0(S, \mathbb{P}^N))$$

should be “very small”.

In this talk, we describe this fundamental group $\pi_1(U_0(S, \mathbb{P}^N))$ by means of Zariski-van Kampen monodromy associated with a Lefschetz pencil on $S$. 
§2. Zariski-van Kampen theorem

We formulate and prove a theorem of Zariski-van Kampen type on the fundamental groups of algebraic fiber spaces.

Let $X$ and $Y$ be smooth quasi-projective varieties, and let $f : X \to Y$ be a dominant morphism.

For simplicity, we assume the following:

The general fiber of $f$ is connected.

For a point $y \in Y$, we put

$$F_y := f^{-1}(y).$$

We then choose general points

$$b \in Y \quad \text{and} \quad \tilde{b} \in F_b \subset X.$$ 

Let

$$\iota : F_b \hookrightarrow X$$

denote the inclusion.
We denote by 
\[ \text{Sing}(f) \subset X \]
the Zariski closed subset consisting of the critical points of \( f \).

The following is Nori’s lemma:

**Proposition.**
If there exists a Zariski closed subset \( \Xi \) of codimension \( \geq 2 \) such that
\[
F_y \setminus (F_y \cap \text{Sing}(f)) \neq \emptyset \quad \text{for all} \quad y \notin \Xi,
\]
then we have an exact sequence
\[
\pi_1(F_b, \tilde{b}) \xrightarrow{\iota_*} \pi_1(X, \tilde{b}) \xrightarrow{f_*} \pi_1(Y, b) \rightarrow 1.
\]

We will investigate
\[
\ker(\pi_1(F_b, \tilde{b}) \xrightarrow{\iota_*} \pi_1(X, \tilde{b})).
\]
We fix, once and for all, a hypersurface $\Sigma$ of $Y$ with the following properties. We put

$$Y^\circ := Y \setminus \Sigma, \quad X^\circ := f^{-1}(Y^\circ),$$

and let

$$f^\circ : X^\circ \to Y^\circ$$

denote the restriction of $f$ to $X^\circ$.

The required property is as follows:

The morphism $f^\circ$ is smooth, and is locally trivial (in the category of topological spaces and continuous maps).

The existence of such a hypersurface $\Sigma$ follows from Hironaka’s resolution of singularities, for example.

We can assume that $b \in Y^\circ$. 

Let $I$ denote the closed interval $[0, 1] \subset \mathbb{R}$. Let 
\[ \tilde{\alpha} : I \rightarrow X^\circ \]
be a loop with the base point $\tilde{b} \in F_b \subset X^\circ$. Then the family of pointed spaces 
\[(F_{f(\tilde{\alpha}(t))}, \tilde{\alpha}(t))\]
is trivial over $I$, and hence we obtain an automorphism 
\[ \tilde{\mu}([\tilde{\alpha}]) : \pi_1(F_b, \tilde{b}) \cong \pi_1(F_b, \tilde{b}), \quad g \mapsto g^{\tilde{\mu}([\tilde{\alpha}])}, \]
which depends only on the homotopy class of the loop $\tilde{\alpha}$ in $X^\circ$. We thus obtain a homomorphism 
\[ \tilde{\mu} : \pi_1(X^\circ, \tilde{b}) \rightarrow \text{Aut}(\pi_1(F_b, \tilde{b})), \]
which is called the monodromy on $\pi_1(F_b)$. 

Our main purpose is to describe the kernel of 
\[ \iota_* : \pi_1(F_b, \tilde{b}) \rightarrow \pi_1(X, \tilde{b}) \]
in terms of the monodromy $\tilde{\mu}$. 

Remark. 
The classical Zariski-van Kampen theorem deals with the situation where there exists a continuous section 
\[ s : Y \rightarrow X \]
of $f$ so that we have a monodromy 
\[ \mu := \tilde{\mu} \circ s_* : \pi_1(Y^\circ, b) \rightarrow \text{Aut}(\pi_1(F_b, \tilde{b})). \]
Definition.
Let $G$ be a group, and let $S$ be a subset of $G$. We denote by
\[
\langle \langle S \rangle \rangle_G \triangleleft G
\]
the smallest \textit{normal} subgroup of $G$ containing $S$.

Let $\Gamma$ be a subgroup of $\text{Aut}(G)$. For $\gamma \in \Gamma$ and $g \in G$, we put
\[
R(G, \Gamma) := \{ g^{-1}g^\gamma \mid g \in G, \gamma \in \Gamma \} \subset G.
\]
We then put
\[
G \sslash \Gamma := G/\langle \langle R(G, \Gamma) \rangle \rangle_G,
\]
and call $G \sslash \Gamma$ the \textit{Zariski-van Kampen quotient} of $G$ by $\Gamma$.

Definition.
An element
\[
g^{-1}g^{\tilde{\mu}([\tilde{\alpha}])} \quad (g \in \pi_1(F_b, \tilde{b}), \ [\tilde{\alpha}] \in \pi_1(X^\circ, \tilde{b}))
\]
of $\pi_1(F_b, \tilde{b})$ is called a \textit{monodromy relation}.
We consider the following conditions.

(C1) \( \text{Sing}(f) \) is of codimension \( \geq 2 \) in \( X \).

(C2) There exists a Zariski closed subset
\[
\Xi \subset Y
\]
with codimension \( \geq 2 \) such that \( F_y \) is non-empty and irreducible for any \( y \in Y \setminus \Xi \).

(C3) There exist a subspace \( Z \subset Y \) and a continuous section
\[
s_Z : Z \to f^{-1}(Z)
\]
of \( f \) over \( Z \) such that \( Z \ni b \), that \( Z \hookrightarrow Y \) induces a surjective homomorphism
\[
\pi_2(Z, b) \twoheadrightarrow \pi_2(Y, b),
\]
and that \( s_Z(Z) \cap \text{Sing}(f) = \emptyset \) and \( s_Z(b) = \tilde{b} \).

Our generalized Zariski-van Kampen theorem is as follows:

**Theorem.**

We put
\[
\tilde{K} := \text{Ker}(\pi_1(X^\circ, \tilde{b}) \to \pi_1(X, \tilde{b})),
\]
where \( \pi_1(X^\circ, \tilde{b}) \to \pi_1(X, \tilde{b}) \) is induced by the inclusion. Under the above conditions (C1)-(C3), the kernel of
\[
\iota_* : \pi_1(F_b, \tilde{b}) \to \pi_1(X, \tilde{b})
\]
is equal to the normal subgroup
\[
\langle \langle R(\pi_1(F_b, \tilde{b}), \tilde{\mu}((\tilde{\alpha})) \rangle \rangle
\]
\[
= \langle \langle \{ g^{-1}g^{\tilde{\mu}([\tilde{\alpha}])} \mid g \in \pi_1(F_b, \tilde{b}), [\tilde{\alpha}] \in \tilde{K} \} \rangle \rangle
\]
normally generated by the monodromy relations coming from the elements of \( \tilde{K} \).
Theorem.
Assume the following:
(C1) \( \text{Sing}(f) \) is of codimension \( \geq 2 \) in \( X \).
(C2) There exists a Zariski closed subset \( \Xi \subset Y \) with codimension \( \geq 2 \) such that \( F_y \) is non-empty and irreducible for any \( y \in Y \setminus \Xi \).
(C4) There exist an irreducible smooth curve \( C \subset Y \) passing through \( b \) and a continuous section
\[ s_C : C \to f^{-1}(C) \]
of \( f \) over \( C \) with the following properties:
(i) \( \pi_1(C^\circ) \to \pi_1(Y^\circ) \), where \( C^\circ := C \cap Y^\circ \).
(ii) \( \pi_2(C) \to \pi_2(Y) \).
(iii) \( C \) intersects each irreducible component of \( \Sigma \) transversely at least at one point.
(iv) \( s_C(C) \cap \text{Sing}(f) = \emptyset \) and \( s_C(b) = \tilde{b} \).

We put
\[ K_C := \text{Ker}(\pi_1(C^\circ, b) \to \pi_1(C, b)) \]

By the section \( s_C \), we have a monodromy action
\[ \mu_C : \pi_1(C^\circ, b) \to \text{Aut}(\pi_1(F_b, \tilde{b})) \]
Then we have
\[ \text{Ker}(\iota_*) = \langle \langle R(\pi_1(F_b), \mu_C(K_C)) \rangle \rangle \].
Remark.
The main difference from the classical Zariski-van Kampen theorem is that we assume the existence of a section $s_Z$ of $f$ only over a subspace $Z \subset Y$ such that $\pi_2(Z) \rightarrow \pi_2(Y)$.

The necessity of the existence of such a section is shown by the following example.

Example.
Let $L \rightarrow \mathbb{P}^1$ be the total space of a line bundle of degree $d > 0$ on $\mathbb{P}^1$, and let $L^\times$ be the complement of the zero section with the natural projection

$$f : X := L^\times \rightarrow Y := \mathbb{P}^1,$$

so that $\pi_1(F_b) \cong \mathbb{Z}$. Then we have $\Sigma = \emptyset$, $X^\circ = X$ and hence $\tilde{K} = \text{Ker}(\pi_1(X^\circ) \rightarrow \pi_1(X))$ is trivial. In particular, we have

$$R(\pi_1(F_b), \tilde{\mu}(\tilde{K})) = \{1\}.$$

On the other hand, the kernel of

$$\iota_* : \pi_1(F_b) \cong \mathbb{Z} \rightarrow \pi_1(X) \cong \mathbb{Z}/d\mathbb{Z}$$

is non-trivial, and equal to the image of the boundary homomorphism

$$\pi_2(Y) \cong \mathbb{Z} \rightarrow \pi_1(F_b) \cong \mathbb{Z}.$$

Remark.
The condition (C3) or (C4-(ii)) is vacuous if $\pi_2(Y) = 0$ (for example, if $Y$ is an abelian variety).
§2. Grassmannian dual varieties

A Zariski closed subset of a projective space is said to be non-degenerate if it is not contained in any hyperplane.

We denote by $G^c(\mathbb{P}^N)$ the Grassmannian variety of linear subspaces of the projective space $\mathbb{P}^N$ with codimension $c$.

Definition.
Let $W$ be a closed subscheme of $\mathbb{P}^N$ such that every irreducible component is of dimension $n$. For a positive integer $c \leq n$, the Grassmannian dual variety of $W$ in $G^c(\mathbb{P}^N)$ is the locus

$$\left\{ L \in G^c(\mathbb{P}^N) \mid W \cap L \text{ fails to be smooth of dimension } n - c \right\}.$$

For a non-negative integer $k \leq n$, we denote by

$$U_k(W, \mathbb{P}^N) \subset G^{n-k}(\mathbb{P}^N)$$

the complement of the Grassmannian dual variety of $W$ in $G^{n-k}(\mathbb{P}^N)$; that is, $U_k(W, \mathbb{P}^N)$ is

$$\left\{ L \in G^{n-k}(\mathbb{P}^N) \mid L \text{ intersects } W \text{ along a smooth scheme of dimension } k \right\}.$$

Remark.
When $n - k = 1$, the variety $U_{n-1}(W, \mathbb{P}^N)$ is the complement of the usual dual variety of $W$ in $G^1(\mathbb{P}^N) = (\mathbb{P}^N)^\vee$. 

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Let
\[ X \subset \mathbb{P}^N \]
be a smooth non-degenerate projective variety of dimension \( n \geq 2 \). We choose a general line
\[ \Lambda \subset (\mathbb{P}^N)^\vee, \]
and a general point
\[ 0 \in \Lambda. \]
Let \( H_t \ (t \in \Lambda) \) denote the pencil of hyperplanes corresponding to \( \Lambda \), and let
\[ A \cong \mathbb{P}^{N-2} \]
denote the axis of the pencil. We then put
\[ Y_t := X \cap H_t \quad \text{and} \quad Z_\Lambda := X \cap A. \]
Then \( Z_\Lambda \) is smooth, and every irreducible component of \( Z_\Lambda \) is of dimension \( n - 2 \). (In fact, \( Z_\Lambda \) is irreducible if \( n > 2 \).)

We have natural inclusions
\[ G^{c-2}(A) \hookrightarrow G^{c-1}(H_t) \hookrightarrow G^c(\mathbb{P}^N). \]
Hence, for \( k = 0, \ldots, n - 2 \), we have natural inclusions
\[ U_k(Z_\Lambda, A) \hookrightarrow U_k(Y_t, H_t) \hookrightarrow U_k(X, \mathbb{P}^N). \]
Indeed, we have
\[ U_k(Z_\Lambda, A) = \{ L \in U_k(X, \mathbb{P}^N) \mid L \subset A \}, \]
\[ U_k(Y_t, H_t) = \{ L \in U_k(X, \mathbb{P}^N) \mid L \subset H_t \}. \]
Let \( k \) be an integer such that \( 0 \leq k \leq n - 2 \). Then \( U_k(Z_\Lambda, A) \) is non-empty. We choose a base point
\[
L_0 \in U_k(Z_\Lambda, A),
\]
which serves also as a base point of \( U_k(X, \mathbb{P}^N) \) and of \( U_k(Y_t, H_t) \) by the natural inclusions.

We then consider the family
\[
f : U_k(Y, \Lambda) \to \Lambda
\]
of the varieties \( U_k(Y_t, H_t) \), where
\[
U_k(Y, \Lambda) := \{ (L, t) \in U_k(X, \mathbb{P}^N) \times \Lambda \mid L \subset H_t \}
\]
\[
= \bigsqcup_{t \in \Lambda} U_k(Y_t, H_t),
\]
and \( f \) is the natural projection.

The point \( L_0 \) yields a holomorphic section
\[
s_o : \Lambda \to U_k(Y, \Lambda)
\]
of \( f \).

There exists a proper Zariski closed subset
\[
\Sigma_\Lambda \subset \Lambda
\]
such that \( f \) is locally trivial (in the category of topological spaces and continuous maps) over \( \Lambda \setminus \Sigma_\Lambda \). By the section \( s_o \), we have the monodromy action
\[
\pi_1(\Lambda \setminus \Sigma_\Lambda, 0) \to \text{Aut}(\pi_1(U_k(Y_0, H_0), L_o)).
\]
We have the following theorem of Lefschetz type.

Theorem. Consider the homomorphism

\[ \iota_* : \pi_1(U_k(Y_0, H_0), L_0) \to \pi_1(U_k(X, \mathbb{P}^N), L_0) \]

induced by the inclusion

\[ \iota : U_k(Y_0, H_0) \hookrightarrow U_k(X, \mathbb{P}^N). \]

(1) If \( k < n - 2 \), then \( \iota_* \) is an isomorphism.
(2) If \( k = n - 2 \), then \( \iota_* \) is surjective and induces an isomorphism

\[ \pi_1(U_k(Y_0, H_0)) \cong \pi_1(U_k(X, \mathbb{P}^N)). \]

Compare this theorem with the following classical hyperplane section theorem of Lefschetz on homotopy groups:

Theorem. Let \( b \) be a point of \( Y_0 \), and let

\[ j_k : \pi_k(Y_0, b) \to \pi_k(X, b) \]

be the homomorphism of the \( k \)th homotopy groups induced by the inclusion.

(1) If \( k < n - 1 \), then \( j_k \) is an isomorphism.
(2) If \( k = n - 1 \), then \( j_k \) is surjective.

Remark. The description of Zariski-van Kampen type of the kernel of \( j_{n-1} \) is also given by Chéniot-Libgober (2003) and Chéniot-Eyral (2006).
Sketch of the proof.

We put
\[ U_k(Y) := \{(L, H) \in U_k(X, \mathbb{P}^N) \times (\mathbb{P}^N)^\vee \mid L \subset H\}, \]
and consider the diagram
\[ U_k(Y) \to U_k(X, \mathbb{P}^N) \]
\[ \downarrow \]
\[ (\mathbb{P}^N)^\vee \]
of the natural projections. The morphism \( U_k(Y) \to U_k(X, \mathbb{P}^N) \) is locally trivial (in the holomorphic category) with a fiber being a linear subspace of \((\mathbb{P}^N)^\vee\). Hence we obtain
\[ \pi_1(U_k(Y)) \cong \pi_1(U_k(X, \mathbb{P}^N)). \]

By definition, we have
\[
\begin{align*}
U_k(Y_0, H_0) & \hookrightarrow U_k(Y, \Lambda) \hookrightarrow U_k(Y) \\
& \downarrow \quad \square \quad \downarrow \quad \square \quad \downarrow \\
H_0 & \in \quad \Lambda \quad \hookrightarrow (\mathbb{P}^N)^\vee,
\end{align*}
\]
and we have a section for \( U_k(Y, \Lambda) \to \Lambda \). Moreover we have
\[ \pi_2(\Lambda) \cong \pi_2((\mathbb{P}^N)^\vee). \]

By the generalized Zariski-van Kampen theorem, we obtain
\[ \pi_1(U_k(Y_0, H_0)) \parallel \pi_1(\Lambda \setminus \Sigma_\Lambda) \cong \pi_1(U_k(Y)). \]

If \( k < n - 2 \), then we have a surjection
\[ \pi_1(U_k(Z_\Lambda, A)) \twoheadrightarrow \pi_1(U_k(Y_0, H_0)). \]

Because \( \pi_1(\Lambda \setminus \Sigma_\Lambda) \) acts on \( \pi_1(U_k(Z_\Lambda, A)) \) trivially, it acts on \( \pi_1(U_k(Y_0, H_0)) \) trivially.
§3. Simple braid groups

We study the case where \( k = 0 \).

Let \( X \subset \mathbb{P}^N \) be a smooth non-degenerate projective variety of dimension \( n \) and degree \( d \). Then we have

\[
U_0(X, \mathbb{P}^N) = \left\{ L \in G^n(\mathbb{P}^N) \mid L \text{ intersects } X \text{ at distinct } d \text{ points} \right\}.
\]

By the previous theorem of Lefschetz type, it is enough to consider the case where \( \dim X = 2 \) in order to study \( \pi_1(U_0(X, \mathbb{P}^N)) \). Hence, from now on, we assume

\[
\dim X = 2,
\]

and study the monodromy

\[
\pi_1(\Lambda \setminus \Sigma_\Lambda) \rightarrow \text{Aut}(\pi_1(U_0(Y_0, H_0)))
\]

associated with a Lefschetz pencil on \( X \) corresponding to the line \( \Lambda \subset (\mathbb{P}^N)^\vee \). In this case,

\[
Y_0 = X \cap H_0
\]

is a compact Riemann surface embedded in \( H_0 \cong \mathbb{P}^{N-1} \) as a non-degenerate curve of degree \( d \). Note that \( U_0(Y_0, H_0) \) is the complement of the dual hypersurface

\[
(Y_0)^\vee \subset H_0^\vee \cong (\mathbb{P}^{N-1})^\vee
\]

of \( Y_0 \).
First we define the simple braid group $SB^d_g$ of $d$ strings on a compact Riemann surface $C$ of genus $g > 0$.

We denote by

$$\text{Div}^d(C) := (C \times \cdots \times C)/S_d$$

the variety of effective divisors of degree $d$ on $C$, and by

$$\text{rDiv}^d(C) := \text{Div}^d(C) \setminus \text{the big diagonal} \subset \text{Div}^d(C)$$

the Zariski open subset consisting of reduced divisors (that is, $\text{rDiv}^d(C)$ is the configuration space of distinct $d$ points on $C$). We fix a base point

$$D_0 = p_1 + \cdots + p_d \in \text{rDiv}^d(C).$$

**Definition.**

The *braid group*

$$B^d_g = B(C, D_0)$$

is defined to be the fundamental group $\pi_1(\text{rDiv}^d(C), D_0)$.

The *simple braid group*

$$SB^d_g = SB(C, D_0)$$

is defined to be the kernel of the homomorphism

$$B(C, D_0) = \pi_1(\text{rDiv}^d(C), D_0) \to \pi_1(\text{Div}^d(C), D_0)$$

induced by the inclusion

$$\text{rDiv}^d(C) \subseteq \text{Div}^d(C).$$
A braid on $C$ is called *simple* if it interchanges two points $p_i$ and $p_j$ of $D_0$ around a simple path connecting $p_i$ and $p_j$, and does not move other points.

Figure

It is easy to see that $SB^d_g$ is the subgroup of $B^d_g$ generated by simple braids, whence the name.
Next we introduce the notion of Plücker generality.

**Definition.**
Suppose that $C$ is embedded in $\mathbb{P}^M$ as a non-degenerate smooth curve. We say that $C \subset \mathbb{P}^M$ is *Plücker general* if the dual curve 
\[
\rho(C)^\vee \subset (\mathbb{P}^2)^\vee
\]
of the image of a general projection 
\[
\rho : C \to \mathbb{P}^2
\]
has only ordinary nodes and ordinary cusps as its singularities.

Our second main result is as follows:

**Theorem.**
Let $C \subset \mathbb{P}^M$ be a smooth non-degenerate projective curve of degree $d$ and genus $g > 0$. Suppose that 
\[
d \geq g + 4
\]
and that $C$ is Plücker general in $\mathbb{P}^M$. Let $D_0 = C \cap H_0$ be a general hyperplane section of $C$. Then 
\[
\pi_1(U_0(C, \mathbb{P}^M), D_0) = \pi_1((\mathbb{P}^M)^\vee \setminus C^\vee, H_0)
\]
is canonically isomorphic to 
\[
SB(C, D_0).
\]
For the proof, we use the following.

- We apply the generalized Zariski-van Kampen theorem to the natural morphism

\[ \text{Div}^d(C) \to \text{Pic}^d(C), \]

where \(\text{Pic}^d(C)\) is the Picard variety. Note that

\[ \pi_2(\text{Pic}^d(C)) = 0. \]

Then we can show that, under the assumption \(d \geq g + 4\),

\[ \pi_1(\text{Div}^d(C)) \cong \pi_1(\text{Pic}^d(C)) = H_1(C, \mathbb{Z}). \]

- We then apply the generalized Zariski-van Kampen theorem to the natural morphism

\[ \text{rDiv}^d(C) \to \text{Pic}^d(C). \]

If \(L\) is a very ample line bundle of degree \(d\) on \(C\) that embeds \(C\) into \(\mathbb{P}^m\), then the fiber of \(\text{rDiv}^d(C) \to \text{Pic}^d(C)\) over \([L] \in \text{Pic}^d(C)\) is canonically isomorphic to

\[ (\mathbb{P}^m)^\vee \setminus (C_L)^\vee = U_0(C_L, \mathbb{P}^m), \]

where \(C_L \subset \mathbb{P}^m\) is the image of \(C\) by the embedding by \(L\). In particular, \(\pi_1(U_0(C_L, \mathbb{P}^m))\) is isomorphic to

\[ SB^d_g = \text{Ker}(\pi_1(\text{rDiv}^d(C)) \to \pi_1(\text{Pic}^d(C))), \]

if \([L] \in \text{Pic}^d(C)\) is a general point.

- Finally, we use Harris’ result on Severi problem, which asserts that the moduli of irreducible nodal plane curves of degree \(d\) and genus \(g\) is irreducible. By the assumption of Plücker generality, we conclude that

\[ \pi_1(U_0(C, \mathbb{P}^M)) \cong \pi_1(U_0(C_L, \mathbb{P}^m)), \]

where \([L] \in \text{Pic}^d(C)\) is a general point.
Let
\[ X \subset \mathbb{P}^N \]
be a smooth non-degenerate projective surface of degree \( d \), and let
\[ \{Y_t\}_{t \in \Lambda} \]
be a general pencil of hyperplane sections of \( X \) parameterized by a line
\[ \Lambda \subset (\mathbb{P}^N)^\vee. \]
Let
\[ \varphi : \mathcal{Y}_\Lambda := \{ (x, t) \in X \times \Lambda \mid x \in H_t \} \to \Lambda \]
be the fibration of the pencil. We denote by
\[ \Sigma'_\Lambda \subset \Lambda \]
the set of critical values of \( \varphi \). Then \( \varphi \) is locally trivial over \( \Lambda \setminus \Sigma'_\Lambda \). Let 0 be a general point of \( \Lambda \). The corresponding member \( Y_0 \) is a compact Riemann surface of genus
\[ g := (d + H_0 \cdot K_X)/2 + 1. \]
Consider the base locus
\[ Z_\Lambda := X \cap A \]
of the pencil, where \( A \cong \mathbb{P}^{N-2} \) is the axis of the pencil \( \{H_t\} \).

Note that
\[ U_0(Z_\Lambda, A) = \{A\} \quad \text{and} \quad Z_\Lambda \in \text{rDiv}^d(Y_0), \]
and each point of \( Z_\Lambda \) yields a holomorphic section of
\[ \varphi : \mathcal{Y}_\Lambda \to \Lambda. \]
Let
\[ \mathcal{M}_g^d = \mathcal{M}(Y_0, Z_\Lambda) \]
be the group of orientation-preserving diffeomorphisms \( \gamma \) of \( Y_0 \) acting from right such that
\[ p_i \gamma = p_i \quad \text{for each point } p_i \text{ of } Z_\Lambda. \]
We put
\[ \Gamma_g^d = \Gamma(Y_0, Z_\Lambda) := \pi_0(\mathcal{M}(Y_0, Z_\Lambda)) \]
the group of isotopy classes of elements of \( \mathcal{M}_g^d = \mathcal{M}(Y_0, Z_\Lambda) \).
Then \( \Gamma_g^d = \Gamma(Y_0, Z_\Lambda) \) acts on the simple braid group
\[ SB_g^d = SB(Y_0, Z_\Lambda) \]
in a natural way.

By the monodromy action, we obtain a homomorphism
\[ \pi_1(\Lambda \setminus \Sigma_{\Lambda}', 0) \to \Gamma_g^d = \Gamma(Y_0, Z_\Lambda) = \pi_0(\mathcal{M}(Y_0, Z_\Lambda)). \]
We denote by
\[ \Gamma_\Lambda \subset \Gamma_g^d = \Gamma(Y_0, Z_\Lambda) \]
the image of the this monodromy homomorphism.
Combining the results above, we obtain the following:

Corollary.

Let \( X, \{Y_t\}_{t \in \Lambda}, Z_\Lambda = X \cap A \) and \( \Gamma_\Lambda \) be as above. Suppose that
\[
g > 0, \quad d \geq g + 4,
\]
and that a general hyperplane section of \( X \) is Plücker general. Then we have a natural isomorphism
\[
\pi_1(U_0(X, \mathbb{P}^N), A) \cong SB(Y_0, Z_\Lambda) \sslash \Gamma_\Lambda.
\]

Remark.

Let \( L \) be an ample line bundle of a smooth projective surface \( S \), and let \( X_m \subset \mathbb{P}^{N(m)} \) be the image of \( S \) by the embedding given by the complete linear system \( |L^\otimes m| \). If \( m \) is sufficiently large, then \( X_m \subset \mathbb{P}^{N(m)} \) satisfies \( d \geq g + 4 \).

According to this corollary, the conjecture that \( \pi_1(U_0(X, \mathbb{P}^N)) \) is “very small” is rephrased as the conjecture that \( \Gamma_\Lambda \subset \Gamma_g^d \) is “large”. As for the largeness of \( \Gamma_\Lambda \), we have the following result due to I. Smith (2001).

Theorem.

The vanishing cycles of the Lefschetz fibration \( \mathcal{Y}_\Lambda \to \Lambda \) fill up the fiber \( Y_0 \); that is, their complement is a bunch of discs. Moreover distinct points of \( Z_\Lambda \) are on distinct discs.