# Fundamental groups of complements of dual varieties in Grassmannian

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**Ichiro Shimada**
**(Hokkaido University, Sapporo, JAPAN)**

## §1. Introduction

This work is motivated by the conjecture in the paper

> [ADKY]
> D. Auroux, S. K. Donaldson, L. Katzarkov, and M. Yotov.
> Fundamental groups of complements of plane curves and symplectic invariants.
> Topology, 43(6): 1285-1318, 2004,

on the fundamental group

$$\pi_1(\mathbb{P}^2 \setminus B),$$

where $B$ is the branch curve of a general projection $S \longrightarrow \mathbb{P}^2$ from a smooth projective surface

$$S \subset \mathbb{P}^N.$$

By the previous work of Moishezon-Teicher-Robb and by their own new examples, they conjectured in [ADKY] that $\pi_1(\mathbb{P}^2 \setminus B)$ is "small".

Let $\mathrm{Gr}^2(\mathbb{P}^N)$ be the Grassmannian variety of linear subspaces in $\mathbb{P}^N$ with codimension 2. We put

$$U_0(S, \mathbb{P}^N) := \{\ L \in \mathrm{Gr}^2(\mathbb{P}^N) \ \mid\ L \cap S \text{ is smooth of dimension } 0\ \},$$

which is a Zariski open subset of the Grassmannian $\mathrm{Gr}^2(\mathbb{P}^N)$.

It is easy to see that there exists a natural inclusion

$$\mathbb{P}^2 \setminus B \ \hookrightarrow\ U_0(S, \mathbb{P}^N),$$

which induces a surjective homomorphism

$$\pi_1(\mathbb{P}^2 \setminus B) \ \twoheadrightarrow\ \pi_1(U_0(S, \mathbb{P}^N)).$$

Hence, if the conjecture is true, the fundamental group

$$\pi_1(U_0(S, \mathbb{P}^N))$$

should be "very small".

In this talk, we describe this fundamental group $\pi_1(U_0(S, \mathbb{P}^N))$ by means of *Zariski-van Kampen monodromy* associated with a Lefschetz pencil on $S$.

## §2. Zariski-van Kampen theorem

We formulate and prove a theorem of Zariski-van Kampen type on the fundamental groups of algebraic fiber spaces.

Let $X$ and $Y$ be smooth quasi-projective varieties, and let

$$f : X \to Y$$

be a dominant morphism.

For simplicity, we assume the following:

The general fiber of $f$ is connected.

For a point $y \in Y$, we put

$$F_y := f^{-1}(y).$$

We then choose general points

$$b \in Y \quad \text{and} \quad \tilde{b} \in F_b \subset X.$$

Let

$$\iota : F_b \hookrightarrow X$$

denote the inclusion.

We denote by

$$\mathrm{Sing}(f) \subset X$$

the Zariski closed subset consisting of the critical points of $f$.

The following is Nori's lemma:

**Proposition.**
If there exists a Zariski closed subset $\Xi \subset Y$ of codimension $\geq 2$ such that

$$F_y \setminus (F_y \cap \mathrm{Sing}(f)) \neq \emptyset \quad \text{for all} \quad y \notin \Xi,$$

then we have an exact sequence

$$\pi_1(F_b, \tilde{b}) \xrightarrow{\iota_*} \pi_1(X, \tilde{b}) \xrightarrow{f_*} \pi_1(Y, b) \to 1.$$

We will investigate

$$\mathrm{Ker}(\,\pi_1(F_b, \tilde{b}) \xrightarrow{\iota_*} \pi_1(X, \tilde{b})\,).$$

We fix, once and for all, a hypersurface $\Sigma$ of $Y$ with the following properties. We put

$$Y^\circ := Y \setminus \Sigma, \quad X^\circ := f^{-1}(Y^\circ),$$

and let

$$f^\circ : X^\circ \to Y^\circ$$

denote the restriction of $f$ to $X^\circ$.

The required property is as follows:

The morphism $f^\circ$ is smooth, and is locally trivial (in the category of topological spaces and continuous maps).

The existence of such a hypersurface $\Sigma$ follows from Hironaka's resolution of singularities, for example.

We can assume that $b \in Y^\circ$.

Let $I$ denote the closed interval $[0,1] \subset \mathbb{R}$. Let

$$\tilde{\alpha} \, : \, I \, \longrightarrow \, X^{\circ}$$

be a loop with the base point $\tilde{b} \in F_b \subset X^{\circ}$.

Then the family of pointed spaces

$$(F_{f(\tilde{\alpha}(t))}, \tilde{\alpha}(t))$$

is trivial over $I$, and hence we obtain an automorphism

$$\tilde{\mu}([\tilde{\alpha}]) : \pi_1(F_b, \tilde{b}) \xrightarrow{\sim} \pi_1(F_b, \tilde{b}), \qquad g \mapsto g^{\tilde{\mu}([\tilde{\alpha}])},$$

which depends only on the homotopy class of the loop $\tilde{\alpha}$ in $X^{\circ}$.

We thus obtain a homomorphism

$$\tilde{\mu} \, : \, \pi_1(X^{\circ}, \tilde{b}) \, \longrightarrow \, \mathrm{Aut}(\pi_1(F_b, \tilde{b})),$$

which is called the *monodromy on* $\pi_1(F_b)$.

Our main purpose is to describe the kernel of

$$\iota_* \, : \, \pi_1(F_b, \tilde{b}) \, \longrightarrow \, \pi_1(X, \tilde{b})$$

in terms of the monodromy $\tilde{\mu}$.

**Definition.**
Let $G$ be a group, and let $S$ be a subset of $G$. We denote by

$$\langle\langle S\rangle\rangle_G \ \triangleleft \ G$$

the smallest *normal* subgroup of $G$ containing $S$.

Let $\Gamma$ be a subgroup of $\mathrm{Aut}(G)$. We put

$$R(G,\Gamma) := \{\, g^{-1}g^{\gamma} \mid g \in G, \gamma \in \Gamma \,\} \ \subset \ G.$$

We then put

$$G/\!/\Gamma := G/\,\langle\langle R(G,\Gamma)\rangle\rangle_G,$$

and call $G/\!/\Gamma$ the *Zariski-van Kampen quotient* of $G$ by $\Gamma$

**Definition.**
An element

$$g^{-1}g^{\tilde{\mu}([\tilde{\alpha}])} \qquad (g \in \pi_1(F_b,\tilde{b}),\ [\tilde{\alpha}] \in \pi_1(X^{\circ},\tilde{b}))$$

of $\pi_1(F_b,\tilde{b})$ is called a *monodromy relation.*

We consider the following conditions.

(C1) $\mathrm{Sing}(f)$ is of codimension $\geq 2$ in $X$.

(C2) There exists a Zariski closed subset

$$\Xi \subset Y$$

with codimension $\geq 2$ such that $F_y$ is non-empty and irreducible for any $y \in Y \setminus \Xi$.

(C3) There exist a subspace $Z \subset Y$ and a continuous section

$$s_Z : Z \longrightarrow f^{-1}(Z)$$

of $f$ over $Z$ such that $Z \ni b$, that $Z \hookrightarrow Y$ induces a surjective homomorphism

$$\pi_2(Z, b) \twoheadrightarrow \pi_2(Y, b),$$

and that $s_Z(Z) \cap \mathrm{Sing}(f) = \emptyset$ and $s_Z(b) = \tilde{b}$.

Our generalized Zariski-van Kampen theorem is as follows:

**Theorem.**
We put

$$\tilde{K} := \mathrm{Ker}(\pi_1(X^\circ, \tilde{b}) \to \pi_1(X, \tilde{b})),$$

where $\pi_1(X^\circ, \tilde{b}) \longrightarrow \pi_1(X, \tilde{b})$ is induced by the inclusion. Under the above conditions (C1)-(C3), the kernel of

$$\iota_* : \pi_1(F_b, \tilde{b}) \longrightarrow \pi_1(X, \tilde{b})$$

is equal to the normal subgroup

$$\langle\langle R(\pi_1(F_b, \tilde{b}), \tilde{\mu}(\tilde{K}))\rangle\rangle = \langle\langle\{ g^{-1} g^{\tilde{\mu}([\tilde{\alpha}])} \mid g \in \pi_1(F_b, \tilde{b}),\ [\tilde{\alpha}] \in \tilde{K} \}\rangle\rangle$$

normally generated by the monodromy relations coming from the elements of $\tilde{K}$.

**Theorem.**
**Assume the following:**

(C1) $\mathrm{Sing}(f)$ is of codimension $\geq 2$ in $X$.

(C2) There exists a Zariski closed subset $\Xi \subset Y$ with codimension $\geq 2$ such that $F_y$ is non-empty and irreducible for any $y \in Y \setminus \Xi$.

(C4) There exist an irreducible smooth curve $C \subset Y$ passing through $b$ and a continuous section

$$s_C : C \to f^{-1}(C)$$

of $f$ over $C$ with the following properties:

(i) $\pi_1(C^\circ) \twoheadrightarrow \pi_1(Y^\circ)$, where $C^\circ := C \cap Y^\circ$.

(ii) $\pi_2(C) \twoheadrightarrow \pi_2(Y)$.

(iii) $C$ intersects each irreducible component of $\Sigma$ transversely at least at one point.

(iv) $s_C(C) \cap \mathrm{Sing}(f) = \emptyset$ and $s_C(b) = \tilde{b}$.

We put

$$K_C \;:=\; \mathrm{Ker}(\pi_1(C^\circ, b) \to \pi_1(C, b)).$$

By the section $s_C$, we have a monodromy action

$$\mu_C \;:\; \pi_1(C^\circ, b) \;\to\; \mathrm{Aut}(\pi_1(F_b, \tilde{b})).$$

Then we have

$$\mathrm{Ker}(\iota_*) \;=\; \langle\langle R(\pi_1(F_b), \mu_C(K_C))\rangle\rangle.$$

**Remark.**
The classical Zariski-van Kampen theorem deals with the situation where there exists a continuous section

$$s : Y \rightarrow X$$

of $f$ so that we have a monodromy

$$\mu := \tilde{\mu} \circ s_* \; : \; \pi_1(Y^\circ, b) \longrightarrow \mathrm{Aut}(\pi_1(F_b, \tilde{b})).$$

The main difference from the classical Zariski-van Kampen theorem is that we assume the existence of a section $s_Z$ of $f$ only over a subspace $Z \subset Y$ such that $\pi_2(Z) \twoheadrightarrow \pi_2(Y)$.

The necessity of the existence of such a section is shown by the following example.

**Example.**
Let $L \to \mathbb{P}^1$ be the total space of a line bundle of degree $d > 0$ on $\mathbb{P}^1$, and let $L^\times$ be the complement of the zero section with the natural projection

$$f \;:\; X := L^\times \;\to\; Y := \mathbb{P}^1,$$

so that $\pi_1(F_b) \cong \mathbb{Z}$. Then we have $\Sigma = \emptyset$, $X^\circ = X$ and hence $\tilde{K} = \operatorname{Ker}(\pi_1(X^\circ) \to \pi_1(X))$ is trivial. In particular, we have

$$R(\pi_1(F_b), \tilde{\mu}(\tilde{K})) = \{1\}.$$

On the other hand, the kernel of

$$\iota_* \;:\; \pi_1(F_b) \cong \mathbb{Z} \;\to\; \pi_1(X) \cong \mathbb{Z}/d\mathbb{Z}$$

is non-trivial, and equal to the image of the boundary homomorphism

$$\pi_2(Y) \cong \mathbb{Z} \;\to\; \pi_1(F_b) \cong \mathbb{Z}.$$

**Remark.**
The condition (C3) or (C4-(ii)) is vacuous if $\pi_2(Y) = 0$ (for example, if $Y$ is an abelian variety).

## §2. Grassmannian dual varieties

A Zariski closed subset of a projective space is said to be *non-degenerate* if it is not contained in any hyperplane.

We denote by $\mathrm{Gr}^c(\mathbb{P}^N)$ the Grassmannian variety of linear subspaces of the projective space $\mathbb{P}^N$ with codimension $c$.

Definition.
Let $W$ be a closed subscheme of $\mathbb{P}^N$ such that every irreducible component is of dimension $n$. For a positive integer $c \le n$, the *Grassmannian dual variety* of $W$ in $\mathrm{Gr}^c(\mathbb{P}^N)$ is the locus

$$\{\ L \in \mathrm{Gr}^c(\mathbb{P}^N)\ \mid\ W \cap L \text{ \emph{fails} to be smooth of dimension } n - c\ \}.$$

For a non-negative integer $k \le n$, we denote by

$$U_k(W, \mathbb{P}^N) \ \subset\ \mathrm{Gr}^{n-k}(\mathbb{P}^N)$$

the complement of the Grassmannian dual variety of $W$ in $\mathrm{Gr}^{n-k}(\mathbb{P}^N)$; that is, $U_k(W, \mathbb{P}^N)$ is

$$\left\{\ L \in \mathrm{Gr}^{n-k}(\mathbb{P}^N)\ \middle|\ \begin{array}{l} L \text{ intersects } W \text{ along a smooth} \\ \text{scheme of dimension } k \end{array}\ \right\}.$$

**Remark.**

When $n-k=1$, the variety $U_{n-1}(W,\mathbb{P}^N)$ is the complement of the usual dual variety

$$\left\{\; H \in (\mathbb{P}^N)^\vee \;\middle|\; \begin{array}{l} H \text{ fails to intersect } W \text{ along a} \\ \text{smooth scheme of dimension } n-1 \end{array} \right\}.$$

of $W$ in $\mathrm{Gr}^1(\mathbb{P}^N) = (\mathbb{P}^N)^\vee$.

Let

$$X \subset \mathbb{P}^N$$

be a smooth non-degenerate projective variety of dimension $n \geq 2$. We choose a general line

$$\Lambda \subset (\mathbb{P}^N)^\vee,$$

and a general point

$$0 \in \Lambda.$$

Let $H_t$ ($t \in \Lambda$) denote the pencil of hyperplanes corresponding to $\Lambda$, and let

$$A \cong \mathbb{P}^{N-2}$$

denote the axis of the pencil. We then put

$$Y_t := X \cap H_t \quad \text{and} \quad Z_\Lambda := X \cap A.$$

Then $Z_\Lambda$ is smooth, and every irreducible component of $Z_\Lambda$ is of dimension $n-2$. (In fact, $Z_\Lambda$ is irreducible if $n > 2$.)

We have natural inclusions

$$\mathrm{Gr}^{c-2}(A) \hookrightarrow \mathrm{Gr}^{c-1}(H_t) \hookrightarrow \mathrm{Gr}^c(\mathbb{P}^N).$$

Hence, for $k = 0, \dots, n-2$, we have natural inclusions

$$U_k(Z_\Lambda, A) \hookrightarrow U_k(Y_t, H_t) \hookrightarrow U_k(X, \mathbb{P}^N).$$

Indeed, we have

$$\begin{aligned} U_k(Z_\Lambda, A) &= \{\, L \in U_k(X, \mathbb{P}^N) \mid L \subset A \,\}, \\ U_k(Y_t, H_t) &= \{\, L \in U_k(X, \mathbb{P}^N) \mid L \subset H_t \,\}. \end{aligned}$$

Let $k$ be an integer such that $0 \le k \le n-2$. Then $U_k(Z_\Lambda, A)$ is non-empty. We choose a base point

$$L_o \in U_k(Z_\Lambda, A),$$

which serves also as a base point of $U_k(X, \mathbb{P}^N)$ and of $U_k(Y_t, H_t)$ by the natural inclusions.

We then consider the family

$$f : \mathcal{U}_k(\mathcal{Y}, \Lambda) \to \Lambda$$

of the varieties $U_k(Y_t, H_t)$, where

$$\mathcal{U}_k(\mathcal{Y}, \Lambda) := \{ (L, t) \in U_k(X, \mathbb{P}^N) \times \Lambda \mid L \subset H_t \} = \bigsqcup_{t \in \Lambda} U_k(Y_t, H_t),$$

and $f$ is the natural projection.

The point $L_o$ yields a holomorphic section

$$s_o : \Lambda \to \mathcal{U}_k(\mathcal{Y}, \Lambda)$$

of $f$. In fact, we have

$$L_o \in U_k(Z_\Lambda, A) \subset U_k(Y_t, H_t)$$

for all $t \in \Lambda$.

There exists a proper Zariski closed subset

$$\Sigma_\Lambda \subset \Lambda$$

such that $f$ is locally trivial (in the category of topological spaces and continuous maps) over $\Lambda \setminus \Sigma_\Lambda$. By the section $s_o$, we have the monodromy action

$$\pi_1(\Lambda \setminus \Sigma_\Lambda, 0) \to \mathrm{Aut}(\pi_1(U_k(Y_0, H_0), L_o)).$$

We have the following theorem of Lefschetz type.

**Theorem.**
Consider the homomorphism

$$\iota_* : \pi_1(U_k(Y_0, H_0), L_o) \to \pi_1(U_k(X, \mathbb{P}^N), L_o)$$

induced by the inclusion

$$\iota : U_k(Y_0, H_0) \hookrightarrow U_k(X, \mathbb{P}^N).$$

(1) If $k < n - 2$, then $\iota_*$ is an isomorphism.
(2) If $k = n - 2$, then $\iota_*$ is surjective and induces an isomorphism

$$\pi_1(U_k(Y_0, H_0))//\pi_1(\Lambda \setminus \Sigma_\Lambda) \cong \pi_1(U_k(X, \mathbb{P}^N)).$$

Compare this theorem with the following classical hyperplane section theorem of Lefschetz on homotopy groups:

**Theorem.**
Let $b$ be a point of $Y_0$, and let

$$j_k : \pi_k(Y_0, b) \longrightarrow \pi_k(X, b)$$

be the homomorphism of the $k$th homotopy groups induced by the inclusion.

(1) If $k < n - 1$, then $j_k$ is an isomorphism.
(2) If $k = n - 1$, then $j_k$ is surjective.

**Remark.**
The description of Zariski-van Kampen type of the kernel of $j_{n-1}$ is also given by Chéniot-Libgober (2003) and Chéniot- Eyral (2006).

**Sketch of the proof.**

We put

$$\mathcal{U}_k(\mathcal{Y}) \;:=\; \{\, (L, H) \in U_k(X, \mathbb{P}^N) \times (\mathbb{P}^N)^\vee \;\mid\; L \subset H \,\},$$

and consider the diagram

$$\begin{array}{ccc} \mathcal{U}_k(\mathcal{Y}) & \longrightarrow & U_k(X, \mathbb{P}^N) \\ \downarrow & & \\ (\mathbb{P}^N)^\vee & & \end{array}$$

of the natural projections. The morphism $\mathcal{U}_k(\mathcal{Y}) \to U_k(X, \mathbb{P}^N)$ is locally trivial (in the holomorphic category) with a fiber being a linear subspace of $(\mathbb{P}^N)^\vee$. Hence we obtain

$$\pi_1(\mathcal{U}_k(\mathcal{Y})) \;\cong\; \pi_1(U_k(X, \mathbb{P}^N)).$$

By definition, we have

$$\begin{array}{ccccc} U_k(Y_0, H_0) & \hookrightarrow & \mathcal{U}_k(\mathcal{Y}, \Lambda) & \hookrightarrow & \mathcal{U}_k(\mathcal{Y}) \\ \downarrow & \square & \downarrow & \square & \downarrow \\ H_0 & \in & \Lambda & \hookrightarrow & (\mathbb{P}^N)^\vee, \end{array}$$

and we have a section for $\mathcal{U}_k(\mathcal{Y}, \Lambda) \to \Lambda$. Moreover we have

$$\pi_2(\Lambda) \cong \pi_2((\mathbb{P}^N)^\vee).$$

By the generalized Zariski-van Kampen theorem, we obtain

$$\pi_1(U_k(Y_0, H_0))//\pi_1(\Lambda \setminus \Sigma_\Lambda) \cong \pi_1(\mathcal{U}_k(\mathcal{Y})).$$

If $k < n - 2$, then we have a surjection

$$\pi_1(U_k(Z_\Lambda, A)) \twoheadrightarrow \pi_1(U_k(Y_0, H_0)).$$

Because $\pi_1(\Lambda \setminus \Sigma_\Lambda)$ acts on $\pi_1(U_k(Z_\Lambda, A))$ trivially, it acts on $\pi_1(U_k(Y_0, H_0))$ trivially.

## §3. Simple braid groups

We study the case where $k = 0$.

Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective variety of dimension $n$ and degree $d$. Then we have

$$U_0(X, \mathbb{P}^N) = \{ \; L \in \mathrm{Gr}^n(\mathbb{P}^N) \;\; | \;\; L \text{ intersects } X \text{ at distinct } d \text{ points} \;\; \}.$$

By the previous theorem of Lefschetz type, it is enough to consider the case where $\dim X = 2$ in order to study $\pi_1(U_0(X, \mathbb{P}^N))$.

Hence, from now on, we assume

$$\dim X = 2,$$

and study the monodromy

$$\pi_1(\Lambda \setminus \Sigma_\Lambda) \ \longrightarrow \ \mathrm{Aut}(\pi_1(U_0(Y_0, H_0)))$$

associated with a Lefschetz pencil on $X$ corresponding to a general line $\Lambda \subset (\mathbb{P}^N)^\vee$. In this case,

$$Y_0 = X \cap H_0$$

is a compact Riemann surface embedded in $H_0 \cong \mathbb{P}^{N-1}$ as a non-degenerate curve of degree $d$.

Note that

$$U_0(Y_0, H_0) = \left\{ \ L \in \mathrm{Gr}^1(H_0) \ \middle| \ \begin{array}{l} L \text{ intersects the curve } Y_0 \text{ at} \\ \text{distinct } d \text{ points} \end{array} \right\}$$

is the complement of the dual hypersurface

$$(Y_0)^\vee \ \subset \ H_0^\vee \cong (\mathbb{P}^{N-1})^\vee$$

of $Y_0$.

First we define the simple braid group $SB_g^d$ of $d$ strings on a compact Riemann surface $C$ of genus $g > 0$.

We denote by

$$\mathrm{Div}^d(C) := (C \times \cdots \times C)/S_d$$

the variety of effective divisors of degree $d$ on $C$, and by

$$\mathrm{rDiv}^d(C) := \mathrm{Div}^d(C) \setminus \text{the big diagonal} \quad \subset \ \mathrm{Div}^d(C)$$

the Zariski open subset consisting of reduced divisors (that is, $\mathrm{rDiv}^d(C)$ is the configuration space of distinct $d$ points on $C$). We fix a base point

$$D_0 = p_1 + \cdots + p_d \ \in \ \mathrm{rDiv}^d(C).$$

Definition.
The *braid group*

$$B_g^d = B(C, D_0)$$

is defined to be the fundamental group $\pi_1(\mathrm{rDiv}^d(C), D_0)$.

The *simple braid group*

$$SB_g^d = SB(C, D_0)$$

is defined to be the kernel of the homomorphism

$$B(C, D_0) = \pi_1(\mathrm{rDiv}^d(C), D_0) \ \to \ \pi_1(\mathrm{Div}^d(C), D_0)$$

induced by the inclusion

$$\mathrm{rDiv}^d(C) \ \hookrightarrow \ \mathrm{Div}^d(C).$$

A braid on $C$ is called *simple* if it interchanges two points $p_i$ and $p_j$ of $D_0$ around a simple path connecting $p_i$ and $p_j$, and does not move other points.

It is easy to see that $SB_g^d$ is the subgroup of $B_g^d$ generated by simple braids, whence the name.

**Figure**

**Definition.**
Suppose that $C$ is embedded in $\mathbb{P}^M$ as a non-degenerate smooth curve. We say that $C \subset \mathbb{P}^M$ is *Plücker general* if the dual curve

$$\rho(C)^\vee \subset (\mathbb{P}^2)^\vee$$

of the image of a general projection

$$\rho : C \to \mathbb{P}^2$$

has only ordinary nodes and ordinary cusps as its singularities.

Our second main result is as follows:

**Theorem.**
Let $C \subset \mathbb{P}^M$ be a smooth non-degenerate projective curve of degree $d$ and genus $g > 0$. Suppose that

$$d \geq g + 4,$$

and that $C$ is Plücker general in $\mathbb{P}^M$. Let $D_0 = C \cap H_0$ be a general hyperplane section of $C$. Then

$$\pi_1(U_0(C, \mathbb{P}^M), D_0) = \pi_1((\mathbb{P}^M)^\vee \setminus C^\vee, H_0)$$

is canonically isomorphic to

$$SB(C, D_0).$$

For the proof, we use the following.

- We apply the generalized Zariski-van Kampen theorem to the natural morphism
$$\mathrm{Div}^d(C) \;\to\; \mathrm{Pic}^d(C),$$
where $\mathrm{Pic}^d(C)$ is the Picard variety. Note that
$$\pi_2(\mathrm{Pic}^d(C)) = 0.$$
Then we can show that, under the assumption $d \geq g + 4$,
$$\pi_1(\mathrm{Div}^d(C)) \;\cong\; \pi_1(\mathrm{Pic}^d(C)) \;=\; H_1(C, \mathbb{Z}).$$
- We then apply the generalized Zariski-van Kampen theorem to the natural morphism
$$\mathrm{rDiv}^d(C) \;\to\; \mathrm{Pic}^d(C).$$
If $L$ is a very ample line bundle of degree $d$ on $C$ that embeds $C$ into $\mathbb{P}^m$, then the fiber of $\mathrm{rDiv}^d(C) \to \mathrm{Pic}^d(C)$ over $[L] \in \mathrm{Pic}^d(C)$ is canonically isomorphic to
$$(\mathbb{P}^m)^\vee \setminus (C_L)^\vee \;=\; U_0(C_L, \mathbb{P}^m),$$
where $C_L \subset \mathbb{P}^m$ is the image of $C$ by the embedding by $L$. In particular, $\pi_1(U_0(C_L, \mathbb{P}^m))$ is isomorphic to
$$SB_g^d \;=\; \mathrm{Ker}(\pi_1(\mathrm{rDiv}^d(C)) \to \pi_1(\mathrm{Pic}^d(C))),$$
if $[L] \in \mathrm{Pic}^d(C)$ is a general point.

- Finally, we use Harris' result on Severi problem, which asserts that the moduli of irreducible nodal plane curves of degree $d$ and genus $g$ is irreducible. By the assumption of Plücker generality, we conclude that

$$\pi_1(U_0(C, \mathbb{P}^M)) \cong \pi_1(U_0(C_L, \mathbb{P}^m)),$$

where $[L] \in \mathrm{Pic}^d(C)$ is a general point.

Let

$$X \subset \mathbb{P}^N$$

be a smooth non-degenerate projective surface of degree $d$, and let

$$\{Y_t\}_{t\in\Lambda}$$

be a general pencil of hyperplane sections of $X$ parameterized by a line

$$\Lambda \subset (\mathbb{P}^N)^\vee.$$

Let

$$\varphi : \mathcal{Y}_\Lambda := \{ (x,t) \in X \times \Lambda \mid x \in H_t \} \to \Lambda$$

be the fibration of the pencil. We denote by

$$\Sigma'_\Lambda \subset \Lambda$$

the set of critical values of $\varphi$. Then $\varphi$ is locally trivial over $\Lambda \setminus \Sigma'_\Lambda$. Let $0 \in \Lambda$ be a general point of $\Lambda$. The corresponding member $Y_0$ is a compact Riemann surface of genus

$$g := (d + H_0 \cdot K_X)/2 + 1.$$

Consider the base locus

$$Z_\Lambda := X \cap A$$

of the pencil, where $A \cong \mathbb{P}^{N-2}$ is the axis of the pencil $\{H_t\}$.

Note that

$$U_0(Z_\Lambda, A) = \{A\} \quad \text{and} \quad Z_\Lambda \in \mathrm{rDiv}^d(Y_0),$$

and each point of $Z_\Lambda$ yields a holomorphic section of

$$\varphi : \mathcal{Y}_\Lambda \to \Lambda.$$

Let

$$\boldsymbol{\mathcal{M}_g^d = \mathcal{M}(Y_0, Z_\Lambda)}$$

be the group of orientation-preserving diffeomorphisms $\gamma$ of $Y_0$ acting from right such that

$$p_i{}^\gamma = p_i \quad \text{for each point } p_i \text{ of } Z_\Lambda.$$

We put

$$\Gamma_g^d = \Gamma(Y_0, Z_\Lambda) := \pi_0(\mathcal{M}(Y_0, Z_\Lambda))$$

the group of isotopy classes of elements of $\mathcal{M}_g^d = \mathcal{M}(Y_0, Z_\Lambda)$. Then $\Gamma_g^d = \Gamma(Y_0, Z_\Lambda)$ acts on the simple braid group

$$SB_g^d = SB(Y_0, Z_\Lambda)$$

in a natural way.

By the monodromy action, we obtain a homomorphism

$$\pi_1(\Lambda \setminus \Sigma'_\Lambda, 0) \ \longrightarrow\ \Gamma_g^d = \Gamma(Y_0, Z_\Lambda) = \pi_0(\mathcal{M}(Y_0, Z_\Lambda)).$$

We denote by

$$\Gamma_\Lambda \ \subset\ \Gamma_g^d = \Gamma(Y_0, Z_\Lambda)$$

the image of the this monodromy homomorphism.

Combining the results above, we obtain the following:

**Corollary.**
Let $X$, $\{Y_t\}_{t\in\Lambda}$, $Z_\Lambda = X \cap A$ and $\Gamma_\Lambda$ be as above. Suppose that

$$g > 0, \quad d \geq g + 4,$$

and that a general hyperplane section of $X$ is Plücker general. Then we have a natural isomorphism

$$\pi_1(U_0(X, \mathbb{P}^N), A) \;\cong\; SB(Y_0, Z_\Lambda)//\Gamma_\Lambda.$$

**Remark.**
Let $L$ be an ample line bundle of a smooth projective surface $S$, and let $X_m \subset \mathbb{P}^{N(m)}$ be the image of $S$ by the embedding given by the complete linear system $|L^{\otimes m}|$. If $m$ is sufficiently large, then $X_m \subset \mathbb{P}^{N(m)}$ satisfies $d \geq g + 4$.

According to this corollary, the conjecture that $\pi_1(U_0(X, \mathbb{P}^N))$ is "very small" is rephrased as the conjecture that $\Gamma_\Lambda \subset \Gamma_g^d$ is "large". As for the largeness of $\Gamma_\Lambda$, we have the following result due to I. Smith (2001).

**Theorem.**
The vanishing cycles of the Lefschetz fibration $\mathcal{Y}_\Lambda \to \Lambda$ fill up the fiber $Y_0$; that is, their complement is a bunch of discs. Moreover distinct points of $Z_\Lambda$ are on distinct discs.