Fundamental groups of complements of dual varieties in Grassmannian

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§1. Introduction

This work is motivated by the conjecture in the paper

[ADKY]
Fundamental groups of complements of plane curves and symplectic invariants.
Topology, 43(6): 1285-1318, 2004,

on the fundamental group

$$\pi_1(\mathbb{P}^2 \setminus B),$$

where $B$ is the branch curve of a general projection $S \to \mathbb{P}^2$ from a smooth projective surface

$$S \subset \mathbb{P}^N.$$  

By the previous work of Moishezon-Teicher-Robb and by their own new examples, they conjectured in [ADKY] that $\pi_1(\mathbb{P}^2 \setminus B)$ is “small”.


Let $\text{Gr}^2(\mathbb{P}^N)$ be the Grassmannian variety of linear subspaces in $\mathbb{P}^N$ with codimension 2. We put

$$U_0(S, \mathbb{P}^N) := \{ L \in \text{Gr}^2(\mathbb{P}^N) \mid L \cap S \text{ is smooth of dimension } 0 \},$$

which is a Zariski open subset of the Grassmannian $\text{Gr}^2(\mathbb{P}^N)$.

It is easy to see that there exists a natural inclusion

$$\mathbb{P}^2 \setminus B \hookrightarrow U_0(S, \mathbb{P}^N),$$

which induces a surjective homomorphism

$$\pi_1(\mathbb{P}^2 \setminus B) \longrightarrow \pi_1(U_0(S, \mathbb{P}^N)).$$

Hence, if the conjecture is true, the fundamental group

$$\pi_1(U_0(S, \mathbb{P}^N))$$

should be “very small”.

In this talk, we describe this fundamental group $\pi_1(U_0(S, \mathbb{P}^N))$ by means of Zariski-van Kampen monodromy associated with a Lefschetz pencil on $S$. 
§2. Zariski-van Kampen theorem

We formulate and prove a theorem of Zariski-van Kampen type on the fundamental groups of algebraic fiber spaces.

Let $X$ and $Y$ be smooth quasi-projective varieties, and let $f : X \to Y$ be a dominant morphism.

For simplicity, we assume the following:

The general fiber of $f$ is connected.

For a point $y \in Y$, we put $F_y := f^{-1}(y)$.

We then choose general points $b \in Y$ and $\tilde{b} \in F_b \subset X$.

Let $\iota : F_b \hookrightarrow X$ denote the inclusion.
We denote by
\[ \text{Sing}(f) \subset X \]
the Zariski closed subset consisting of the critical points of \( f \).

The following is Nori’s lemma:

**Proposition.**
If there exists a Zariski closed subset \( \Xi \subset Y \) of codimension \( \geq 2 \) such that
\[ F_y \setminus (F_y \cap \text{Sing}(f)) \neq \emptyset \quad \text{for all} \quad y \notin \Xi, \]
then we have an exact sequence
\[ \pi_1(F_b, \tilde{b}) \xrightarrow{\iota_*} \pi_1(X, \tilde{b}) \xrightarrow{f_*} \pi_1(Y, b) \to 1. \]

We will investigate
\[ \text{Ker}(\pi_1(F_b, \tilde{b}) \xrightarrow{\iota_*} \pi_1(X, \tilde{b})). \]
We fix, once and for all, a hypersurface $\Sigma$ of $Y$ with the following properties. We put

$$Y^\circ := Y \setminus \Sigma, \quad X^\circ := f^{-1}(Y^\circ),$$

and let

$$f^\circ : X^\circ \to Y^\circ$$

denote the restriction of $f$ to $X^\circ$.

The required property is as follows:

The morphism $f^\circ$ is smooth, and is locally trivial (in the category of topological spaces and continuous maps).

The existence of such a hypersurface $\Sigma$ follows from Hironaka’s resolution of singularities, for example.

We can assume that $b \in Y^\circ$. 
Let $I$ denote the closed interval $[0, 1] \subset \mathbb{R}$. Let

$$\tilde{\alpha} : I \to X^\circ$$

be a loop with the base point $\tilde{b} \in F_b \subset X^\circ$.

Then the family of pointed spaces

$$(F_{f(\tilde{\alpha}(t))}, \tilde{\alpha}(t))$$

is trivial over $I$, and hence we obtain an automorphism

$$\tilde{\mu}([\tilde{\alpha}]) : \pi_1(F_b, \tilde{b}) \xrightarrow{\sim} \pi_1(F_b, \tilde{b}), \quad g \mapsto g^{\tilde{\mu}([\tilde{\alpha}])},$$

which depends only on the homotopy class of the loop $\tilde{\alpha}$ in $X^\circ$.

We thus obtain a homomorphism

$$\tilde{\mu} : \pi_1(X^\circ, \tilde{b}) \to \text{Aut}(\pi_1(F_b, \tilde{b})), $$

which is called the *monodromy on* $\pi_1(F_b)$.

Our main purpose is to describe the kernel of

$$\iota_* : \pi_1(F_b, \tilde{b}) \to \pi_1(X, \tilde{b})$$

in terms of the monodromy $\tilde{\mu}$. 
Definition.
Let $G$ be a group, and let $S$ be a subset of $G$. We denote by
$$\langle \langle S \rangle \rangle_G \triangleleft G$$
the smallest normal subgroup of $G$ containing $S$.

Let $\Gamma$ be a subgroup of $\text{Aut}(G)$. We put
$$R(G, \Gamma) := \{ g^{-1} g^\gamma \mid g \in G, \gamma \in \Gamma \} \subset G.$$  

We then put
$$G//\Gamma := G/\langle \langle R(G, \Gamma) \rangle \rangle_G,$$
and call $G//\Gamma$ the Zariski-van Kampen quotient of $G$ by $\Gamma$.

Definition.
An element
$$g^{-1} g^\tilde{\mu}([\tilde{\alpha}]) \quad (g \in \pi_1(F_b, \tilde{b}), \ [\tilde{\alpha}] \in \pi_1(X^\circ, \tilde{b}))$$
of $\pi_1(F_b, \tilde{b})$ is called a monodromy relation.
We consider the following conditions.

(C1) $\text{Sing}(f)$ is of codimension $\geq 2$ in $X$.

(C2) There exists a Zariski closed subset

$$\Xi \subset Y$$

with codimension $\geq 2$ such that $F_y$ is non-empty and irreducible for any $y \in Y \setminus \Xi$.

(C3) There exist a subspace $Z \subset Y$ and a continuous section

$$s_Z : Z \to f^{-1}(Z)$$

of $f$ over $Z$ such that $Z \ni b$, that $Z \hookrightarrow Y$ induces a surjective homomorphism

$$\pi_2(Z, b) \rightarrow \pi_2(Y, b),$$

and that $s_Z(Z) \cap \text{Sing}(f) = \emptyset$ and $s_Z(b) = \tilde{b}$. 
Our generalized Zariski-van Kampen theorem is as follows:

Theorem.
We put
\[ \tilde{K} := \text{Ker}(\pi_1(X^\circ, \tilde{b}) \to \pi_1(X, \tilde{b})), \]
where \( \pi_1(X^\circ, \tilde{b}) \to \pi_1(X, \tilde{b}) \) is induced by the inclusion. Under the above conditions (C1)-(C3), the kernel of
\[ \iota_* : \pi_1(F_b, \tilde{b}) \to \pi_1(X, \tilde{b}) \]
is equal to the normal subgroup
\[ \langle \langle R(\pi_1(F_b, \tilde{b}), \tilde{\mu}(\tilde{K})) \rangle \rangle = \langle \langle \{ g^{-1}g^{\tilde{\mu}(\tilde{\alpha})} \mid g \in \pi_1(F_b, \tilde{b}), [\tilde{\alpha}] \in \tilde{K} \} \rangle \rangle \]

normally generated by the monodromy relations coming from the elements of \( \tilde{K} \).
Theorem.
Assume the following:
(C1) $\text{Sing}(f)$ is of codimension $\geq 2$ in $X$.
(C2) There exists a Zariski closed subset $\Xi \subset Y$ with codimension $\geq 2$ such that $F_y$ is non-empty and irreducible for any $y \in Y \setminus \Xi$.
(C4) There exist an irreducible smooth curve $C \subset Y$ passing through $b$ and a continuous section
$$s_C : C \to f^{-1}(C)$$
of $f$ over $C$ with the following properties:
(i) $\pi_1(C^\circ) \to \pi_1(Y^\circ)$, where $C^\circ := C \cap Y^\circ$.
(ii) $\pi_2(C) \to \pi_2(Y)$.
(iii) $C$ intersects each irreducible component of $\Sigma$ transversely at least at one point.
(iv) $s_C(C) \cap \text{Sing}(f) = \emptyset$ and $s_C(b) = \tilde{b}$.

We put
$$K_C := \ker(\pi_1(C^\circ, b) \to \pi_1(C, b)).$$
By the section $s_C$, we have a monodromy action
$$\mu_C : \pi_1(C^\circ, b) \to \text{Aut}(\pi_1(F_b, \tilde{b})).$$
Then we have
$$\ker(\iota_*) = \langle \langle R(\pi_1(F_b), \mu_C(K_C)) \rangle \rangle.$$
Remark.
The classical Zariski-van Kampen theorem deals with the situation where there exists a continuous section
\[ s : Y \to X \]
of \( f \) so that we have a monodromy
\[ \mu := \tilde{\mu} \circ s_* : \pi_1(Y^\circ, b) \to Aut(\pi_1(F_b, \tilde{b})). \]
The main difference from the classical Zariski-van Kampen theorem is that we assume the existence of a section \( s_Z \) of \( f \) only over a subspace \( Z \subset Y \) such that \( \pi_2(Z) \to \pi_2(Y) \).
The necessity of the existence of such a section is shown by the following example.

Example.
Let \( L \to \mathbb{P}^1 \) be the total space of a line bundle of degree \( d > 0 \) on \( \mathbb{P}^1 \), and let \( L^\times \) be the complement of the zero section with the natural projection
\[
f : X := L^\times \to Y := \mathbb{P}^1,\]
so that \( \pi_1(F_b) \cong \mathbb{Z} \). Then we have \( \Sigma = \emptyset, \ X^\circ = X \) and hence \( \widetilde{K} = \ker(\pi_1(X^\circ) \to \pi_1(X)) \) is trivial. In particular, we have
\[
R(\pi_1(F_b), \tilde{\mu}(\widetilde{K})) = \{1\}.
\]
On the other hand, the kernel of
\[
\iota_* : \pi_1(F_b) \cong \mathbb{Z} \to \pi_1(X) \cong \mathbb{Z}/d\mathbb{Z}
\]
is non-trivial, and equal to the image of the boundary homomorphism
\[
\pi_2(Y) \cong \mathbb{Z} \to \pi_1(F_b) \cong \mathbb{Z}.
\]

Remark.
The condition (C3) or (C4-(ii)) is vacuous if \( \pi_2(Y) = 0 \) (for example, if \( Y \) is an abelian variety).
§2. Grassmannian dual varieties

A Zariski closed subset of a projective space is said to be non-degenerate if it is not contained in any hyperplane.

We denote by $\text{Gr}^c(\mathbb{P}^N)$ the Grassmannian variety of linear subspaces of the projective space $\mathbb{P}^N$ with codimension $c$.

Definition.
Let $W$ be a closed subscheme of $\mathbb{P}^N$ such that every irreducible component is of dimension $n$. For a positive integer $c \leq n$, the Grassmannian dual variety of $W$ in $\text{Gr}^c(\mathbb{P}^N)$ is the locus

$$\{ \ L \in \text{Gr}^c(\mathbb{P}^N) \mid W \cap L \text{ fails to be smooth of dimension } n - c \ \}.$$ 

For a non-negative integer $k \leq n$, we denote by

$$U_k(W, \mathbb{P}^N) \subset \text{Gr}^{n-k}(\mathbb{P}^N)$$

the complement of the Grassmannian dual variety of $W$ in $\text{Gr}^{n-k}(\mathbb{P}^N)$; that is, $U_k(W, \mathbb{P}^N)$ is

$$\{ \ L \in \text{Gr}^{n-k}(\mathbb{P}^N) \mid \ L \text{ intersects } W \text{ along a smooth scheme of dimension } k \ \}.$$
Remark.
When $n - k = 1$, the variety $U_{n-1}(W, \mathbb{P}^N)$ is the complement of the usual dual variety

$$\left\{ H \in (\mathbb{P}^N)^\vee \mid H \text{ fails to intersect } W \text{ along a smooth scheme of dimension } n - 1 \right\}.$$

of $W$ in $\text{Gr}^1(\mathbb{P}^N) = (\mathbb{P}^N)^\vee$. 
Let
\[ X \subset \mathbb{P}^N \]
be a smooth non-degenerate projective variety of dimension \( n \geq 2 \). We choose a general line
\[ \Lambda \subset (\mathbb{P}^N)^\vee, \]
and a general point
\[ 0 \in \Lambda. \]
Let \( H_t \ (t \in \Lambda) \) denote the pencil of hyperplanes corresponding to \( \Lambda \), and let
\[ A \cong \mathbb{P}^{N-2} \]
denote the axis of the pencil. We then put
\[ Y_t := X \cap H_t \quad \text{and} \quad Z_\Lambda := X \cap A. \]
Then \( Z_\Lambda \) is smooth, and every irreducible component of \( Z_\Lambda \) is of dimension \( n - 2 \). (In fact, \( Z_\Lambda \) is irreducible if \( n > 2 \).)

We have natural inclusions
\[ \text{Gr}^{c-2}(A) \hookrightarrow \text{Gr}^{c-1}(H_t) \hookrightarrow \text{Gr}^c(\mathbb{P}^N). \]
Hence, for \( k = 0, \ldots, n - 2 \), we have natural inclusions
\[ U_k(Z_\Lambda, A) \hookrightarrow U_k(Y_t, H_t) \hookrightarrow U_k(X, \mathbb{P}^N). \]
Indeed, we have
\[ U_k(Z_\Lambda, A) = \{ L \in U_k(X, \mathbb{P}^N) \mid L \subset A \}, \]
\[ U_k(Y_t, H_t) = \{ L \in U_k(X, \mathbb{P}^N) \mid L \subset H_t \}. \]
Let $k$ be an integer such that $0 \leq k \leq n - 2$. Then $U_k(Z_\Lambda, A)$ is non-empty. We choose a base point

$$L_o \in U_k(Z_\Lambda, A),$$

which serves also as a base point of $U_k(X, \mathbb{P}^N)$ and of $U_k(Y_t, H_t)$ by the natural inclusions.

We then consider the family

$$f : U_k(\mathcal{Y}, \Lambda) \to \Lambda$$

of the varieties $U_k(Y_t, H_t)$, where

$$U_k(\mathcal{Y}, \Lambda) := \{ (L, t) \in U_k(X, \mathbb{P}^N) \times \Lambda \mid L \subset H_t \} = \bigsqcup_{t \in \Lambda} U_k(Y_t, H_t),$$

and $f$ is the natural projection.

The point $L_o$ yields a holomorphic section

$$s_o : \Lambda \to U_k(\mathcal{Y}, \Lambda)$$

of $f$. In fact, we have

$$L_o \in U_k(Z_\Lambda, A) \subset U_k(Y_t, H_t)$$

for all $t \in \Lambda$. 

There exists a proper Zariski closed subset
\[ \Sigma_\Lambda \subset \Lambda \]
such that \( f \) is locally trivial (in the category of topological spaces and continuous maps) over \( \Lambda \setminus \Sigma_\Lambda \). By the section \( s_o \), we have the monodromy action
\[ \pi_1(\Lambda \setminus \Sigma_\Lambda, 0) \to \text{Aut}(\pi_1(U_k(Y_0, H_0), L_o)). \]

We have the following theorem of Lefschetz type.

**Theorem.**
Consider the homomorphism
\[ \iota_* : \pi_1(U_k(Y_0, H_0), L_o) \to \pi_1(U_k(X, \mathbb{P}^N), L_o) \]
induced by the inclusion
\[ \iota : U_k(Y_0, H_0) \hookrightarrow U_k(X, \mathbb{P}^N). \]

(1) If \( k < n - 2 \), then \( \iota_* \) is an isomorphism.
(2) If \( k = n - 2 \), then \( \iota_* \) is surjective and induces an isomorphism
\[ \pi_1(U_k(Y_0, H_0))/\pi_1(\Lambda \setminus \Sigma_\Lambda) \cong \pi_1(U_k(X, \mathbb{P}^N)). \]
Compare this theorem with the following classical hyperplane section theorem of Lefschetz on homotopy groups:

**Theorem.**
Let $b$ be a point of $Y_0$, and let

$$j_k : \pi_k(Y_0, b) \to \pi_k(X, b)$$

be the homomorphism of the $k$th homotopy groups induced by the inclusion.

(1) If $k < n - 1$, then $j_k$ is an isomorphism.

(2) If $k = n - 1$, then $j_k$ is surjective.

**Remark.**
The description of Zariski-van Kampen type of the kernel of $j_{n-1}$ is also given by Chéniot-Libgober (2003) and Chéniot-Eyral (2006).
Sketch of the proof.

We put

\[ U_k(Y) := \{ (L, H) \in U_k(X, \mathbb{P}^N) \times (\mathbb{P}^N)^\vee \mid L \subset H \} , \]

and consider the diagram

\[ \begin{array}{ccc}
U_k(Y) & \to & U_k(X, \mathbb{P}^N) \\
\downarrow & & \downarrow \\
(\mathbb{P}^N)^\vee & & (\mathbb{P}^N)^\vee
\end{array} \]

of the natural projections. The morphism \( U_k(Y) \to U_k(X, \mathbb{P}^N) \) is locally trivial (in the holomorphic category) with a fiber being a linear subspace of \((\mathbb{P}^N)^\vee\). Hence we obtain

\[ \pi_1(U_k(Y)) \cong \pi_1(U_k(X, \mathbb{P}^N)). \]

By definition, we have

\[ \begin{array}{ccc}
U_k(Y_0, H_0) & \hookrightarrow & U_k(Y, \Lambda) \hookrightarrow U_k(Y) \\
\downarrow & & \square \downarrow & \square \downarrow \\
H_0 & \in & \Lambda & \hookrightarrow (\mathbb{P}^N)^\vee,
\end{array} \]

and we have a section for \( U_k(Y, \Lambda) \to \Lambda \). Moreover we have

\[ \pi_2(\Lambda) \cong \pi_2((\mathbb{P}^N)^\vee). \]
By the generalized Zariski-van Kampen theorem, we obtain
\[ \pi_1(U_k(Y_0, H_0))//\pi_1(\Lambda \setminus \Sigma) \cong \pi_1(U_k(Y)). \]
If \( k < n - 2 \), then we have a surjection
\[ \pi_1(U_k(Z, A)) \rightarrow \pi_1(U_k(Y_0, H_0)). \]
Because \( \pi_1(\Lambda \setminus \Sigma) \) acts on \( \pi_1(U_k(Z, A)) \) trivially, it acts on \( \pi_1(U_k(Y_0, H_0)) \) trivially.
§3. Simple braid groups

We study the case where $k = 0$.

Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective variety of dimension $n$ and degree $d$. Then we have

$$U_0(X, \mathbb{P}^N) = \{ L \in \text{Gr}^n(\mathbb{P}^N) \mid L \text{ intersects } X \text{ at distinct } d \text{ points} \}.$$ 

By the previous theorem of Lefschetz type, it is enough to consider the case where $\dim X = 2$ in order to study $\pi_1(U_0(X, \mathbb{P}^N))$. 
Hence, from now on, we assume
\[ \dim X = 2, \]
and study the monodromy
\[ \pi_1(\Lambda \setminus \Sigma_\Lambda) \to \text{Aut}(\pi_1(U_0(Y_0, H_0))) \]
associated with a Lefschetz pencil on \( X \) corresponding to a general line \( \Lambda \subset (\mathbb{P}^N)^\vee \). In this case,
\[ Y_0 = X \cap H_0 \]
is a compact Riemann surface embedded in \( H_0 \cong \mathbb{P}^{N-1} \) as a non-degenerate curve of degree \( d \).

Note that
\[ U_0(Y_0, H_0) = \left\{ L \in \text{Gr}^1(H_0) \mid \begin{array}{c} L \text{ intersects the curve } Y_0 \text{ at } \\ \text{distinct } d \text{ points} \end{array} \right\} \]
is the complement of the dual hypersurface
\[ (Y_0)^\vee \subset H_0^\vee \cong (\mathbb{P}^{N-1})^\vee \]
of \( Y_0 \).
First we define the simple braid group $SB^d_g$ of $d$ strings on a compact Riemann surface $C$ of genus $g > 0$.

We denote by

$$\text{Div}^d(C) := (C \times \cdots \times C)/S_d$$

the variety of effective divisors of degree $d$ on $C$, and by

$$\text{rDiv}^d(C) := \text{Div}^d(C) \setminus \text{the big diagonal} \subset \text{Div}^d(C)$$

the Zariski open subset consisting of reduced divisors (that is, $\text{rDiv}^d(C)$ is the configuration space of distinct $d$ points on $C$). We fix a base point

$$D_0 = p_1 + \cdots + p_d \in \text{rDiv}^d(C).$$

**Definition.**

The *braid group*

$$B^d_g = B(C, D_0)$$

is defined to be the fundamental group $\pi_1(\text{rDiv}^d(C), D_0)$.

The *simple braid group*

$$SB^d_g = SB(C, D_0)$$

is defined to be the kernel of the homomorphism

$$B(C, D_0) = \pi_1(\text{rDiv}^d(C), D_0) \to \pi_1(\text{Div}^d(C), D_0)$$

induced by the inclusion

$$\text{rDiv}^d(C) \hookrightarrow \text{Div}^d(C).$$
A braid on $C$ is called \textit{simple} if it interchanges two points $p_i$ and $p_j$ of $D_0$ around a simple path connecting $p_i$ and $p_j$, and does not move other points.

It is easy to see that $SB^d_g$ is the subgroup of $B^d_g$ generated by simple braids, whence the name.
Definition.
Suppose that $C$ is embedded in $\mathbb{P}^M$ as a non-degenerate smooth curve. We say that $C \subset \mathbb{P}^M$ is Plücker general if the dual curve
\[ \rho(C)^\vee \subset (\mathbb{P}^2)^\vee \]
of the image of a general projection
\[ \rho : C \to \mathbb{P}^2 \]
has only ordinary nodes and ordinary cusps as its singularities.

Our second main result is as follows:

Theorem.
Let $C \subset \mathbb{P}^M$ be a smooth non-degenerate projective curve of degree $d$ and genus $g > 0$. Suppose that
\[ d \geq g + 4, \]
and that $C$ is Plücker general in $\mathbb{P}^M$. Let $D_0 = C \cap H_0$ be a general hyperplane section of $C$. Then
\[ \pi_1(U_0(C, \mathbb{P}^M), D_0) = \pi_1((\mathbb{P}^M)^\vee \setminus C^\vee, H_0) \]
is canonically isomorphic to
\[ SB(C, D_0). \]
For the proof, we use the following.

- We apply the generalized Zariski-van Kampen theorem to the natural morphism
  \[ \text{Div}^d(C) \to \text{Pic}^d(C), \]
  where \( \text{Pic}^d(C) \) is the Picard variety. Note that
  \[ \pi_2(\text{Pic}^d(C)) = 0. \]
  Then we can show that, under the assumption \( d \geq g + 4 \),
  \[ \pi_1(\text{Div}^d(C)) \cong \pi_1(\text{Pic}^d(C)) = H_1(C, \mathbb{Z}). \]
- We then apply the generalized Zariski-van Kampen theorem to the natural morphism
  \[ \text{rDiv}^d(C) \to \text{Pic}^d(C). \]
  If \( L \) is a very ample line bundle of degree \( d \) on \( C \) that embeds \( C \) into \( \mathbb{P}^m \), then the fiber of \( \text{rDiv}^d(C) \to \text{Pic}^d(C) \) over \([L] \in \text{Pic}^d(C)\) is canonically isomorphic to
  \[ (\mathbb{P}^m)^\vee \setminus (C_L)^\vee = U_0(C_L, \mathbb{P}^m), \]
  where \( C_L \subset \mathbb{P}^m \) is the image of \( C \) by the embedding by \( L \). In particular, \( \pi_1(U_0(C_L, \mathbb{P}^m)) \) is isomorphic to
  \[ SB_g^d = \text{Ker}(\pi_1(\text{rDiv}^d(C)) \to \pi_1(\text{Pic}^d(C))), \]
  if \([L] \in \text{Pic}^d(C)\) is a general point.
Finally, we use Harris’ result on Severi problem, which asserts that the moduli of irreducible nodal plane curves of degree $d$ and genus $g$ is irreducible. By the assumption of Plücker generality, we conclude that

$$\pi_1(U_0(C, \mathbb{P}^M)) \cong \pi_1(U_0(C_L, \mathbb{P}^m)),$$

where $[L] \in \text{Pic}^d(C)$ is a general point.
Let \( X \subset \mathbb{P}^N \) be a smooth non-degenerate projective surface of degree \( d \), and let \( \{Y_t\}_{t \in \Lambda} \) be a general pencil of hyperplane sections of \( X \) parameterized by a line \( \Lambda \subset (\mathbb{P}^N)^\vee \).

Let \( \varphi : \mathcal{Y}_\Lambda := \{ (x, t) \in X \times \Lambda \mid x \in H_t \} \to \Lambda \) be the fibration of the pencil. We denote by \( \Sigma'_\Lambda \subset \Lambda \) the set of critical values of \( \varphi \). Then \( \varphi \) is locally trivial over \( \Lambda \setminus \Sigma'_\Lambda \).

Let 0 \in \Lambda be a general point of \( \Lambda \). The corresponding member \( Y_0 \) is a compact Riemann surface of genus

\[
g := (d + H_0 \cdot K_X)/2 + 1.
\]

Consider the base locus \( Z_\Lambda := X \cap A \) of the pencil, where \( A \cong \mathbb{P}^{N-2} \) is the axis of the pencil \( \{H_t\} \).

Note that

\[
U_0(Z_\Lambda, A) = \{A\} \quad \text{and} \quad Z_\Lambda \in \text{rDiv}^d(Y_0),
\]

and each point of \( Z_\Lambda \) yields a holomorphic section of

\[
\varphi : \mathcal{Y}_\Lambda \to \Lambda.
\]
Let
\[ M^d_g = \mathcal{M}(Y_0, Z_\Lambda) \]
be the group of orientation-preserving diffeomorphisms \( \gamma \) of \( Y_0 \) acting from right such that
\[ p_i \gamma = p_i \quad \text{for each point } p_i \text{ of } Z_\Lambda. \]
We put
\[ \Gamma^d_g = \Gamma(Y_0, Z_\Lambda) := \pi_0(\mathcal{M}(Y_0, Z_\Lambda)) \]
the group of isotopy classes of elements of \( M^d_g = \mathcal{M}(Y_0, Z_\Lambda) \). Then \( \Gamma^d_g = \Gamma(Y_0, Z_\Lambda) \) acts on the simple braid group
\[ SB^d_g = SB(Y_0, Z_\Lambda) \]
in a natural way.

By the monodromy action, we obtain a homomorphism
\[ \pi_1(\Lambda \setminus \Sigma'_\Lambda, 0) \to \Gamma^d_g = \Gamma(Y_0, Z_\Lambda) = \pi_0(\mathcal{M}(Y_0, Z_\Lambda)). \]
We denote by
\[ \Gamma_\Lambda \subset \Gamma^d_g = \Gamma(Y_0, Z_\Lambda) \]
the image of the this monodromy homomorphism.
Combining the results above, we obtain the following:

Corollary.
Let $X, \{Y_t\}_{t \in \Lambda}, Z_\Lambda = X \cap A$ and $\Gamma_\Lambda$ be as above. Suppose that
\[ g > 0, \quad d \geq g + 4, \]
and that a general hyperplane section of $X$ is Plücker general. Then
we have a natural isomorphism
\[ \pi_1(U_0(X, \mathbb{P}^N), A) \cong SB(Y_0, Z_\Lambda)/\Gamma_\Lambda. \]

Remark.
Let $L$ be an ample line bundle of a smooth projective surface $S$, and let
$X_m \subset \mathbb{P}^{N(m)}$ be the image of $S$ by the embedding given by the complete
linear system $|L^\otimes m|$. If $m$ is sufficiently large, then $X_m \subset \mathbb{P}^{N(m)}$ satisfies
$d \geq g + 4$. 
According to this corollary, the conjecture that $\pi_1(U_0(X, \mathbb{P}^N))$ is “very small” is rephrased as the conjecture that $\Gamma_\Lambda \subset \Gamma_d^g$ is “large”. As for the largeness of $\Gamma_\Lambda$, we have the following result due to I. Smith (2001).

Theorem.
The vanishing cycles of the Lefschetz fibration $\mathcal{Y}_\Lambda \to \Lambda$ fill up the fiber $Y_0$; that is, their complement is a bunch of discs. Moreover distinct points of $Z_\Lambda$ are on distinct discs.