Dessins d’enfants and transcendental lattices of singular $K3$ surfaces

\[ \Downarrow \]

Dessins d’enfants and transcendental lattices of extremal elliptic surfaces

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- By a lattice, we mean a finitely generated free $\mathbb{Z}$-module $\Lambda$ equipped with a non-degenerate symmetric bilinear form

$$\Lambda \times \Lambda \rightarrow \mathbb{Z}.$$
§1. Introduction of the theory of dessins

Definition. A dessin d’enfant (a dessin, for short) is a connected graph that is bi-colored (i.e., each vertex is colored by black or while, and every edge connects a black vertex and a white vertex) and oriented (i.e., for each vertex, a cyclic ordering is given to the set of edges emitting from the vertex). Two dessins are isomorphic if there exists an isomorphism of graphs between them that preserves the coloring and the orientation.

We denote by $\mathcal{D}(n)$ the set of isomorphism classes of dessins with $n$ edges.
Definition. A permutation pair is a pair \((\sigma_0, \sigma_1)\) of elements of the symmetric group \(S_n\) such that the subgroup \(<\sigma_0, \sigma_1> \subset S_n\) is a transitive permutation group. Two permutation pairs \((\sigma_0, \sigma_1)\) and \((\sigma'_0, \sigma'_1)\) are isomorphic if there exists \(g \in S_n\) such that \(\sigma'_0 = g^{-1}\sigma_0 g\) and \(\sigma'_1 = g^{-1}\sigma_1 g\) hold.

We denote by \(\mathcal{P}(n)\) the set of isomorphism classes \([\sigma_0, \sigma_1]\) of permutation pairs \((\sigma_0, \sigma_1)\) of elements of \(S_n\).

Definition. A Belyĭ pair is a pair \((C, \beta)\) of a compact connected Riemann surface \(C\) and a finite morphism \(C \to \mathbb{P}^1\) that is étale over \(\mathbb{P}^1 \setminus \{0, 1, \infty\}\). Two Belyĭ pairs \((C, \beta)\) and \((C', \beta')\) are isomorphic if there exists an isomorphism \(\phi : C \cong C'\) such that \(\phi \circ \beta' = \beta\).

We denote by \(\mathcal{B}(n)\) the set of isomorphism classes of Belyĭ pairs of degree \(n\).
Proposition. For each $n$, there exist canonical bijections

$$
\mathcal{D}(n) \sim \mathcal{P}(n) \sim \mathcal{B}(n).
$$

Proof. First we define $f_{\mathcal{DP}} : \mathcal{D}(n) \to \mathcal{P}(n)$. Let $D \in \mathcal{D}(n)$ be given. We number the edges of $D$ by $1, \ldots, n$, and let $\sigma_0 \in S_n$ (resp. $\sigma_1 \in S_n$) be the product of the cyclic permutations of the edges at the black (resp. white) vertices coming from the cyclic ordering. Since $D$ is connected, $\langle \sigma_0, \sigma_1 \rangle$ is transitive. The isomorphism class $[\sigma_0, \sigma_1]$ does not depend on the choice of the numbering of edges. Hence $f_{\mathcal{DP}}(D) := [\sigma_0, \sigma_1]$ is well-defined.
Next, we define \( f_{PB} : \mathcal{P}(n) \to \mathcal{B}(n) \). We choose a base point \( b_0 \in \mathbb{P}^1 \setminus \{0, 1, \infty\} \) on the real open segment \((0, 1) \subset \mathbb{R}\), and consider the fundamental group \( \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, b_0) \), which is a free group generated by the homotopy classes \( \gamma_0 \) and \( \gamma_1 \) of the loops depicted below:

Let \([\sigma_0, \sigma_1] \in \mathcal{P}(n)\) be given. Then we have an étale covering of degree \( n \)

\[
\beta^0 : C^0 \to \mathbb{P}^1 \setminus \{0, 1, \infty\}
\]
corresponding the homomorphim \( \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, b_0) \to S_n \) defined by \( \gamma_0 \mapsto \sigma_0 \) and \( \gamma_1 \mapsto \sigma_1 \). Compactifying \((C^0, \beta^0)\), we obtain a Belyĭ pair \( f_{PB}([\sigma_0, \sigma_1]) := (C, \beta)\).
Finally, we define $f_{BD} : \mathcal{B}(n) \rightarrow \mathcal{D}(n)$. Suppose that a Belyï pair $(C, \beta) \in \mathcal{B}(n)$ be given. Let $D$ be the bi-colored graph such that the black vertices are $\beta^{-1}(0)$, the white vertices are $\beta^{-1}(1)$, and the edges are $\beta^{-1}(I)$, where $I := [0, 1] \subset \mathbb{R}$ is the closed interval. Then $D$ is connected, since $C$ is connected. We then give a cyclic ordering on the set of edges emitting from each vertex by means of the orientation of $C$ induced by the complex structure of $C$.

These three maps $f_{DP}$, $f_{PB}$ and $f_{BD}$ yield the bijections 
\[
\mathcal{D}(n) \xrightarrow{\sim} \mathcal{P}(n) \xrightarrow{\sim} \mathcal{B}(n).
\]
Proposition. (1) If $(C,\beta)$ is a Belyĭ pair, then $(C,\beta)$ can be defined over $\overline{\mathbb{Q}} \subset \mathbb{C}$.
(2) If Belyĭ pairs $(C,\beta)$ and $(C',\beta')$ over $\overline{\mathbb{Q}}$ are isomorphic, then the isomorphism is defined over $\overline{\mathbb{Q}}$.

Corollary. For each $n$, the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathcal{D}(n) \cong \mathcal{P}(n) \cong \mathcal{B}(n)$.

Theorem (Belyĭ). A non-singular curve $C$ over $\mathbb{C}$ is defined over $\overline{\mathbb{Q}}$ if there exists a finite morphism $\beta : C \to \mathbb{P}^1$ such that $(C,\beta)$ is a Belyĭ pair.

Corollary. We put $\mathcal{B} := \bigcup_n \mathcal{B}(n)$. Then the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathcal{B}$ is faithful.

Indeed, considering the $j$-invariants of elliptic curves over $\overline{\mathbb{Q}}$, we see that the action is faithful on a subset $\mathcal{B}_1 \subset \mathcal{B}$ of Belyĭ pairs of genus 1. In fact, the action is faithful on a subset $\mathcal{B}_{0,\text{tree}} \subset \mathcal{B}$ of Belyĭ pairs of genus 0 whose dessins are trees (L. Schneps, H. W. Lenstra, Jr).
§2. Elliptic surfaces of Belyĭ type

The goal is to introduce an invariant of dessins by means of elliptic surfaces.

By an elliptic surface, we mean a non-singular compact complex relatively-minimal elliptic surface $\varphi : X \to C$ with a section $O_\varphi : C \to X$. We denote by

$$\Sigma_\varphi \subset C$$

the finite set of points $v \in C$ such that $\varphi^{-1}(v)$ is singular, by

$$J_\varphi : C \to \mathbb{P}^1$$

the functional invariant of $\varphi : X \to C$, and by

$$h_\varphi : \pi_1(C \setminus \Sigma_\varphi, b) \to \text{Aut}(H_1(E_b)) \cong SL_2(\mathbb{Z})$$

the homological invariant of $\varphi : X \to C$, where $b \in C \setminus \Sigma_\varphi$ is a base point, and $H_1(E_b)$ is the first homology group $H_1(E_b, \mathbb{Z})$ of $E_b := \varphi^{-1}(b)$ with the intersection pairing.
Definition. An elliptic surface $\varphi : X \to C$ is of Belyĭ type if $(C, J_\varphi)$ is a Belyĭ pair and $\Sigma_\varphi \subset J_\varphi^{-1}(\{0, 1, \infty\})$.

Consider the homomorphism

$$\tilde{h} : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, b_0) = \langle \gamma_0, \gamma_1 \rangle \to PSL_2(\mathbb{Z})$$

given by

$$\tilde{h}(\gamma_0) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \mod \pm I_2, \quad \tilde{h}(\gamma_1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mod \pm I_2.$$

Let $(C, \beta)$ be a Belyĭ pair, and let $b \in C$ be a point such that $\beta(b) = b_0$. Then the elliptic surfaces $\varphi : X \to C$ of Belyĭ type with $J_\varphi = \beta$ are in one-to-one correspondence with the homomorphisms

$$h : \pi_1(C \setminus \beta^{-1}(\{0, 1, \infty\}), b) \to SL_2(\mathbb{Z})$$

that make the following diagram commutative:

$$
\begin{array}{ccc}
\pi_1(C \setminus \beta^{-1}(\{0, 1, \infty\}), b) & \xrightarrow{h} & SL_2(\mathbb{Z}) \\
\beta_* \downarrow & & \downarrow \\
\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, b_0) & \xrightarrow{\tilde{h}} & PSL_2(\mathbb{Z}).
\end{array}
$$
We denote by

$$\text{NS}(X) := (H^2(X, \mathbb{Z})/\text{torsion}) \cap H^{1,1}(X)$$

the Néron-Severi lattice of $X$, and by $P_\varphi$ the sublattice of $\text{NS}(X)$ generated by the classes of the section $O_\varphi$ and the irreducible components of singular fibers.

Definition. An elliptic surface $\varphi : X \to C$ is \textit{extremal} if

$$P_\varphi \otimes \mathbb{C} = \text{NS}(X) \otimes \mathbb{C} = H^{1,1}(X);$$

(that is, the Picard number of $X$ is equal to $h^{1,1}(X)$, and the Mordell-Weil rank is 0.)
Theorem (Mangala Nori). Let \( \varphi : X \to C \) be an elliptic surface. Suppose that \( J_\varphi \) is non-constant. Then \( \varphi : X \to C \) is extremal if and only if the following hold:

- \( \varphi : X \to C \) is of Belyǐ type,
- the dessin of \((C, J_\varphi)\) has valencies \( \leq 3 \) at the black vertices, and valencies \( \leq 2 \) at the white vertices, and
- there are no singular fibers of type \( I_0^*, II, III \) or \( IV \).

Example. A \( K3 \) surface of Picard number 20 with the transcendental lattice

\[
\begin{bmatrix}
4 & 2 \\
2 & 4 \\
\end{bmatrix}
\]

has a structure of the extremal elliptic surface with singular fibers of the type \( I_0^*, II^*, IV^* \). The \( J \)-invariant of this elliptic \( K3 \) surface is therefore constant 0.
We define a topological invariant $Q_\varphi$ of an elliptic surface $\varphi : X \to C$. We put
\[
X_\varphi^0 := X \setminus (\varphi^{-1}(\Sigma_\varphi) \cup O_\varphi(C)),
\]
and let
\[
H_2(X_\varphi^0) := H_2(X_\varphi^0, \mathbb{Z})/\text{torsion}
\]
be the second homology group modulo the torsion with the intersection pairing
\[
(\ , \ ) : H_2(X_\varphi^0) \times H_2(X_\varphi^0) \to \mathbb{Z}.
\]
We then put
\[
I(X_\varphi^0) := \{ x \in H_2(X_\varphi^0) \mid (x, y) = 0 \text{ for all } y \},
\]
and
\[
Q_\varphi := H_2(X_\varphi^0)/I(X_\varphi^0).
\]
Then $Q_\varphi$ is torsion-free, and $(\ , \ )$ induces a non-degenerate symmetric bilinear form on $Q_\varphi$. Thus $Q_\varphi$ is a lattice.
Proposition. The invariant $Q_\varphi$ is isomorphic to the orthogonal complement of

$$P_\varphi = \langle O_\varphi, \text{the irred. components in fibers} \rangle \subset H^2(X)$$

in $H^2(X)$.

Corollary. If $\varphi : X \to C$ is an extremal elliptic surface, then $Q_\varphi$ is isomorphic to the transcendental lattice of $X$. 
We can calculate $Q_\varphi$ from the homological invariant

$$h_\varphi : \pi_1(C \setminus \Sigma_\varphi, b) \to \text{Aut}(H_1(E_b)).$$

For simplicity, we assume that $r := |\Sigma_\varphi| > 0$. We choose loops

$$\lambda_i : I \to C \setminus \Sigma_\varphi \quad (i = 1, \ldots, N := 2g(C) + r - 1)$$

with the base point $b$ such that their union is a strong deformation retract of $C \setminus \Sigma_\varphi$. Then $\pi_1(C \setminus \Sigma_\varphi, b)$ is a free group generated by $[\lambda_1], \ldots, [\lambda_N]$. Then $X_\varphi^0$ is homotopically equivalent to a topological space obtained from

$$E_b \setminus \{O_\varphi(b)\} \sim S^1 \vee S^1$$

by attaching $2N$ tubes $S^1 \times I$, two of which lying over each loop $\lambda^i$. 
We prepare $N$ copies of $H_1(E_b) \cong \mathbb{Z}^2$, and consider the homomorphism

$$
\partial : \bigoplus_{i=1}^{N} H_1(E_b) \rightarrow H_1(E_b)
$$

defined by

$$
\partial(x_1, \ldots, x_N) := \sum_{i=1}^{N} (h_{\varphi}([\lambda_i])x_i - x_i).
$$

Then $H_2(X^0_\varphi)$ is isomorphic to $\text{Ker} \partial$. The intersection pairing on $H_2(X^0_\varphi)$ is calculated by perturbing the loops $\lambda_i$ to the loops $\lambda'_i$ with the base point $b' \neq b$. 

§3. An invariant of dessins

Let \((C, \beta)\) be a Belyï pair. We put

\[
\begin{align*}
\beta^{-1}(0) &= \beta^{-1}(0)_0(3) \sqcup \beta^{-1}(0)_1(3) \sqcup \beta^{-1}(0)_2(3), \\
\beta^{-1}(1) &= \beta^{-1}(1)_0(2) \sqcup \beta^{-1}(1)_1(2), \\
\beta^{-1}(\infty) &= \beta^{-1}(\infty)_1 \sqcup \beta^{-1}(\infty)_2 \sqcup \beta^{-1}(\infty)_3 \sqcup \ldots,
\end{align*}
\]

where

\[
\beta^{-1}(p)_{a(m)} = \left\{ x \in \beta^{-1}(p) \mid \text{the ramification index of } \beta \text{ at } x \equiv a \mod m \right\},
\]

\[
\beta^{-1}(\infty)_b = \left\{ x \in \beta^{-1}(\infty) \mid \beta \text{ has a pole of order } b \text{ at } x \right\}.
\]
A *type-specification* is a list

\[ s = [s_{00}, s_{01}, s_{02}, s_{10}, s_{11}, s_{\infty b} \ (b = 1, 2, \ldots)] \]

of maps, where

\[
\begin{align*}
    s_{00} & : \beta^{-1}(0)_{0(3)} \rightarrow \{I_0, I_0^*\}, \\
    s_{01} & : \beta^{-1}(0)_{1(3)} \rightarrow \{II, IV^*\}, \\
    s_{02} & : \beta^{-1}(0)_{2(3)} \rightarrow \{II^*, IV\}, \\
    s_{10} & : \beta^{-1}(1)_{0(2)} \rightarrow \{I_0^*, I_0^*\}, \\
    s_{11} & : \beta^{-1}(1)_{1(2)} \rightarrow \{III, III^*\}, \\
    s_{\infty b} & : \beta^{-1}(\infty)_b \rightarrow \{I_b, I_b^*\}.
\end{align*}
\]
If \( g(C) > 0 \), then, for each type-specification \( s \), there exist exactly \( 2^{2g(C)-1} \) elliptic surfaces \( \varphi : X \to C \) of Belyĭ type such that \( J_{\varphi} = \beta \) and that the types of singular fibers are \( s \).

If \( g(C) = 0 \), then, for exactly half of all the type-specifications \( s \), there exists an elliptic surface \( \varphi : X \to C \) of Belyĭ type (unique up to isomorphism) such that \( J_{\varphi} = \beta \) and that the types of singular fibers are \( s \).

The set of pairs of the type-specification \( s \) and the invariant \( Q_{\varphi} \) of the corresponding elliptic surface \( \varphi : X \to C \) of Belyĭ type is an invariant of the Belyĭ pair \( (C, \beta) \).
Example. Consider the simplest dessin

<table>
<thead>
<tr>
<th>$\beta^{-1}(0)$</th>
<th>$\beta^{-1}(1)$</th>
<th>$\beta^{-1}(\infty)$</th>
<th>$X$</th>
<th>$Q_\varphi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$II$</td>
<td>$III$</td>
<td>$I_1$</td>
<td>none</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$I_1^*$</td>
<td>rational</td>
<td>0</td>
</tr>
<tr>
<td>$III^*$</td>
<td>$I_1$</td>
<td>rational</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$I_1^*$</td>
<td></td>
<td>none</td>
<td>$-$</td>
<td></td>
</tr>
<tr>
<td>$IV^*$</td>
<td>$III$</td>
<td>$I_1$</td>
<td>rational</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$I_1^*$</td>
<td>none</td>
<td>$-$</td>
</tr>
<tr>
<td>$III^*$</td>
<td>$I_1$</td>
<td>none</td>
<td>$-$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$I_1^*$</td>
<td>$K3$</td>
<td>$\begin{bmatrix} 2 &amp; 0 \ 0 &amp; 12 \end{bmatrix}$</td>
<td></td>
</tr>
</tbody>
</table>
If the dessin has valencies $\leq 3$ at black vertices and $\leq 2$ at white vertices, then we can restrict ourselves to type-specifications $s$ that yields extremal elliptic surfaces.
Example. Consider the dessins of degree 2:

For each of them, there exists a unique type-specification that yields an extremal non-rational elliptic surface.

<table>
<thead>
<tr>
<th></th>
<th>$\beta^{-1}(0)$</th>
<th>$\beta^{-1}(1)$</th>
<th>$\beta^{-1}(\infty)$</th>
<th>$X$</th>
<th>$Q_\varphi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1$</td>
<td>$II^*$</td>
<td>$I_0$</td>
<td>$I_1^* \times 2$</td>
<td>$K3$</td>
<td>$\begin{bmatrix} 4 &amp; 0 \ 0 &amp; 4 \end{bmatrix}$</td>
</tr>
<tr>
<td>$D_2$</td>
<td>$IV^* \times 2$</td>
<td>$I_0$</td>
<td>$I_2^*$</td>
<td>$K3$</td>
<td>$\begin{bmatrix} 6 &amp; 0 \ 0 &amp; 6 \end{bmatrix}$</td>
</tr>
<tr>
<td>$D_3$</td>
<td>$II^*$</td>
<td>$III^* \times 2$</td>
<td>$I_2^*$</td>
<td>$\chi = 36$</td>
<td>$\begin{bmatrix} 6 &amp; 2 &amp; 2 &amp; 2 \ 2 &amp; 2 &amp; 0 &amp; 2 \ 2 &amp; 0 &amp; 2 &amp; 0 \ 2 &amp; 2 &amp; 0 &amp; 4 \end{bmatrix}$</td>
</tr>
</tbody>
</table>
§4. Examples

In 1989, Miranda and Persson classified all semi-stable extremal elliptic $K3$ surfaces.

In 2001, S.- and Zhang classified all (not necessarily semi-stable) extremal elliptic $K3$ surfaces, and calculated their transcendental lattices.

In 2007, Beukers and Montanus determined the Belyi pairs (or dessins) $(\mathbb{P}^1, J_\varphi)$ associated with the semi-stable extremal elliptic $K3$ surfaces $\varphi : X \to \mathbb{P}^1$.

We will use the dessins by Beukers and Montanus as examples for our invariant.
Definition. A dessin is said to be of MPBM-type if it is of genus 0, of degree 24 and has valency 3 at every black vertices and valency 2 at the white vertices.

Proposition. Let $\varphi : X \to \mathbb{P}^1$ be a semi-stable extremal elliptic $K3$ surface. Then the Belyĭ pair $(\mathbb{P}^1, J_\varphi)$ is of MPBM-type.

By the valencies at $\infty$ of the Belyĭ pair $(C, \beta)$, we mean the orders of poles of $\beta$. 
When we write a dessin of MPBM-type, we omit the white vertices. For example, we write the dessin of MPBM-type with valency \([8, 8, 2, 2, 2, 2]\) at \(\infty\)
Example. The dessins of MPBM-type with valency $[11, 5, 3, 3, 1, 1]$ at $\infty$:

Beukers and Montanus showed that they are defined over $\mathbb{Q}(\sqrt{5})$, and are conjugate by $\text{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$. By the semi-stable extremal type-specification, the invariants $Q_\varphi$ of the associated elliptic $K3$ surfaces are

$$\begin{bmatrix} 6 & 3 \\ 3 & 84 \end{bmatrix} \text{ for } D_1, \quad \begin{bmatrix} 24 & 9 \\ 9 & 24 \end{bmatrix} \text{ for } D_2.$$  

They are not isomorphic.
Example (continued). By the calculation of the transcendental lattices, we have known that the transcendental lattice of an extremal elliptic \( K3 \) surface of type

\[
I_{11} + I_5 + I_3 + I_3 + I_1 + I_1
\]
is either

\[
\begin{bmatrix}
6 & 3 \\
3 & 84
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
24 & 9 \\
9 & 24
\end{bmatrix},
\]

and both lattices actually occur. However, we have not known which lattice corresponds to which dessin.
Example. The dessins of MPBM-type with valency $[6, 6, 5, 5, 1, 1]$ at $\infty$:

Beukers and Montanus showed that they are defined over $\mathbb{Q}(\sqrt{3})$, and are conjugate by $\text{Gal}(\mathbb{Q}(\sqrt{3})/\mathbb{Q})$. By the semi-stable extremal type-specification, the invariants $Q_\varphi$ of $D_1$ and $D_2$ are both

$$\begin{bmatrix} 30 & 0 \\ 0 & 30 \end{bmatrix}.$$
Example (continued). By the extremal type-specification

\[ I_6 + I_6 + I_5 + I_5 + I_1^* + I_1^* \] over \( \beta^{-1}(\infty) \),

the invariants \( Q_\varphi \) of \( D_1 \) and \( D_2 \) are

\[
Q_1 = \begin{bmatrix}
580 & -3944 & -7196 & -1440 \\
-3944 & 26846 & 48964 & 9800 \\
-7196 & 48964 & 89326 & 17880 \\
-1440 & 9800 & 17880 & 3580
\end{bmatrix}, \quad \text{and}
\]

\[
Q_2 = \begin{bmatrix}
260 & 456 & 2232 & 1748 \\
456 & 876 & 4092 & 3048 \\
2232 & 4092 & 19574 & 14966 \\
1748 & 3048 & 14966 & 11764
\end{bmatrix},
\]

respectively.

Both are even, positive-definite, and of discriminant 14400. They are not isomorphic, because there are four vectors \( v \) such that \( Q_1(v, v) = 6 \), while there are no vectors \( v \) such that \( Q_2(v, v) = 6 \).
§5. Galois-invariant part

For a lattice $\Lambda$, we put

$$\Lambda^\vee := \text{Hom}(\Lambda, \mathbb{Z}).$$

Then there is a canonical embedding $\Lambda \hookrightarrow \Lambda^\vee$. Then there is a canonical embedding $\Lambda \hookrightarrow \Lambda^\vee$ with the finite cokernel

$$D(\Lambda) := \Lambda^\vee / \Lambda$$

of order $|\text{disc } \Lambda|$. We have a symmetric bilinear form

$$\Lambda^\vee \times \Lambda^\vee \to \mathbb{Q}$$

that extends the symmetric bilinear form on $\Lambda$. We consider the natural non-degenerate quadratic form

$$q(\Lambda) : D(\Lambda) \times D(\Lambda) \to \mathbb{Q}/\mathbb{Z}.$$

The finite quadratic form $(D(\Lambda), q(\Lambda))$ is called the discriminant form of $\Lambda$. 
Since $H^2(X)$ is a unimodular lattice, we obtain the following:

**Proposition.** Let $\varphi : X \to C$ be an elliptic surface. Then the finite quadratic forms $(D(Q_\varphi), q(Q_\varphi))$ and $(D(P_\varphi), -q(P_\varphi))$ are isomorphic.

For an embedding $\sigma : \mathbb{C} \hookrightarrow \mathbb{C}$, we denote by $\varphi^\sigma : X^\sigma \to C^\sigma$ the pull-back of

$$
\begin{array}{ccc}
X & \to & C \\
\downarrow & & \downarrow \\
\text{Spec } \mathbb{C}
\end{array}
$$

by $\sigma^* : \text{Spec } \mathbb{C} \to \text{Spec } \mathbb{C}$. Since the lattice $P_\varphi$ is defined algebraically, we see that $P_\varphi$ and $P_\varphi^\sigma$ are isomorphic.

**Corollary.** For any $\sigma : \mathbb{C} \hookrightarrow \mathbb{C}$, the finite quadratic forms $(D(Q_\varphi), q(Q_\varphi))$ and $(D(Q_{\varphi^\sigma}), q(Q_{\varphi^\sigma}))$ are isomorphic.

Thus the finite quadratic form $(D(Q_\varphi), q(Q_\varphi))$ is Galois-invariant. Therefore we can use $(D(Q_\varphi), q(Q_\varphi))$ to distinguish distinct Galois orbits in the set of dessins $\mathcal{D}(n)$. 
Remark. If Λ is an even lattice, then we can refine the discriminant form \( q(\Lambda) : D(\Lambda) \times D(\Lambda) \to \mathbb{Q}/\mathbb{Z} \) to

\[
q(\Lambda) : D(\Lambda) \times D(\Lambda) \to \mathbb{Q}/2\mathbb{Z}.
\]

Example. Consider again the two dessins of MPBM-type with valency \([11, 5, 3, 3, 1, 1]\) at \(\infty\), which are conjugate by \(\text{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})\):

Their invariants under the semi-stable extremal type-specification

\[
\begin{pmatrix} 6 & 3 \\ 3 & 84 \end{pmatrix} \quad \text{for } D_1 \quad \text{and} \quad \begin{pmatrix} 24 & 9 \\ 9 & 24 \end{pmatrix} \quad \text{for } D_2
\]

are not isomorphic, but in the same genus, and hence they have isomorphic (\(\mathbb{Q}/2\mathbb{Z}\)-valued) discriminant forms.
Example. The dessins of MPBM-type with valency $[8, 8, 3, 3, 1, 1]$ at $\infty$:

By the semi-stable extremal type-specification, the invariants $Q_\varphi$ are

$$Q_1 = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix} \quad \text{for } D_1, \quad Q_2 = \begin{bmatrix} 24 & 0 \\ 0 & 24 \end{bmatrix} \quad \text{for } D_2.$$  

Since $|D(Q_1)| = 144$ and $|D(Q_2)| = 576$, we see that these dessins are not Galois conjugate.

In fact, the Mordell-Weil group of the semi-stable extremal elliptic $K3$ surface over the dessin $D_1$ has a torsion of order 2, while that of $D_2$ is torsion-free.
Problem. Let $\varphi : X \to C$ be an extremal elliptic surface defined over $\overline{Q}$. Let $Q'$ be a lattice that is in the same genus as $Q_\varphi$. Is there $\sigma \in \text{Gal}(\overline{Q}/Q)$ such that $Q_\varphi^\sigma \cong Q'$?

Remark. YES, if $X$ is a $K3$ surface.

Remark. By using prescribed type-specifications, we can use $(D(Q_\varphi), q(Q_\varphi))$ to distinguish the Galois orbits in the set of marked dessins.