On lattice-theoretic invariants of curves on a surface

Ichiro Shimada

Hiroshima University

August 12, 2009, MSJ-SI, Sapporo
By a *lattice*, we mean a free $\mathbb{Z}$-module $L$ of finite rank with a nondegenerate symmetric bilinear form

$$L \times L \to \mathbb{Z}.$$ 

A lattice $L$ is naturally embedded into $\text{Hom}(L, \mathbb{Z})$. The *discriminant group* of $L$ is the finite abelian group

$$\text{disc } L := \text{Hom}(L, \mathbb{Z})/L.$$
We fix a simply connected complex projective surface $S$, and consider reduced (possibly reducible) curves $B$ on $S$.

We define several invariants of $(S, B)$ by means of abelian coverings of $S$ branching along $B$, and apply them for the construction of examples of Zariski couples.

Let $B$ and $B'$ be curves on $S$.

**Definition**

We say that a homeomorphism $f : B \sim B'$ *preserves the classes of irreducible components* if we have $[B_i] = [f(B_i)]$ in $H^2(S, \mathbb{Z})$ for any irreducible component $B_i$ of $B$. 

**Definition**

We say that $B$ and $B'$ have the *same embedding topology* and write

$$B \sim_{\text{top}} B'$$

if there is a homeomorphism $h : (S, B) \cong (S, B')$ such that $h|B : B \cong B'$ preserves the classes of irreducible components.

**Definition**

We say that $B$ and $B'$ are *of the same configuration type* and write

$$B \sim_{\text{cfg}} B'$$

if there are

- tubular neighborhoods $T \subset S$ of $B$ and $T' \subset S$ of $B'$,
- a homeomorphism $\tau : (T, B) \cong (T', B')$

such that $\tau|B : B \cong B'$ preserves the classes of irreducible components.
It is obvious that

\[ B \sim_{\text{top}} B' \implies B \sim_{\text{cfg}} B' \]

According to Artal-Bartolo (1994), we define as follows:

**Definition**

A couple \([B, B']\) of curves on \(S\) is said to be a Zariski couple if \(B \sim_{\text{cfg}} B'\) but \(B \not\sim_{\text{top}} B'\).

The first example was discovered by Zariski in 1930’s.

**Example**

There are irreducible curves \(B\) and \(B'\) of degree 6 on \(\mathbb{P}^2\) with six ordinary cusps as their only singularities such that

\[ \pi_1(\mathbb{P}^2 \setminus B) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, \text{ and} \]
\[ \pi_1(\mathbb{P}^2 \setminus B') \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}. \]

Hence \([B, B']\) is a Zariski couple of simple sextics.
Definition

A plane curve $B \subset \mathbb{P}^2$ of degree 6 is a simple sextic if it has only simple singularities:

\begin{align*}
A_n & \quad x^{n+1} + y^2 = 0 \quad (n \geq 1) \\
D_n & \quad x^{n-1} + xy^2 = 0 \quad (n \geq 4) \\
E_6 & \quad x^4 + y^3 = 0 \\
E_7 & \quad x^3 y + y^3 = 0 \\
E_8 & \quad x^5 + y^3 = 0
\end{align*}

For simple sextics, we have

\begin{align*}
\# \text{ of config types} &= 11159 \\
&< \# \text{ of emb-top types} = ? \\
&< \# \text{ of connected componets of equi-sing families} = ?
\end{align*}
Example

We have three plane curves of degree 6

\[ B_1 = C_1 + Q_1, \quad B_2 = C_2 + Q_2, \quad B_4 = C_4 + Q_4, \]

where \( Q_i \) is a quartic with one tacnode (\( A_3 \) point) and \( C_i \) is a smooth conic tangent to \( Q_i \) at two points with multiplicity 4 (two \( A_7 \)-points).

Let \( E_i \to Q_i \) be the normalization of \( Q_i \). Then \( E_i \) is of genus 1 and has four special points

\[ p, q \text{ the pull-back of } A_3, \quad s, t \text{ the pull-back of } 2A_7. \]

Then the order of \([p + q − s − t]\) in \( \text{Pic}^0(E_i) \) is 1, 2 and 4 according to \( i = 1, 2, 4 \). Their emb-top types are different, and hence \([B_1, B_2, B_4]\) is a Zariski triple.
Example

Consider two simple sextics

\[ B_\pm : z \cdot (G(x, y, z) \pm \sqrt{5}H(x, y, z)) = 0, \quad \text{where} \]

\[ G = -9x^4z - 14x^3yz + 58x^3z^2 - 48x^2y^2z - 64x^2yz^2 + \\
10x^2z^3 + 108xy^3z - 20xy^2z^2 - 44y^5 + 10y^4z, \]

\[ H = 5x^4z + 10x^3yz - 30x^3z^2 + 30x^2y^2z + \\
20x^2yz^2 - 40xy^3z + 20y^5. \]

The quintic \( G \pm \sqrt{5}H = 0 \) has a \( A_{10} \)-singular point and intersects the line \( z = 0 \) at only one point that is a \( A_9 \)-singular point. They are \textit{conjugate} under the action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), but have the different embedding topology. Hence \([B_+, B_-]\) is an arithmetic Zariski couple.
We fix a finite abelian group $A$ once and for all.

Let $B$ be a curve on $S$ with the irreducible components $B_1, \ldots, B_m$. Since $S$ is simply connected, all étale Galois coverings of $S \setminus B$ with the Galois group $A$ are in one-to-one correspondence with the set

$$ C_A(S, B) := \left\{ \gamma \mid \gamma \text{ is a surjective homomorphism } \quad \begin{array}{c} H^2(B) = \bigoplus \mathbb{Z}[B_i] \to A \\ \text{such that the image of the restriction map } r : H^2(S) \to H^2(B) = \bigoplus \mathbb{Z}[B_i] \\ \text{is contained in Ker } \gamma \end{array} \right\}. $$

For $\gamma \in C_A(S, B)$, we denote the corresponding covering by

$$ \varphi_\gamma : W_\gamma \to S \setminus B. $$
**Definition**

A *smooth projective completion* of \( \varphi_\gamma : W_\gamma \to S \setminus B \) is a morphism

\[
\phi : X \to S
\]

from a smooth projective surface \( X \) such that

- \( X \) contains \( W_\gamma \) as a Zariski open dense subset, and
- \( \phi \) extends \( \varphi_\gamma : W_\gamma \to S \setminus B \).

We choose a smooth projective completion \( \phi : X \to S \) of \( \varphi_\gamma \) (not necessarily \( A \)-equivariant), and put

\[
\mathcal{E}(X) := \left\{ E \subset X \mid E \text{ is an irreducible curve on } X \text{ such that } \phi(E) \text{ is a point on } S \right\}.
\]
We consider

\[ H^2(X)' := H^2(X)/(\text{the torsion part}) \]

as a lattice by the cup-product. In this lattice, we have two

submodules

\[
\phi^*\text{NS}(S) = \langle [\phi^*C] \mid C \text{ is a curve on } S \rangle,
\]

and

\[
\langle \mathcal{E}(X) \rangle = \langle [E] \mid E \in \mathcal{E}(X) \rangle.
\]

The cup-product is non-degenerate on each of them. Moreover we have

\[
\phi^*\text{NS}(S) \perp \langle \mathcal{E}(X) \rangle.
\]

Hence the cup-product is non-degenerate on

\[
\Sigma(X) := \phi^*\text{NS}(S) \oplus \langle \mathcal{E}(X) \rangle,
\]

that is, \( \Sigma(X) \) is a sublattice of \( H^2(X)' \).
We denote by

\[ \overline{\Sigma}(X) := (\Sigma(X) \otimes \mathbb{Q}) \cap H^2(X)' \]

the \textit{primitive closure} of \( \Sigma(X) = \phi^* \text{NS}(S) \oplus \langle \mathcal{E}(X) \rangle \) in \( H^2(X)' \).

**Definition**

We put

\[
F_A(S, B, \gamma) := \overline{\Sigma}(X)/\Sigma(X), \\
\sigma_A(S, B, \gamma) := \text{disc} \Sigma(X) = \text{Hom}(\Sigma(X), \mathbb{Z})/\Sigma(X),
\]

which are finite abelian groups, and

\[
T_A(S, B, \gamma) := \Sigma(X)^\perp = \overline{\Sigma}(X)^\perp \subset H^2(X)',
\]

which is a primitive sublattice of \( H^2(X)' \).
Proposition

Neither $F_A(S, B, \gamma)$ nor $\sigma_A(S, B, \gamma)$ nor $T_A(S, B, \gamma)$ does depend on the choice of the smooth projective completion $\phi : X \to S$ of $\varphi_\gamma : W_\gamma \to S \setminus B$.

The proof is very easy. Suppose that $\phi' : X' \to S$ is another smooth projective completion. Then we have a commutative diagram

\[
\begin{array}{ccc}
X'' & \xrightarrow{\sim} & X' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\sim} & X' \\
\downarrow & & \downarrow \\
S & & \end{array}
\]

where $X''$ is a smooth projective surface, and $X'' \to X$ and $X'' \to X'$ are birational morphisms that are isom over $S \setminus B$. 


Since a birational morphism between smooth surfaces are composite of blowing-ups at points, we obtain orthogonal direct-sum decompositions

\[ \Sigma(X') = \Sigma(X) \oplus \langle e_1 \rangle \oplus \cdots \oplus \langle e_N \rangle \quad \text{and} \]
\[ H^2(X')' = H^2(X)' \oplus \langle e_1 \rangle \oplus \cdots \oplus \langle e_N \rangle, \]

where \( e_1, \ldots, e_N \) are classes with \( e_i^2 = -1 \). Hence we obtain

\[ \bar{\Sigma}(X) / \Sigma(X) \cong \bar{\Sigma}(X'') / \Sigma(X''), \]
\[ \text{Hom}(\Sigma(X), \mathbb{Z}) / \Sigma(X) \cong \text{Hom}(\Sigma(X''), \mathbb{Z}) / \Sigma(X''), \]
\[ \Sigma(X)^\perp \cong \Sigma(X'')^\perp. \]

The same isomorphisms hold between \( X' \) and \( X'' \).\]
By means of the celebrated theorem of Villamayor on the existence of the equivariant resolution of singularities, we can prove the following:

**Proposition**

There exists an $A$-equivariant smooth projective completion $\phi : X \to S$ for $\varphi_\gamma : W_\gamma \to S \setminus B$.

Hence the finite abelian groups $F_A(S, B, \gamma)$, $\sigma_A(S, B, \gamma)$ and the lattice $T_A(S, B, \gamma)$ are $A$-modules.
Consider the case $S = \mathbb{P}^2$ and $A = \mathbb{Z}/2\mathbb{Z}$.

**Example**

The example of Zariski revisited:
Let $B$ be a plane curve of degree 6 defined by

$$f^3 + g^2 = 0, \quad \deg f = 2, \quad \deg g = 3, \quad \text{general.}$$

Then $B$ is irreducible and $\text{Sing } B$ consists of six cusps. The conic $Q : f = 0$ passes through $\text{Sing } B$. On the other hand, there is an irreducible sextic $B'$ with $\text{Sing } B'$ consisting of 6 cusps such that there are no conics passing through them. (Del Pezzo, Segre, Zariski.)

- $F_A(\mathbb{P}^2, B, \gamma) \cong \mathbb{Z}/3\mathbb{Z}$, $\pi_1(\mathbb{P}^2 \setminus B) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$,
- $F_A(\mathbb{P}^2, B', \gamma) = 0$, $\pi_1(\mathbb{P}^2 \setminus B') \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

In fact, $F_A(\mathbb{P}^2, B, \gamma) \cong \mathbb{Z}/3\mathbb{Z}$ is generated by the class of the lift of the conic $Q$. (The conic $Q$ splits in $X$ into $\tilde{Q}^{\pm} \cup \tilde{Q}^{-}$.)

Consider again the case $S = \mathbb{P}^2$ and $A = \mathbb{Z}/2\mathbb{Z}$.

**Example**

Recall the conjugate plane sextics $B_+$ and $B_-$ with $A_9 + A_{10}$-singular points:

$$B_\pm : z \cdot (G(x, y, z) \pm \sqrt{5} H(x, y, z)) = 0.$$  

We have

$$T_A(\mathbb{P}^2, B_+, \gamma) \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}, \quad T_A(\mathbb{P}^2, B_-, \gamma) \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}.$$
The invariant $T_A(S, B, \gamma)$ is a topological invariant. Let

$$h: (S, B) \cong (S, B')$$

be a homeomorphism. Since $h$ induces a homeomorphism $S \setminus B \cong S \setminus B'$, we obtain a bijection

$$h^*: C_A(S, B') \cong C_A(S, B).$$

For $\gamma \in C_A(S, B')$, the covering $\varphi_{h^*\gamma}: W_{h^*\gamma} \to S \setminus B$ is obtained by pulling back $\varphi_{\gamma}: W_{\gamma} \to S \setminus B'$ by $h: S \setminus B \cong S \setminus B'$.

**Theorem**

*Suppose that the classes $[B_i]$ span $\text{NS}(S) \otimes \mathbb{Q}$ over $\mathbb{Q}$. Then*

$$T_A(S, B, h^*\gamma) \cong T_A(S, B', \gamma).$$
On lattice-theoretic invariants of curves on a surface

Properties of $T_A(S, B, \gamma)$

Proof Since $W_{h^*\gamma}$ is homeomorphic to $W_\gamma$, it is enough to show that $T_A(S, B, \gamma)$ is determined by the homeo-type of $W_\gamma$. Let

$$\iota_W : H_2(W_\gamma) \times H_2(W_\gamma) \rightarrow \mathbb{Z}$$

be the intersection pairing. We put

$${\text{Ker}}(\iota_W) := \{ x \in H_2(W_\gamma) \mid \iota_W(x, y) = 0 \text{ for all } y \in H_2(W_\gamma) \}.$$ 

Then $\iota_W$ induces a non-degenerate symmetric bilinear form

$$\overline{\iota}_W : H_2(W_\gamma)/{\text{Ker}}(\iota_W) \times H_2(W_\gamma)/{\text{Ker}}(\iota_W) \rightarrow \mathbb{Z}.$$ 

The lattice $(H_2(W_\gamma)/{\text{Ker}}(\iota_W), \overline{\iota}_W)$ is determined by the homeomorphism type of $W_\gamma$. Hence the proof is completed by showing that the homomorphism

$$H_2(W_\gamma) \rightarrow H_2(X)$$

induced by the inclusion $W_\gamma \hookrightarrow X$ induces

$$H_2(W_\gamma)/{\text{Ker}}(\iota_W) \cong T_A(S, B, \gamma) \subset H_2(X)'.$$
**Definition**

A *map of equi-configuration* is a homeomorphism \( \tau : (\mathcal{T}, B) \cong (\mathcal{T}', B') \), where \( \mathcal{T}, \mathcal{T}' \subset S \) are tubular nbds of \( B, B' \) respectively, such that the induced homeomorphic \( B \cong B' \) preserves the classes of irreducible components.

If \( \tau : (\mathcal{T}, B) \cong (\mathcal{T}', B') \) is a map of equi-configuration, then we have the following commutative diagram:

\[
\begin{array}{ccc}
H^2(S) & \xrightarrow{r} & H^2(B') \\
\| & & \downarrow \tau^* \\
H^2(S) & \xrightarrow{r} & H^2(B).
\end{array}
\]

Therefore \( \tau \) induces a bijection

\[ \tau_* : \mathcal{C}_A(S, B) \cong \mathcal{C}_A(S, B'). \]
Corollary

Let $\tau : (\mathcal{T}, B) \sim (\mathcal{T}', B')$ be a map of equi-configuration. If $T_A(S, B, \gamma) \not\sim T_A(S, B', \tau_*\gamma)$, then $[B, B']$ is a Zariski couple.

By this corollary, we can obtain many examples of Zariski couples.

We can also prove the following for

$$\sigma_A(S, B, \gamma) := \text{disc } \Sigma(X)$$

by means of the minimal good embedded resolution of $B \subset S$.

Proposition

Let $\tau : (\mathcal{T}, B) \sim (\mathcal{T}', B')$ be a map of equi-configuration. Then $\sigma_A(S, B, \gamma) \cong \sigma_A(S, B', \tau_*\gamma)$. 
Since the lattice $H^2(X)'$ is unimodular, the three invariants
\[
F_A(S, B, \gamma) = \frac{\Sigma(X)}{\Sigma(X)},
\]
\[
\sigma_A(S, B, \gamma) = \text{disc } \Sigma(X) = \text{Hom}(\Sigma(X), \mathbb{Z})/\Sigma(X)
\]
\[
T_A(S, B, \gamma) = \Sigma(X)^\perp
\]
satisfy
\[
| \text{disc}(T_A(S, B, \gamma)) | = \frac{|\sigma_A(S, B, \gamma)|}{|F_A(S, B, \gamma)|^2}.
\]

**Corollary**

Let $\tau : (T, B) \cong (T', B')$ be a map of equi-configuration, so that $\sigma_A(S, B, \gamma) \cong \sigma_A(S, B', \tau_* \gamma)$. If $|F_A(S, B, \gamma)| \neq |F_A(S, B', \tau_* \gamma)|$, then $[B, B']$ is a Zariski couple.
Example

For the Zariski triple

\[ B_1 = C_1 + Q_1, \quad B_2 = C_2 + Q_2, \quad B_4 = C_4 + Q_4 \]

described above, we have

\[ F_A(\mathbb{P}^2, B_4) \cong \mathbb{Z}/2\mathbb{Z}, \quad F_A(\mathbb{P}^2, B_2) \cong \mathbb{Z}/4\mathbb{Z}, \quad F_A(\mathbb{P}^2, B_1) \cong \mathbb{Z}/8\mathbb{Z}. \]

These cyclic groups are generated by the classes of

- the reduced part of the proper transform of \( C_4 \),
- the lift of a conic passing through \( \text{Sing } B_2 \),
- the lift of a line passing through \( \text{Sing } B_1 \),

respectively.
**Definition**

We say that a complex $K3$ surface $X$ is *singular* if $\text{rank}(\text{NS}(X))$ attains the possible maximum 20.

Shioda and Inose showed that the isomorphism class of a singular $K3$ surface is determined by its transcendental lattice

$$T(X) := \text{NS}(X)^\perp$$

with its Hodge decomposition $T(X) \otimes \mathbb{C} = H^{2,0} \oplus H^{0,2}$.

We denote by

$$\tilde{T}(X)$$

the oriented transcendental lattice of $X$, which is a positive definite lattice of rank 2.
Shioda and Inose also gave an explicit way of constructing a singular $K3$ surface $X$ from the oriented transcendental lattice $\tilde{T}(X)$. The singular $K3$ surface $X$ is obtained as a certain double cover of the Kummer surface $\text{Km}(E \times E')$, where $E$ and $E'$ are elliptic curves with CM by some orders of $\mathbb{Q}(\sqrt{-|\text{disc} \ T(X)|})$. In particular, we have

**Theorem**

*Every singular $K3$ surface is defined over a number field.*
S.- and Schütt (2007) proved the following by means of the class field theory of imaginary quadratic fields:

**Theorem**

Let $X$ and $X'$ be singular $K3$ surfaces defined over $\overline{\mathbb{Q}}$. If their transcendental lattices $T(X)$ and $T(X')$ are contained in the same genus, then there exists $\sigma \in \text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q})$ such that $X' \cong X^\sigma$. 
Definition

A plane curve $B \subset \mathbb{P}^2$ of degree 6 is called a *maximizing sextic* if $B$ has only simple singularities and the sum of the Milnor numbers attains the possible maximum 19.

Let $B$ be a maximizing sextic. Consider the double covering $W_\gamma \to \mathbb{P}^2 \setminus B$ branching along every irreducible component of $B$. Then we obtain a singular $K3$ surface $X$ as a smooth projective completion $X \to \mathbb{P}^2$ of $W_\gamma \to \mathbb{P}^2 \setminus B$, and the lattice invariant $T_A(\mathbb{P}^2, B, \gamma)$ is the transcendental lattice $T(X)$.

By searching for maximizing sextics $B$ such that the genus of the lattice $T_A(\mathbb{P}^2, B, \gamma)$ contains more than one isomorphism classes, we obtain many examples of arithmetic Zariski couples of maximizing sextics:
<table>
<thead>
<tr>
<th>No.</th>
<th>sing – type</th>
<th>$T_A(\mathbb{P}^2, B, \gamma)$, $T_A(\mathbb{P}^2, B^\sigma, \gamma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$E_8 + A_{10} + A_1$</td>
<td>$L[6, 2, 8]$, $L[2, 0, 22]$</td>
</tr>
<tr>
<td>2</td>
<td>$E_8 + A_6 + A_4 + A_1$</td>
<td>$L[8, 2, 18]$, $L[2, 0, 70]$</td>
</tr>
<tr>
<td>3</td>
<td>$E_6 + D_5 + A_6 + A_2$</td>
<td>$L[12, 0, 42]$, $L[6, 0, 84]$</td>
</tr>
<tr>
<td>4</td>
<td>$E_6 + A_{10} + A_3$</td>
<td>$L[12, 0, 22]$, $L[4, 0, 66]$</td>
</tr>
<tr>
<td>5</td>
<td>$E_6 + A_{10} + A_2 + A_1$</td>
<td>$L[18, 6, 24]$, $L[6, 0, 66]$</td>
</tr>
<tr>
<td>6</td>
<td>$E_6 + A_7 + A_4 + A_2$</td>
<td>$L[24, 0, 30]$, $L[6, 0, 120]$</td>
</tr>
<tr>
<td>7</td>
<td>$E_6 + A_6 + A_4 + A_2 + A_1$</td>
<td>$L[30, 0, 42]$, $L[18, 6, 72]$</td>
</tr>
<tr>
<td>8</td>
<td>$D_8 + A_{10} + A_1$</td>
<td>$L[6, 2, 8]$, $L[2, 0, 22]$</td>
</tr>
<tr>
<td>9</td>
<td>$D_8 + A_6 + A_4 + A_1$</td>
<td>$L[8, 2, 18]$, $L[2, 0, 70]$</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>34</td>
<td>$A_7 + A_5 + A_4 + A_2 + A_1$</td>
<td>$L[24, 0, 30]$, $L[6, 0, 120]$</td>
</tr>
</tbody>
</table>

where

$$L[a, b, c] := \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$
Question:

Are there any arithmetic Zariski couple $[B, B']$ such that $\pi_1(\mathbb{P}^2 \setminus B)$ is not isomorphic to $\pi_1(\mathbb{P}^2 \setminus B')$?

Note that $\pi_1(\mathbb{P}^2 \setminus B)$ and $\pi_1(\mathbb{P}^2 \setminus B')$ have isomorphic pro-finite completions.
On lattice-theoretic invariants of curves on a surface

Arithmetic Zariski couples of maximizing sextics

Digression:

Recall the arithmetic Zariski couple $[B_+, B_-]$ with

$$TA(\mathbb{P}^2, B_+, \gamma) \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}, \quad TA(\mathbb{P}^2, B_-, \gamma) \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}.$$ 

Consider the double coverings of the complements:

$$W_\pm : w^2 \cdot (G(x, y, 1) \pm \sqrt{5} \cdot H(x, y, 1)) = 1.$$ 

Both of them are smooth affine surfaces in $\mathbb{C}^3$. They are not homeomorphic.

Remark

Many examples of conjugate but non-homeomorphic complex varieties have been constructed since Serre (1960).
The invariant $F_A(S, B, \gamma)$ is related to $\pi_1(S \setminus B)$.

We denote by $N\gamma \subset \pi_1(S \setminus B)$ the kernel of the homomorphism

$$\pi_1(S \setminus B) \rightarrow A$$

corresponding to $\varphi_\gamma : W_\gamma \rightarrow S \setminus B$. Then the action of $A$ on $H_1(W_\gamma) = N_\gamma/[N_\gamma, N_\gamma]$ is associated with

$$0 \rightarrow H_1(W_\gamma) \rightarrow \pi_1(S \setminus B)/[N_\gamma, N_\gamma] \rightarrow A \rightarrow 0 :$$

that is, $a \in A$ acts on $x \in H_1(W_\gamma)$ by

$$a(x) = \tilde{a} \cdot x \cdot \tilde{a}^{-1},$$

where $\tilde{a} \in \pi_1(S \setminus B)/[N_\gamma, N_\gamma]$ is the pre-image of $a \in A$, and we regard $H_1(W_\gamma)$ as a normal subgroup of $\pi_1(S \setminus B)/[N_\gamma, N_\gamma]$.

Hence, if $A$ acts on $H_1(W_\gamma)$ non-trivially, $\pi_1(S \setminus B)$ is non-abelian.
We can prove the following:

**Theorem**

*Assume the following:*

(a) \( A \) is a cyclic group of prime order \( l \),

(b) the classes \([B_1], \ldots, [B_m]\) span \( \text{NS}(S) \otimes \mathbb{Q} \) over \( \mathbb{Q} \), and

(c) \( \gamma([B_i]) \neq 0 \) for \( i = 1, \ldots, m \).

*If the \( p \)-part \( F_{A}(S, B, \gamma)_p \) of \( F_{A}(S, B, \gamma) \) is non-trivial for some \( p \neq l \), then \( \pi_1(S \setminus B) \) acts on \( H_1(W_\gamma) \) non-trivially and hence is non-abelian.*
The invariant $\sigma_A(S, B, \gamma)$ can be computed from the configuration data of $B$.

We have developed a general method of Zariski-van Kampen type to calculate the lattice $T_A(S, B, \gamma)$.

Hence $|F_A(S, B, \gamma)|$ can be also calculated.

However, there seems to be no general method to determine $F_A(S, B, \gamma)$ so far, and it would be more difficult to find algebraic cycles that generate $F_A(S, B, \gamma)$ explicitly.
Thank you!