On lattice-invariants of complex algebraic surfaces and their applications

Ichiro Shimada

Hiroshima University

December 28, 2009, Busan

We work over $\mathbb{C}$.

Abstract

We study some lattice-theoretic topological invariants of complex algebraic surfaces in $\mathbb{P}^3$, and
present an application to the construction of examples of **weak** (arithmetic) Zariski pairs of surfaces with only RDPs in $\mathbb{P}^3$.

This is a joint work with A. Katanaga and M. Oka.

Let S and S' be reduced (possibly reducible) hypersurfaces in $\mathbb{P}^n$.

Definition

(1) We say that S and S' are *of the same configuration type* and write

$$S \sim_{\mathrm{cfg}} S'$$

if there are tubular neighborhoods $\mathcal{T} \subset \mathbb{P}^n$ of S and $\mathcal{T}' \subset \mathbb{P}^n$ of S', and a homeomorphism $(\mathcal{T}, S) \xrightarrow{\sim} (\mathcal{T}', S')$ that preserves the degrees of the irreducible components of S and S'.

(2) We say that S and S' have *the same embedding topology* and write

$$S \sim_{\mathrm{top}} S'$$

if there is a homeomorphism between $(\mathbb{P}^n, S)$ and $(\mathbb{P}^n, S')$.

If two surfaces S and S' in $\mathbb{P}^3$ with only RDPs are of the same configuration type, then

- $\deg S = \deg S'$, and
- the *ADE*-type R_S of $\mathrm{Sing}\, S$ is equal to the *ADE*-type $R_{S'}$ of $\mathrm{Sing}\, S'$.

Definition

We say that two surfaces S and S' in $\mathbb{P}^3$ with only RDPs are *of the weakly same configuration type* and write

$$S \sim_{\mathrm{wcfg}} S'$$

if $\deg S = \deg S'$ and $R_S = R_{S'}$.

It is obvious that $S \sim_{\mathrm{top}} S'$ implies $S \sim_{\mathrm{cfg}} S'$ and $S \sim_{\mathrm{wcfg}} S'$.

Definition

The pair $[S, S']$ of reduced hypersurfaces in $\mathbb{P}^n$ is called a *Zariski pair* if $S \sim_{\mathrm{cfg}} S'$ but $S \not\sim_{\mathrm{top}} S'$.

Definition

The pair $[S, S']$ of surfaces S and S' in $\mathbb{P}^3$ with only RDPs is called a *weak Zariski pair* if $S \sim_{\mathrm{wcfg}} S'$ but $S \not\sim_{\mathrm{top}} S'$.

Many examples of Zariski m-ples of plane curves ($n = 2$) have been constructed.

The first example was discovered by Zariski in 1930's.

Example

Let $B \subset \mathbb{P}^2$ be a plane curve of degree 6 defined by

$$f^3 + g^2 = 0, \quad \deg f = 2, \ \deg g = 3, \ \text{general}.$$

Then B is irreducible and has six cusps as its only singularities. The six cusps are lying on the conic $f = 0$, and we have

$$\pi_1(\mathbb{P}^2 \setminus B) \cong \mathbb{Z}/(2) * \mathbb{Z}/(3).$$

Del Pezzo had observed that there is a plane sextic B' with only six cusps that are *not* lying on a conic. Zariski exhibited such B' and showed that

$$\pi_1(\mathbb{P}^2 \setminus B') \cong \mathbb{Z}/(2) \times \mathbb{Z}/(3).$$

The main problem in these studies is how to distinguish the embedding topologies of plane curves $B \subset \mathbb{P}^2$ of the same configuration type.

The major tool is the fundamental groups $\pi_1(\mathbb{P}^2 \setminus B)$ or its variations like Alexander polynomials.

Aim: Construct Zariski pairs $[S, S']$ of surfaces in $\mathbb{P}^3$ with *only isolated singularities*.

In the construction, we cannot use $\pi_1(\mathbb{P}^3 \setminus S)$:
By Zariski's hyperplane section theorem, we have

$$\pi_1(\mathbb{P}^3 \setminus S) \cong \pi_1(\mathbb{P}^3 \setminus S') \cong \mathbb{Z}/(\deg S).$$

We need new topological invariants.

Let $\mathrm{Aut}(\mathbb{C})$ be the automorphism group of $\mathbb{C}$.

For a scheme $V \to \operatorname{Spec}\mathbb{C}$ and an element $\sigma \in \mathrm{Aut}(\mathbb{C})$, we define $V^\sigma \to \operatorname{Spec}\mathbb{C}$ by the following Cartesian diagram:

$$\begin{array}{ccc} V^\sigma & \longrightarrow & V \\ \downarrow & \square & \downarrow \\ \operatorname{Spec}\mathbb{C} & \underset{\sigma^*}{\longrightarrow} & \operatorname{Spec}\mathbb{C}. \end{array}$$

Two schemes V and V' over $\mathbb{C}$ are said to be *conjugate* if V' is isomorphic over $\mathbb{C}$ to V^σ over $\mathbb{C}$ for some $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Conjugate complex varieties can never be distinguished by any algebraic methods (they are isomorphic over $\mathbb{Q}$), but they can be non-homeomorphic in the classical complex topology.

The first example was given by Serre in 1964.

Other examples have been constructed by:

Abelson (1974),
Grothendieck's dessins d'enfants (1984),
Bartolo, Ruber, and Agustin (2004),
Easton and Vakil (2007),
F. Charles (2009).

Example (S.- and Arima)

Consider two smooth irreducible surfaces $S_\pm$ in $\mathbb{C}^3$ defined by

$$w^2(G(x,y) \pm \sqrt{5} \cdot H(x,y)) = 1, \quad \text{where}$$

$$\begin{aligned} G(x,y) &:= -9\,x^4 - 14\,x^3y + 58\,x^3 - 48\,x^2y^2 - 64\,x^2y \\ &\quad +10\,x^2 + 108\,xy^3 - 20\,xy^2 - 44\,y^5 + 10\,y^4, \\ H(x,y) &:= 5\,x^4 + 10\,x^3y - 30\,x^3 + 30\,x^2y^2 + \\ &\quad +20\,x^2y - 40\,xy^3 + 20\,y^5. \end{aligned}$$

Then S_+ and S_- are not homeomorphic.

Definition

A Zariski pair $[S, S']$ of hypersurfaces in $\mathbb{P}^n$ is called *an arithmetic Zariski pair* if S and S' are conjugate.

Definition

A weak Zariski pair $[S, S']$ of surfaces in $\mathbb{P}^3$ with only RDPs is called a *weak arithmetic Zariski pair* if S and S' are conjugate.

Aim: Construct arithmetic Zariski pairs $[S, S']$ of surfaces in $\mathbb{P}^3$ with *only isolated singularities*.

The first example of arithmetic Zariski pair was given by Bartolo, Ruber, and Agustin (2004) for plane curves.
Their tool was the *braid monodromy*, and cannot be used for surfaces with only isolated singularities.

We need new topological invariants.

Definition

A *quasi-lattice* is a finitely generated $\mathbb{Z}$-module L with a symmetric bilinear form

$$L \times L \to \mathbb{Z}.$$

For a quasi-lattice L, we put

$$\ker L := \{\, x \in L \mid (x, y) = 0 \ \text{for all} \ \ y \in L \,\} = L^{\perp}.$$

Note that $\ker L$ contains the torsion part of L.

Definition

A quasi-lattice L is called a *lattice* if the symmetric bilinear form is non-degenerate (that is, $\ker L = 0$).

For a quasi-lattice L, the free $\mathbb{Z}$-module $L/\ker L$ is a lattice.

Let $S \subset \mathbb{P}^3$ be a surface with only RDPs.

We will define two topological invariants $t(S)$ and $T(S)$ of $(\mathbb{P}^3, S)$, which allow us to construct **weak** (arithmetic) Zariski pairs.

Definition

We put

$$\begin{aligned} t(S) &:= \text{the torsion part of } H^3(\mathbb{P}^3 \setminus S, \mathbb{Z}) \\ &= \text{the torsion part of } H_3(\mathbb{P}^3, S, \mathbb{Z}) \\ &= \text{the torsion part of } H_2(S, \mathbb{Z}). \end{aligned}$$

It is obvious that $S \sim_{\text{top}} S'$ implies $t(S) \cong t(S')$.

We consider the smooth open surface

$$S^\circ := S \setminus \operatorname{Sing} S$$

and the intersection pairing

$$H_2(S^\circ, \mathbb{Z}) \times H_2(S^\circ, \mathbb{Z}) \to \mathbb{Z}.$$

We then put

$$V(S^\circ) := \operatorname{Ker}(H_2(S^\circ) \to H_2(\mathbb{P}^3)).$$

Definition

We define the invariant $T(S)$ by

$$T(S) := V(S^\circ)/\ker V(S^\circ),$$

which is a lattice.

It is obvious that $S \sim_{\text{top}} S'$ implies $T(S) \cong T(S')$.

Calculation of the invariants $t(S)$ and $T(S)$

Let R_S denote the ADE-type of $\mathrm{Sing}(S)$.
Consider the minimal resolution

$$\rho : X \to S$$

of S. We regard $H^2(X, \mathbb{Z})$ as a lattice by the cup-product. Let

$$h \in H^2(X)$$

be the class of the pull-back of a plane section of S.

Let $\mathcal{E}_\rho$ be the set of exceptional curves $E \subset X$ of ρ.
Each $E \in \mathcal{E}_\rho$ is a smooth rational curve with $E^2 = -2$,
and the dual graph of them is a Dynkin diagram of type R_S.

We consider the submodule

$$\langle \mathcal{E}_\rho \rangle \subset H^2(X)$$

generated by the classes of the curves $E \in \mathcal{E}_\rho$. Then $\langle \mathcal{E}_\rho \rangle$ is a sublattice of $H^2(X)$ isomorphic to the negative-definite root lattice of ADE-type R_S. Let

$$\overline{\langle \mathcal{E}_\rho \rangle} := (\langle \mathcal{E}_\rho \rangle \otimes \mathbb{Q}) \cap H^2(X)$$

be the primitive closure of $\langle \mathcal{E}_\rho \rangle$ in $H^2(X)$.

Looking at the topology of the minimal resolution ρ, we obtain the following:

Theorem

The invariant

$$t(S) \quad = \quad \text{the torsion part of } H_2(S,\mathbb{Z})$$

is isomorphic to $\overline{\langle \mathcal{E}_\rho \rangle}/\langle \mathcal{E}_\rho \rangle$.

Theorem

The lattice

$$T(S) := V(S^\circ)/\ker V(S^\circ),$$

where $S^\circ := S \setminus \operatorname{Sing} S$ and $V(S^\circ) := \operatorname{Ker}(H_2(S^\circ) \to H_2(\mathbb{P}^3))$ is isomorphic to the orthogonal complement of $\langle \mathcal{E}_\rho \rangle \oplus \langle h \rangle$ in $H^2(X)$.

Therefore, if we know the data

$$(\langle \mathcal{E}_\rho \rangle, h),$$

then we can calculate $t(S)$ and $T(S)$.

When $\deg S = 4$, X is a $K3$ surface, and $H^2(X)$ is isomorphic to the $K3$ *lattice*

$$\mathbb{L} := (-E_8)^2 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^3.$$

Definition

A *quartic lattice data* is a pair

$$(\Lambda, v)$$

of a negative-definite root sublattice Λ of the $K3$ lattice $\mathbb{L}$ and a vector $v \in \mathbb{L}$ with $v^2 = 4$.

Definition

A quartic lattice data (Λ, v) is *realizable* if there is a quartic surface $S \subset \mathbb{P}^3$ with only RDPs and an isomorphism

$$\phi : H^2(X) \cong \mathbb{L}$$

of lattices such that $\phi(\langle \mathcal{E}_\rho \rangle) = \Lambda$ and $\phi(h) = v$.

If such S exists, then R_S is equal to the ADE-type of the root sublattice Λ.

By the Torelli theorem for $K3$ surfaces, we have the complete list of realizable lattice data.
This task was done by J. G. Yang with an aid of computer.

Example

There is a weak Zariski pair $[S_0, S_1]$ of quartic surfaces such that

- each S_i has 8 nodes as its only singularities, and
- $t(S_0) = 0$, while $t(S_1) \cong \mathbb{Z}/2\mathbb{Z}$.

This pair was already observed by Coble in 1930's:
S_0 is called *azygetic*, while S_1 is called *syzygetic*.
Their difference is also expressed by

$$h^0(\mathbb{P}^3, \mathcal{I}_Q(2)) = \begin{cases} 2 & \text{if } Q = \mathrm{Sing}\, S_0, \\ 3 & \text{if } Q = \mathrm{Sing}\, S_1, \end{cases}$$

where $\mathcal{I}_Q \subset \mathcal{O}_{\mathbb{P}^3}$ is the ideal sheaf of $Q \subset \mathbb{P}^3$.
A syzygetic member S_1 is defined by an equation of the form $\sum a_{ij} A_i A_j = 0$, where A_0, A_1, A_2 are quadratics.

Example

There is a weak Zariski *quartet* $[S_0, S_1, S_2, S_3]$ of quartic surfaces with RDPs of type

$$2A_1 + 2A_2 + 2A_5$$

as their only singularities such that

$$t(S_0) = 0, \quad t(S_1) \cong \mathbb{Z}/2\mathbb{Z}, \quad t(S_2) \cong \mathbb{Z}/3\mathbb{Z}, \quad t(S_3) \cong \mathbb{Z}/6\mathbb{Z}.$$

Definition

A $K3$ surface X is called *singular* if the Picard number of X is 20.

Let X be a singular $K3$ surface. Then the transcendental lattice

$$\mathcal{T}(X) := \mathrm{NS}(X)^{\perp} \quad \text{in } H^2(X, \mathbb{Z})$$

is a positive-definite even lattice of rank 2.
The Hodge decomposition

$$\mathcal{T}(X) \otimes \mathbb{C} = H^{2,0}(X) \oplus H^{0,2}(X)$$

induces an orientation on $\mathcal{T}(X)$. We denote by

$$\tilde{\mathcal{T}}(X)$$

the oriented transcendental lattice of X.
By Torelli theorem, we have

$$\tilde{\mathcal{T}}(X) \cong \tilde{\mathcal{T}}(X') \implies X \cong X'.$$

Construction by Shioda and Inose

Every singular $K3$ surface X is obtained as a certain double cover of the Kummer surface

$$\mathrm{Km}(E \times E'),$$

where E and E' are elliptic curves with CM by some orders of

$$\mathbb{Q}(\sqrt{-|\mathrm{disc}(\mathcal{T}(X))|}).$$

Theorem (Shioda and Inose)

(1) For any positive-definite oriented even lattice $\tilde{\mathcal{T}}$ of rank 2, there exists a singular $K3$ surface X such that $\tilde{\mathcal{T}}(X) \cong \tilde{\mathcal{T}}$.
(2) Every singular $K3$ surface is defined over a number field.

A lattice L is naturally embedded into the dual lattice

$$L^\vee := \mathrm{Hom}(L, \mathbb{Z}).$$

The *discriminant group* of L is the finite abelian group

$$D_L := L^\vee / L.$$

The $\mathbb{Z}$-valued symmetric bilinear form on L extends to

$$L^\vee \times L^\vee \to \mathbb{Q}.$$

A lattice L is said to be even if $x^2 \in 2\mathbb{Z}$ for all $x \in L$. If L is even, then we have a quadratic form

$$q_L : D_L \to \mathbb{Q}/\mathbb{Z}, \;\; \bar{x} \mapsto x^2 \bmod 2\mathbb{Z},$$

which is called the *discriminant form* of L.

We have the following:

Proposition

Let L and L' be even lattices of the same rank.
If L and L' have isomorphic discriminant forms and the same signature, then L and L' belong to the same genus.

Since $H^2(X)$ is unimodular and both of $\mathcal{T}(X)$ and $\mathrm{NS}(X)$ are primitive in $H^2(X)$, we have the following:

Proposition

$$(D_{\mathcal{T}(X)}, q_{\mathcal{T}(X)}) \cong (D_{\mathrm{NS}(X)}, -q_{\mathrm{NS}(X)}).$$

Let X and X' be singular $K3$ surfaces.
It is obvious that, if X and X' are conjugate, then $\mathrm{NS}(X)$ and $\mathrm{NS}(X')$ are isomorphic. Therefore we have the following:

Corollary

If X and X' are conjugate, then $\mathcal{T}(X)$ and $\mathcal{T}(X')$ are in the same genus.

The class field theory of imaginary quadratic fields tells us how the Galois group acts on the j-invariants of elliptic curves with CM. Using this, S.- and Schütt (2007) proved the following converse:

Theorem

If $\mathcal{T}(X)$ and $\mathcal{T}(X')$ are in the same genus,
then X and X' are conjugate.

Definition

A quartic surface $S \subset \mathbb{P}^3$ is *maximizing* if it has only RDPs and its total Milnor number is 19.

If S is a maximizing quartic, then X is a singular $K3$ surface, and we have

$$T(S) \cong \mathcal{T}(X).$$

Hence, if maximizing quartics S and S' are of the weakly same configuration type, and $\mathcal{T}(X)$ and $\mathcal{T}(X')$ are not isomorphic but in the same genus, then $[S, S']$ is a weak arithmetic Zariski pair.

Example

There is a weak arithmetic Zariski pair $[S, S']$ of maximizing quartic surfaces such that

- each of them has RDPs of type $A_1 + A_{18}$ as its only singularities, and
- the minimal resolutions X of S and X' of S' have the transcendental lattices

$$\begin{bmatrix} 4 & 0 \\ 0 & 38 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 6 & 2 \\ 2 & 26 \end{bmatrix},$$

which are in the same genus but are not isomorphic.

Problem: Find the explicit defining equations of S and S'.

Thank you!