Supersingular $K3$ surfaces and lattice theory

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Let $k$ be an algebraically closed field of characteristic $p \geq 0$.

Let $X$ be a smooth projective surface defined over $k$. Two divisors $D_1$ and $D_2$ on $X$ are *numerically equivalent* and denoted by $D_1 \equiv D_2$ if

$$D_1.C = D_2.C$$

holds for any curve $C$ on $X$, where $D.C$ denotes the intersection number of $D$ and $C$. The $\mathbb{Z}$-module

$$\text{NS}(X) := \{ \text{divisors on } X \}/ \equiv$$

is then equipped with a structure of the lattice by the intersection pairing. We call $\text{NS}(X)$ the *Néron-Severi lattice* of $X$. 
The cycle map
\[ \text{NS}(X) \rightarrow H^2_{\text{ét}}(X, \mathbb{Q}_l) \]
is injective. In particular, we have
\[ \text{rank}(\text{NS}(X)) \leq b_2(X). \]

When \( k = \mathbb{C} \), we have
\[ \text{NS}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z}), \]
and hence
\[ \text{rank}(\text{NS}(X)) \leq h^{1,1}(X). \]
Supersingular $K3$ surfaces and lattice theory

Definition

A smooth projective surface $X$ is called a $K3$ surface if

- $\exists$ a nowhere vanishing regular 2-form $\omega_X$ on $X$, and
- $h^1(X, \mathcal{O}_X) = 0$.

$K3$ surfaces are 2-dimensional analogue of elliptic curves.

Examples of $K3$ surfaces:

- double covers of $\mathbb{P}^2$ branching along smooth curves of deg 6,
- smooth complete intersections of degree $(4)$ in $\mathbb{P}^3$, degree $(2, 3)$ in $\mathbb{P}^4$, $(2, 2, 2)$ in $\mathbb{P}^5$,
- minimal resolutions of $A/\langle \iota \rangle$, where $A$ are abelian surfaces and $\iota$ is the involution $x \mapsto -x \ (p \neq 2)$,
If $X$ is a $K3$ surface, then

$$b_2(X) = 22 \quad \text{and} \quad p_g(X) = 1.$$ 

In particular, we have

$$\text{rank}(\text{NS}(X)) \leq 22.$$ 

If $k = \mathbb{C}$, then we have

$$\text{rank}(\text{NS}(X)) \leq 20.$$ 

A complex $K3$ surface $X$ is called singular if $\text{rank}(\text{NS}(X)) = 20$. We have a complete classification and an explicit method of constructing all singular $K3$ surfaces due to Shioda and Inose.
Definition

A $K3$ surface $X$ is called *supersingular* if $\text{rank}(\text{NS}(X)) = 22$.

Problem

*Construct all supersingular $K3$ surfaces.*

More precisely, we want to obtain a theorem of the type:

Every supersingular $K3$ surface is defined by such and such defining equations or constructed by such and such methods.

We have an answer in characteristic 2, but not yet in characteristic $> 2$. 
**Definition**

An algebraic surface $S$ is *unirational* if there is a dominant rational map $\mathbb{P}^2 \cdots \to S$.

If $p = 0$, every unirational surface is rational (birational to $\mathbb{P}^2$). In particular, a complex $K3$ surface can never be unirational.

**Theorem**

If a smooth projective surface $S$ is unirational, then $b_2(S) = \text{rank}(\text{NS}(S))$.

**Corollary**

A unirational $K3$ surface is supersingular.
Consider the Fermat surface of degree $d$ in $\mathbb{P}^3$ 

$$F : w^d + x^d + y^d + z^d = 0$$

in characteristic $p > 0$.

If $d = q + 1$ with $q = p^\nu$, then $F$ is defined by the Hermitian form

$$w\bar{w} + x\bar{x} + y\bar{y} + z\bar{z} = 0 \quad \text{where } \bar{a} = a^q,$$

over $\mathbb{F}_q$, and hence is similar to the quadric surface.

**Theorem (Shioda)**

If $d = q + 1$, then $F$ is unirational.

**Corollary**

If $p \equiv 3 \mod 4$, then the Fermat quartic surface

$$w^4 + x^4 + y^4 + z^4 = 0$$

in characteristic $p$ is a supersingular $K3$ surface.
Suppose that $p = 2$. Let $(x, y, z)$ be the homogeneous coordinates of $\mathbb{P}^2$. Consider the purely inseparable double cover

$$Y : w^2 = f(x, y, z)(= \sum a_{ijk}x^iy^jz^k)$$

of $\mathbb{P}^2$, where $f$ is a general homogeneous polynomial of deg 6. Then $Y$ has 21 ordinary nodes, and the minimal resolution $X$ of $Y$ is a $K3$ surface.

The pull-back of $X \to \mathbb{P}^2$ by the morphism $\mathbb{P}^2 \to \mathbb{P}^2$ given by

$$(\xi, \eta, \zeta) \mapsto (x, y, z) = (\xi^2, \eta^2, \zeta^2)$$

is defined by

$$(w - f^{(1/2)}(\xi, \eta, \zeta))^2 = 0 \quad \text{where} \quad f^{(1/2)}(\xi, \eta, \zeta) = \sum a^{(1/2)}_{ijk}\xi^i\eta^j\zeta^k,$$

and its reduced part $w = f^{(1/2)}(\xi, \eta, \zeta)$ is rational. Therefore $X$ is unirational and hence is supersingular.
By a lattice, we mean a free \( \mathbb{Z} \)-module \( L \) of finite rank with a non-degenerate symmetric bilinear form

\[ L \times L \to \mathbb{Z}. \]

A lattice \( L \) is naturally embedded into the dual lattice

\[ L^\vee := \text{Hom}(L, \mathbb{Z}). \]

The discriminant group of \( L \) is the abelian group

\[ D_L := L^\vee / L, \]

which is finite of order

\[ |\text{disc } L| = |\det M_L|, \]

where \( M_L \) is a symmetric matrix expressing \( L \times L \to \mathbb{Z} \).
**Definition**

A lattice $L$ is *even* if $x^2 \in 2\mathbb{Z}$ holds for any $x \in L$.

**Definition**

Let $p$ be a prime integer. A lattice is *$p$-elementary* if $D_L$ is isomorphic to direct product of the cyclic group of order $p$:

$$D_L \cong (\mathbb{Z}/p\mathbb{Z})^\oplus t.$$  

**Definition**

The signature $\text{sgn}(L) = (s_+, s_-)$ of a lattice $L$ is the pair of numbers of positive and negative eigenvalues of $M_L$ (in $\mathbb{R}$). We say that $L$ is *indefinite* if $s_+ > 0$ and $s_- > 0$. 
Theorem (Rudakov-Shafarevich, Artin)

The Néron-Severi lattice $\mathcal{NS}(X)$ of a supersingular $K3$ surface $X$ in characteristic $p$ is

- even,
- of signature $(1, 21)$ and
- $p$-elementary.

There exists a positive integer $\sigma_X \leq 10$ such that

$$D_{\mathcal{NS}(X)} \cong (\mathbb{Z}/p\mathbb{Z})^\oplus 2\sigma_X.$$ 

Definition

The integer $\sigma_X$ is called the Artin invariant of $X$. 
Theorem

For each $\sigma$, there is only one isomorphism class of lattices $L$ with

- even,
- $\text{rank}(L) = 22$, $\text{sgn}(L) = (1, 21)$ and
- $D_L \cong (\mathbb{Z}/p\mathbb{Z}) \oplus 2\sigma$.

Proof.

Use Eichler’s theorem on isom classes of indefinite lattices.

Corollary

The isomorphism class of the Néron-Severi lattice $\text{NS}(X)$ of a supersingular $K3$ surface $X$ is determined by the characteristic $p$ and the Artin invariant $\sigma_X$. 
The Artin invariant goes down under specialization.

**Theorem (Ogus)**

If $X$ and $X'$ are supersingular $K3$ surfaces in characteristic $p$ with Artin invariant 1, then $X$ and $X'$ are isomorphic.

Recall that we want to obtain a theorem of the type:

*Every supersingular $K3$ surface in characteristic $p$ with Artin invariant $\leq m$ is defined by such and such defining equations or constructed by such and such methods.*

For example:

**Theorem (Ogus)**

Every supersingular $K3$ surface in characteristic $p \neq 2$ with Artin invariant $\leq 2$ is a Kummer surface.
One of the motivations is:

**Conjecture (Artin-Shioda)**

Every supersingular $K3$ surface is unirational.

By the theorem of Ogus, we obtain the following:

**Corollary**

Every supersingular $K3$ surface with Artin invariant $\leq 2$ is unirational.

To show the unirationality, we need the defining equations. We can get some defining equations from the lattice $NS(X)$. 
Theorem (S.-)

Every supersingular $K3$ surface in characteristic 2 is birational to the purely inseparable double cover

$$w^2 = f(x, y, z)$$

of $\mathbb{P}^2$, where $f$ is a homogeneous polynomial of degree 6.

Corollary (Rudakov-Shafarevich)

Every supersingular $K3$ surface in characteristic 2 is unirational.
Theorem (D.Q. Zhang and S.-)  

Every supersingular $K3$ surface $X$ in characteristic 3 with Artin invariant $\leq 6$ is birational to the purely inseparable triple cover of a quadratic surface $\mathbb{P}^1 \times \mathbb{P}^1$.

Corollary (Rudakov-Shafarevich)  

Every supersingular $K3$ surface in characteristic 3 with Artin invariant $\leq 6$ is unirational.
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**Known results**

**Theorem (Pho and S.-)**

Every supersingular $K3$ surface $X$ in characteristic 5 with Artin invariant $\leq 3$ is birational to the double cover of $\mathbb{P}^2$ branching along a curve defined by

$$y^5 z - f(x, z) = 0$$

where $f(x, z)$ is a homogeneous polynomial of degree 6.

**Corollary**

Every supersingular $K3$ surface $X$ in characteristic 5 with Artin invariant $\leq 3$ is unirational.
Known results

Remark

It is difficult to construct supersingular $K3$ surfaces with big Artin invariants *en masse*.

There exist many sporadic examples of supersingular $K3$ surfaces with big Artin invariants, due to Shioda and Goto.
How to obtain the defining equations of $X$ from $\text{NS}(X)$?

**Definition**

An effective divisor $D$ on a $K3$ surface $X$ is a *polarization* if $D^2 > 0$ and the complete linear system $|D|$ has no fixed components.

If $D$ is a polarization, then $|D|$ defines a morphism $\Phi : X \to \mathbb{P}^N$.

Let $X \to Y \to \mathbb{P}^N$ be the Stein factorization of $\Phi$. Then the normal surface $Y$, birational to $X$, has only rational double points.

For a class $h \in \text{NS}(X)$, we put

$$\mathcal{R}(X, h) := \{ x \in \text{NS}(X) \mid (x, h) = 0 \text{ and } x^2 = -2 \}.$$  

Then $\mathcal{R}(X, h)$ is a root system.

**Proposition**

The $ADE$-type of the rational double points of $Y$ is equal to the Dynkin type of the root system $\mathcal{R}(X, [D])$. 
Note that

$$\text{NS}(X) \cong \text{Pic}(X)$$

for a $K3$ surface $X$.

**Proposition**

Let $h = [D] \in \text{NS}(X)$ be a class with $h^2 > 0$. Then $D$ is a polarization if and only if the following hold:

- $D$ is nef (that is, $D.C \geq 0$ for any curve $C$), and
- $\{ x \in \text{NS}(X) \mid x^2 = 0 \text{ and } (x, h) = 1 \}$ is empty.

On the other hand, looking at the nef cone of $\text{NS}(X) \otimes \mathbb{R}$, we obtain the following:

**Proposition**

Let $h \in \text{NS}(X)$ be a class with $h^2 \geq 0$. Then there is an isometry $\varphi$ of $\text{NS}(X)$ such that $\varphi(h)$ is nef.
Therefore we have:

**Theorem**

If there exists a vector \( h \in \text{NS}(X) \) with \( d = h^2 > 0 \) such that

\[
\{ x \in \text{NS}(X) \mid x^2 = 0 \quad \text{and} \quad (x, h) = 1 \} = \emptyset,
\]

then \( X \) has a morphism \( \Phi : X \to \mathbb{P}^N \) with the Stein factorization

\[
X \to Y \to \mathbb{P}^N
\]

such that

- \( (\Phi^* \mathcal{O}(1))^2 = d \), and
- the normal surface \( Y \) has rational double points of the \( ADE \)-type being equal to the Dynkin type of the root system

\[
\mathcal{R}(X, h) := \{ x \in \text{NS}(X) \mid (x, h) = 0 \quad \text{and} \quad x^2 = -2 \}.
\]
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How to obtain the defining equations of $X$ from $\text{NS}(X)$?

We give a proof of the following theorem in char 5:

**Theorem**

Every supersingular $K3$ surface $X$ in char 5 with $\sigma_X \leq 3$ is birational to the double cover of a plane branching along a curve defined by $y^5 - f(x) = 0$ with $\text{deg } f = 6$.

We consider the general case $\sigma_X = 3$. Then $\text{NS}(X)$ is characterized by the properties of being even, $\text{sgn} = (1, 21)$, and

$$D_{\text{NS}(X)} \cong (\mathbb{Z}/5\mathbb{Z})^6.$$ 

We can construct such a lattice $\text{NS}(X)$ lattice-theoretically. For example,

$$\text{NS}(X) \cong (-A_4) \oplus (-A_4) \oplus (-A_4) \oplus (-A_4) \oplus (-A_4) \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix},$$

where $(-A_4)$ denote the negative-definite root lattice of type $A_4$. 
We have a vector \( h \in \text{NS}(X) \) with \( h^2 = 2 \) such that

- \( \{ x \in \text{NS}(X) \mid x^2 = 0 \text{ and } (x, h) = 1 \} = \emptyset \), and
- the root system \( \{ x \in \text{NS}(X) \mid x^2 = -2 \text{ and } (x, h) = 0 \} \) is of type \( 5A_4 \).

Therefore \( X \) is birational to a double cover of \( \mathbb{P}^2 \) branching along a sextic curve \( B \) with \( 5A_4 \) singularities. (Such a curve does not exist in char 0.) Then the Gauss map

\[ B \to B^\vee \]

is inseparable of degree 5. We can show that \( B \) is defined by an equation of the form \( y^5 - f(x) = 0 \).