Supersingular $K3$ surfaces and lattice theory

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Let k be an algebraically closed field of characteristic $p \geq 0$.

Let X be a smooth projective surface defined over k. Two divisors D_1 and D_2 on X are *numerically equivalent* and denoted by $D_1 \equiv D_2$ if

$$D_1.C = D_2.C$$

holds for any curve C on X, where $D.C$ denotes the intersection number of D and C. The $\mathbb{Z}$-module

$$\mathrm{NS}(X) := \{\text{divisors on } X\}/\equiv$$

is then equipped with a structure of the lattice by the intersection pairing. We call $\mathrm{NS}(X)$ the *Néron-Severi lattice* of X.

The cycle map

$$\mathrm{NS}(X) \to H^2_{\text{ét}}(X, \mathbb{Q}_l)$$

is injective. In particular, we have

$$\mathrm{rank}(\mathrm{NS}(X)) \leq b_2(X).$$

When $k = \mathbb{C}$, we have

$$\mathrm{NS}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z}),$$

and hence

$$\mathrm{rank}(\mathrm{NS}(X)) \leq h^{1,1}(X).$$

Definition

A smooth projective surface X is called a $K3$ *surface* if

- $\exists$ a nowhere vanishing regular 2-form ω_X on X, and
- $h^1(X, \mathcal{O}_X) = 0$.

$K3$ surfaces are 2-dimensional analogue of elliptic curves.

Examples of $K3$ surfaces:

- double covers of $\mathbb{P}^2$ branching along smooth curves of deg 6,
- smooth complete intersections of degree (4) in $\mathbb{P}^3$, degree $(2,3)$ in $\mathbb{P}^4$, $(2,2,2)$ in $\mathbb{P}^5$,
- minimal resolutions of $A/\langle\iota\rangle$, where A are abelian surfaces and ι is the involution $x \mapsto -x$ ($p \neq 2$),
- $\ldots$.

If X is a $K3$ surface, then

$$b_2(X) = 22 \quad \text{and} \quad p_g(X) = 1.$$

In particular, we have

$$\mathrm{rank}(\mathrm{NS}(X)) \leq 22.$$

If $k = \mathbb{C}$, then we have

$$\mathrm{rank}(\mathrm{NS}(X)) \leq 20.$$

A complex $K3$ surface X is called *singular* if $\mathrm{rank}(\mathrm{NS}(X)) = 20$. We have a complete classification and an explicit method of constructing all singular $K3$ surfaces due to Shioda and Inose.

Definition

A $K3$ surface X is called *supersingular* if $\mathrm{rank}(\mathrm{NS}(X)) = 22$.

Problem

Construct all supersingular $K3$ surfaces.

More precisely, we want to obtain a theorem of the type:

> Every supersingular $K3$ surface is defined by such and such defining equations or constructed by such and such methods.

We have an answer in characteristic 2,
but not yet in characteristic > 2.

Definition

An algebraic surface S is *unirational* if there is a dominant rational map $\mathbb{P}^2 \cdots \to S$.

If $p = 0$, every unirational surface is rational (birational to $\mathbb{P}^2$).
In particular, a complex $K3$ surface can never be unirational.

Theorem

If a smooth projective surface S is unirational, then
$b_2(S) = \mathrm{rank}(\mathrm{NS}(S))$.

Corollary

A unirational $K3$ surface is supersingular.

Consider the Fermat surface of degree d in $\mathbb{P}^3$

$$F : w^d + x^d + y^d + z^d = 0$$

in characteristic $p > 0$.

If $d = q + 1$ with $q = p^\nu$, then F is defined by the Hermitian form

$$w\bar{w} + x\bar{x} + y\bar{y} + z\bar{z} = 0 \quad \text{where } \bar{a} = a^q,$$

over $\mathbb{F}_q$, and hence is similar to the quadric surface.

Theorem (Shioda)

If $d = q + 1$, then F is unirational.

Corollary

If $p \equiv 3 \bmod 4$, then the Fermat quartic surface

$$w^4 + x^4 + y^4 + z^4 = 0$$

in characteristic p is a supersingular $K3$ surface.

Suppose that $p = 2$. Let (x, y, z) be the homogeneous coordinates of $\mathbb{P}^2$. Consider the purely inseparable double cover

$$Y : w^2 = f(x, y, z)(= \sum a_{ijk} x^i y^j z^k)$$

of $\mathbb{P}^2$, where f is a general homogeneous polynomial of deg 6. Then Y has 21 ordinary nodes, and the minimal resolution X of Y is a $K3$ surface.

The pull-back of $X \to \mathbb{P}^2$ by the morphism $\mathbb{P}^2 \to \mathbb{P}^2$ given by

$$(\xi, \eta, \zeta) \mapsto (x, y, z) = (\xi^2, \eta^2, \zeta^2)$$

is defined by

$$(w - f^{(1/2)}(\xi, \eta, \zeta))^2 = 0 \quad \text{where} \quad f^{(1/2)}(\xi, \eta, \zeta) = \sum a_{ijk}^{(1/2)} \xi^i \eta^j \zeta^k,$$

and its reduced part $w = f^{(1/2)}(\xi, \eta, \zeta)$ is rational. Therefore X is unirational and hence is supersingular.

By a *lattice*, we mean a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form

$$L \times L \to \mathbb{Z}.$$

A lattice L is naturally embedded into the dual lattice

$$L^{\vee} := \mathrm{Hom}(L, \mathbb{Z}).$$

The *discriminant group* of L is the abelian group

$$D_L := L^{\vee}/L,$$

which is finite of order

$$|\operatorname{disc} L| = |\det M_L|,$$

where M_L is a symmetric matrix expressing $L \times L \to \mathbb{Z}$.

Definition

A lattice L is *even* if $x^2 \in 2\mathbb{Z}$ holds for any $x \in L$.

Definition

Let p be a prime integer. A lattice is *p-elementary* if D_L is isomorphic to direct product of the cyclic group of order p:

$$D_L \cong (\mathbb{Z}/p\mathbb{Z})^{\oplus t}.$$

Definition

The signature $\mathrm{sgn}(L) = (s_+, s_-)$ of a lattice L is the pair of numbers of positive and negative eigenvalues of M_L (in $\mathbb{R}$). We say that L is *indefinite* if $s_+ > 0$ and $s_- > 0$.

Theorem (Rudakov-Shafarevich, Artin)

The Néron-Severi lattice $\mathrm{NS}(X)$ of a supersingular $K3$ surface X in characteristic p is

- even,
- of signature $(1, 21)$ and
- p-elementary.

There exists a positive integer $\sigma_X \leq 10$ such that

$$D_{\mathrm{NS}(X)} \cong (\mathbb{Z}/p\mathbb{Z})^{\oplus 2\sigma_X}.$$

Definition

The integer σ_X is called the *Artin invariant* of X.

Theorem

For each σ, there is only one isomorphism class of lattices L with

- even,
- $\mathrm{rank}(L) = 22$, $\mathrm{sgn}(L) = (1, 21)$ and
- $D_L \cong (\mathbb{Z}/p\mathbb{Z})^{\oplus 2\sigma}$.

Proof.

Use Eichler's theorem on isom classes of indefinite lattices. $\square$

Corollary

The isomorphism class of the Néron-Severi lattice $\mathrm{NS}(X)$ of a supersingular $K3$ surface X is determined by the characteristic p and the Artin invariant σ_X.

The Artin invariant goes down under specialization.

Theorem (Ogus)

If X and X' are supersingular $K3$ surfaces in characteristic p with Artin invariant 1, then X and X' are isomorphic.

Recall that we want to obtain a theorem of the type:

Every supersingular $K3$ surface in characteristic p with Artin invariant $\leq m$ is defined by such and such defining equations or constructed by such and such methods.

For example:

Theorem (Ogus)

Every supersingular $K3$ surface in characteristic $p \neq 2$ with Artin invariant ≤ 2 is a Kummer surface.

One of the motivations is:

Conjecture (Artin-Shioda)

Every supersingular $K3$ surface is unirational.

By the theorem of Ogus, we obtain the following:

Corollary

Every supersingular $K3$ surface with Artin invariant ≤ 2 is unirational.

To show the unirationality, we need the defining equations.

We can get some defining equations from the lattice $\mathrm{NS}(X)$.

Theorem (S.-)

Every supersingular $K3$ surface in characteristic 2 is birational to the purely inseparable double cover

$$w^2 = f(x, y, z)$$

of $\mathbb{P}^2$, where f is a homogeneous polynomial of degree 6.

Corollary (Rudakov-Shafarevich)

Every supersingular $K3$ surface in characteristic 2 is unirational.

Theorem (D.Q. Zhang and S.-)

Every supersingular $K3$ surface X in characteristic 3 with Artin invariant ≤ 6 is birational to the purely inseparable triple cover of a quadratic surface $\mathbb{P}^1 \times \mathbb{P}^1$.

Corollary (Rudakov-Shafarevich)

Every supersingular $K3$ surface in characteristic 3 with Artin invariant ≤ 6 is unirational.

Theorem (Pho and S.-)

Every supersingular $K3$ surface X in characteristic 5 with Artin invariant ≤ 3 is birational to the double cover of $\mathbb{P}^2$ branching along a curve defined by

$$y^5 z - f(x, z) = 0$$

where $f(x, z)$ is a homogeneous polynomial of degree 6.

Corollary

Every supersingular $K3$ surface X in characteristic 5 with Artin invariant ≤ 3 is unirational.

Remark

It is difficult to construct supersingular $K3$ surfaces with big Artin invariants *en masse*.

There exist many sporadic examples of supersingular $K3$ surfaces with big Artin invariants, due to Shioda and Goto.

Definition

An effective divisor D on a $K3$ surface X is a *polarization* if $D^2 > 0$ and the complete linear system $|D|$ has no fixed components.

If D is a polarization, then $|D|$ defines a morphism $\Phi : X \to \mathbb{P}^N$. Let $X \to Y \to \mathbb{P}^N$ be the Stein factorization of Φ. Then the normal surface Y, birational to X, has only rational double points.

For a class $h \in \mathrm{NS}(X)$, we put

$$\mathcal{R}(X, h) := \{\, x \in \mathrm{NS}(X) \mid (x, h) = 0 \quad \text{and} \quad x^2 = -2 \,\}.$$

Then $\mathcal{R}(X, h)$ is a root system.

Proposition

The *ADE*-type of the rational double points of Y is equal to the Dynkin type of the root system $\mathcal{R}(X, [D])$.

Note that

$$\mathrm{NS}(X) \cong \mathrm{Pic}(X)$$

for a $K3$ surface X.

Proposition

Let $h = [D] \in \mathrm{NS}(X)$ be a class with $h^2 > 0$. Then D is a polarization if and only if the following hold:

- D is nef (that is, $D.C \geq 0$ for any curve C), and
- $\{ \, x \in \mathrm{NS}(X) \mid x^2 = 0 \quad \text{and} \quad (x, h) = 1 \, \}$ is empty.

On the other hand, looking at the nef cone of $\mathrm{NS}(X) \otimes \mathbb{R}$, we obtain the following:

Proposition

Let $h \in \mathrm{NS}(X)$ be a class with $h^2 \geq 0$. Then there is an isometry φ of $\mathrm{NS}(X)$ such that $\varphi(h)$ is nef.

Therefore we have:

Theorem

If there exists a vector $h \in \mathrm{NS}(X)$ with $d = h^2 > 0$ such that

$$\{\, x \in \mathrm{NS}(X) \mid x^2 = 0 \quad \text{and} \quad (x, h) = 1 \,\} = \emptyset,$$

then X has a morphism $\Phi : X \to \mathbb{P}^N$ with the Stein factorization

$$X \to Y \to \mathbb{P}^N$$

such that

- $(\Phi^* \mathcal{O}(1))^2 = d$, and
- the normal surface Y has rational double points of the ADE-type being equal to the Dynkin type of the root system

$$\mathcal{R}(X, h) := \{\, x \in \mathrm{NS}(X) \mid (x, h) = 0 \quad \text{and} \quad x^2 = -2 \,\}.$$

We give a proof of the following theorem in char 5:

Theorem

Every supersingular $K3$ surface X in char 5 with $\sigma_X \leq 3$ is birational to the double cover of a plane branching along a curve defined by $y^5 - f(x) = 0$ with $\deg f = 6$.

We consider the general case $\sigma_X = 3$. Then $\mathrm{NS}(X)$ is characterized by the properties of being even, sgn $= (1, 21)$, and

$$D_{\mathrm{NS}(X)} \cong (\mathbb{Z}/5\mathbb{Z})^6.$$

We can construct such a lattice $\mathrm{NS}(X)$ lattice-theoretically. For example,

$$\mathrm{NS}(X) \cong (-A_4) \oplus (-A_4) \oplus (-A_4) \oplus (-A_4) \oplus (-A_4) \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix},$$

where $(-A_4)$ denote the negative-definite root lattice of type A_4.

We have a vector $h \in \mathrm{NS}(X)$ with $h^2 = 2$ such that

- $\{\, x \in \mathrm{NS}(X) \mid x^2 = 0 \quad \text{and} \quad (x, h) = 1 \,\} = \emptyset$, and
- the root system $\{\, x \in \mathrm{NS}(X) \mid x^2 = -2 \quad \text{and} \quad (x, h) = 0 \,\}$ is of type $5A_4$.

Therefore X is birational to a double cover of $\mathbb{P}^2$ branching along a sextic curve B with $5A_4$ singularities. (Such a curve does not exist in char 0.) Then the Gauss map

$$B \to B^\vee$$

is inseparable of degree 5. We can show that B is defined by an equation of the form $y^5 - f(x) = 0$.