Transcendental lattices of complex algebraic surfaces

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Let $\text{Aut}(\mathbb{C})$ be the automorphism group of the complex number field $\mathbb{C}$.

For a scheme $V \to \text{Spec} \mathbb{C}$ and an element $\sigma \in \text{Aut}(\mathbb{C})$, we define a scheme $V^\sigma \to \text{Spec} \mathbb{C}$ by the following Cartesian diagram:

$$
\begin{array}{ccc}
V^\sigma & \longrightarrow & V \\
\downarrow & \Box & \downarrow \\
\text{Spec} \mathbb{C} & \underset{\sigma^*}{\longrightarrow} & \text{Spec} \mathbb{C}.
\end{array}
$$

Two schemes $V$ and $V'$ over $\mathbb{C}$ are said to be *conjugate* if $V'$ is isomorphic to $V^\sigma$ over $\mathbb{C}$ for some $\sigma \in \text{Aut}(\mathbb{C})$. 
Conjugate complex varieties can never be distinguished by any algebraic methods (they are isomorphic over $\mathbb{Q}$), but they can be non-homeomorphic in the classical complex topology.

The first example was given by Serre in 1964.

Other examples have been constructed by:
- Abelson (1974),
- Grothendieck’s dessins d’enfants (1984),
- Artal Bartolo, Carmona Ruber, and Cogolludo Agust (2004),

We will construct such examples by means of *transcendental lattices* of complex algebraic surfaces.
Example (S.- and Arima)

Consider two smooth irreducible surfaces $S_{\pm}$ in $\mathbb{C}^3$ defined by

$$w^2(G(x, y) \pm \sqrt{5} \cdot H(x, y)) = 1,$$

where

$$G(x, y) := -9x^4 - 14x^3y + 58x^3 - 48x^2y^2 - 64x^2y + 10x^2 + 108xy^3 - 20xy^2 - 44y^5 + 10y^4,$$

$$H(x, y) := 5x^4 + 10x^3y - 30x^3 + 30x^2y^2 + 20x^2y - 40xy^3 + 20y^5.$$ 

Then $S_+$ and $S_-$ are not homeomorphic.
§ Definition of the transcendental lattice
§ The transcendental lattice is topological
§ The discriminant form is algebraic
§ Fully-rigged surfaces
§ Maximizing curves
§ Arithmetic of fully-rigged $K3$ surfaces
§ Construction of the explicit example
By a lattice, we mean a free $\mathbb{Z}$-module $L$ of finite rank with a non-degenerate symmetric bilinear form

$$L \times L \rightarrow \mathbb{Z}.$$ 

A lattice $L$ is naturally embedded into the dual lattice

$$L^\vee := \text{Hom}(L, \mathbb{Z}).$$

The *discriminant group* of $L$ is the finite abelian group

$$D_L := L^\vee / L.$$ 

A lattice $L$ is called *unimodular* if $D_L$ is trivial.
Let $X$ be a smooth projective surface over $\mathbb{C}$. Then

$$H^2(X) := H^2(X, \mathbb{Z})/(\text{the torsion})$$

is regarded as a unimodular lattice by the cup-product.

The *Néron-Severi lattice*

$$\text{NS}(X) := H^2(X) \cap H^{1,1}(X)$$

of classes of algebraic curves on $X$ is a sublattice of signature

$$\text{sgn}(\text{NS}(X)) = (1, \rho - 1).$$

The *transcendental lattice* of $X$ is defined to be the orthogonal complement of $\text{NS}(X)$ in $H^2(X)$:

$$T(X) := \text{NS}(X)^\perp.$$
Proposition (Shioda)

$T(X)$ is a birational invariant of algebraic surfaces.

Proof.

Suppose that $X$ and $X'$ are birational. There exists a smooth projective surface $X''$ with birational morphisms

$$X'' \to X \quad \text{and} \quad X'' \to X'. $$

Every birational morphism between smooth projective surfaces is a composite of blowing-ups at points. A blowing-up at a point does not change the transcendental lattice.

Hence, for a surface $S$ (possibly singular and possibly open), the transcendental lattice $T(S)$ is well-defined.
Let $X$ be a smooth projective surface over $\mathbb{C}$, and let $C_1, \ldots, C_n \subset X$ be irreducible curves. Suppose that

$$[C_1], \ldots, [C_n] \in \text{NS}(X) \text{ span } \text{NS}(X) \otimes \mathbb{Q} \text{ over } \mathbb{Q}.$$ 

We consider the open surface

$$S := X \setminus (C_1 \cup \cdots \cup C_n).$$

By definition, we have $T(S) = T(X)$.

Consider the intersection pairing

$$\iota : H_2(S) \times H_2(S) \to \mathbb{Z}.$$ 

We put

$$H_2(S) \perp := \{ x \in H_2(S) \mid \iota(x, y) = 0 \text{ for all } y \in H_2(S) \}.$$ 

Then $H_2(S)/H_2(S) \perp$ becomes a lattice.
Proposition

The lattice $T(S) = T(X)$ is isomorphic to $H_2(S)/H_2(S)^\perp$.

Proof.

We put $C := C_1 \cup \cdots \cup C_n$. Consider the diagram:

$$
\begin{array}{ccc}
H_2(S) & \xrightarrow{j_*} & H_2(X) \\
\downarrow \lrcorner & & \downarrow \lrcorner \\
H^2(X, C) & \rightarrow & H^2(X) \xrightarrow{rc} H^2(C) = \bigoplus H^2(C_i),
\end{array}
$$

where $j : S \hookrightarrow X$ is the inclusion. From this, we see that $\text{Im } j_* = T(X)$. Since $T(X)$ is non-degenerate, we have $\text{Ker } j_* = H_2(S)^\perp$. \qed
Let $\sigma$ be an element of $\text{Aut}(\mathbb{C})$. Then

$$[C_1^\sigma], \ldots, [C_n^\sigma] \in \text{NS}(X^\sigma) \text{ span } \text{NS}(X^\sigma) \otimes \mathbb{Q} \text{ over } \mathbb{Q},$$

because the intersection pairing on $\text{NS}(X)$ is defined algebraically.

Since the lattice $H_2(S)/H_2(S)^\perp$ is defined topologically, we obtain the following:

**Corollary**

If $S^\sigma$ and $S$ are homeomorphic, then $T(S^\sigma) = T(X^\sigma)$ is isomorphic to $T(S) = T(X)$.

**Corollary**

If $T(X)$ and $T(X^\sigma)$ are not isomorphic, then there exists a Zariski open subset $S \subset X$ such that $S$ and $S^\sigma$ are not homeomorphic.
Let $L$ be a lattice. The $\mathbb{Z}$-valued symmetric bilinear form on $L$ extends to

$$L^\vee \times L^\vee \to \mathbb{Q}.$$ 

Hence, on the discriminant group $D_L := L^\vee / L$ of $L$, we have a quadratic form

$$q_L : D_L \to \mathbb{Q}/\mathbb{Z}, \quad \bar{x} \mapsto x^2 \mod \mathbb{Z},$$

which is called the discriminant form of $L$.

A lattice $L$ is said to be even if $x^2 \in 2\mathbb{Z}$ for any $x \in L$. If $L$ is even, then $q_L : D_L \to \mathbb{Q}/\mathbb{Z}$ is refined to

$$q_L : D_L \to \mathbb{Q}/2\mathbb{Z}.$$
Since $H^2(X)$ is unimodular and both of $T(X)$ and $\text{NS}(X)$ are primitive in $H^2(X)$, we have the following:

**Proposition**

$$(D_T(X), q_T(X)) \cong (D_{\text{NS}}(X), -q_{\text{NS}}(X)).$$

Since the Néron-Severi lattice is defined algebraically, we obtain the following:

**Corollary**

For any $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$(D_T(X), q_T(X)) \cong (D_T(X^\sigma), q_T(X^\sigma)).$$
Recall that:

Our aim is to construct conjugate open surfaces $S$ and $S^\sigma$ that are not homeomorphic.

For this, it is enough to construct conjugate smooth projective surfaces $X$ and $X^\sigma$ with non-isomorphic transcendental lattices $T(X) \not\cong T(X^\sigma)$.

But $T(X)$ and $T(X^\sigma)$ have isomorphic discriminant forms.

**Problem**

*To what extent does the discriminant form determine the lattice?*
Proposition

Let $L$ and $L'$ be even lattices of the same rank. If $L$ and $L'$ have isomorphic discriminant forms and the same signature, then $L$ and $L'$ belong to the same genus.

Theorem (Eichler)

Suppose that $L$ and $L'$ are indefinite. If $L$ and $L'$ belong to the same spinor-genus, then $L$ and $L'$ are isomorphic.

The difference between genus and spinor-genus is not big. Hence we need to search for $X$ such that $T(X)$ is definite.
**Definition (Katsura)**

Let $S$ be a surface with a smooth projective birational model $X$.

S is fully-rigged

$\iff \text{rank}(\text{NS}(X)) = h^{1,1}(X)$

$\iff \text{rank}(T(S)) = 2p_g(X)$

$\iff T(S)$ is positive-definite.

**Remark**

For abelian surfaces or $K3$ surfaces, fully-rigged surfaces are called “singular”. 
<table>
<thead>
<tr>
<th>Definition (Persson)</th>
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A reduced (possibly reducible) projective plane curve $B \subset \mathbb{P}^2$ of even degree $2m$ is *maximizing* if the following hold:

- $B$ has only simple singularities ($ADE$-singularities), and
- the total Milnor number of $B$ is $3m^2 - 3m + 1$.

Equivalently, $B \subset \mathbb{P}^2$ is maximizing if and only if

- the double cover $Y_B \rightarrow \mathbb{P}^2$ of $\mathbb{P}^2$ branching along $B$ has only RDPs, and
- for the minimal resolution $X_B$ of $Y_B$, the classes of the exceptional divisors span a sublattice of $\text{NS}(X_B)$ with rank $h^{1,1}(X_B) - 1$.

In particular, $X_B$ is fully-rigged.
Persson (1982) found many examples of maximizing curves.

**Example**

The projective plane curve

\[ B : xy(x^n + y^n + z^n)^2 - 4xy((xy)^n + (yz)^n + (zx)^n) = 0 \]

has singular points of type

\[ 2n \times D_{n+2} + n \times A_{n-1} + A_1. \]

It is maximizing.
If $B \subset \mathbb{P}^2$ is of degree 6 and has only simple singularities, then the minimal resolution $X_B$ of the double cover of $\mathbb{P}^2$ branching along $B$ is a $K3$ surface.

In the paper

Yang, Jin-Gen
Sextic curves with simple singularities

Yang classified all sextic curves with only simple singularities by means of Torelli theorem for complex $K3$ surfaces.

His method also gives the transcendental lattices of the fully-rigged $K3$ surfaces $X_B$ obtained as the double plane sextics.
(We use the terminology “fully-rigged $K3$ surfaces” rather than the traditional “singular $K3$ surfaces”.)

Let $X$ be a fully-rigged $K3$ surface; that is, $X$ is a $K3$ surface with the Picard number 20. Then the transcendental lattice $T(X)$ is a positive-definite even lattice of rank 2.

The Hodge decomposition

$$T(X) \otimes \mathbb{C} = H^{2,0}(X) \oplus H^{0,2}(X)$$

induces an orientation on $T(X)$. We denote by

$$\tilde{T}(X)$$

the oriented transcendental lattice of $X$.

By Torelli theorem, we have

$$\tilde{T}(X) \cong \tilde{T}(X') \implies X \cong X'.$$
Construction by Shioda and Inose.

Every fully-rigged $K3$ surface $X$ is obtained as a certain double cover of the Kummer surface

$$\text{Km}(E \times E'),$$

where $E$ and $E'$ are elliptic curves with $CM$ by some orders of

$$\mathbb{Q}(\sqrt{-|\text{disc } T(X)|}).$$

Theorem (Shioda and Inose)

(1) For any positive-definite oriented even lattice $\tilde{T}$ of rank 2, there exists a fully-rigged $K3$ surface $X$ such that $\tilde{T}(X) \cong \tilde{T}$.

(2) Every fully-rigged $K3$ surface is defined over a number field.
The class field theory of imaginary quadratic fields tells us how the Galois group acts on the $j$-invariants of elliptic curves with CM. Using this, S.- and Schütt (2007) proved the following:

**Theorem**

Let $X$ and $X'$ be fully-rigged $K3$ surfaces defined over $\overline{\mathbb{Q}}$. If

$$(D_{T(X)}, q_{T(X)}) \cong (D_{T(X')}, q_{T(X')})$$

(that is, if $T(X)$ and $T(X')$ are in the same genus), then $X$ and $X'$ are conjugate.

Therefore, if the genus contains more than one isomorphism class, then we can construct non-homeomorphic conjugate surfaces as Zariski open subsets of fully-rigged $K3$ surfaces.
From Yang’s table, we know that there exists a sextic curve

\[ B = L + Q, \]

where \( Q \) is a quintic curve with one \( A_{10} \)-singular point, and \( L \) is a line intersecting \( Q \) at only one smooth point of \( Q \). Hence \( B \) has \( A_9 + A_{10} \).

The Néron-Severi lattice \( \text{NS}(X_B) \) is an overlattice of

\[ RA_{9}+A_{10} \oplus \langle h \rangle \]

with index 2,

where \( RA_{9}+A_{10} \) is the negative-definite root lattice of type \( A_9 + A_{10} \), and \( h \) is the vector \([\mathcal{O}_{\mathbb{P}^2}(1)]\) with \( h^2 = 2 \).

(The extension comes from the fact that \( B \) is reducible.)
The genus of even positive-definite lattices of rank 2 corresponding to the discriminant form

$$(D_{NS}(x_B), -q_{NS}(x_B)) \cong (\mathbb{Z}/55\mathbb{Z}, \lfloor 2/55 \rfloor \mod 2)$$

consists of two isomorphism classes:

$$\begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}, \quad \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}. \quad \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}, \quad \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}. \quad$$
On the other hand, the maximizing sextic $B$ is defined by the normal form

$$B_{\pm} : z \cdot (G(x, y, z) \pm \sqrt{5}H(x, y, z)) = 0,$$

where

$$G = -9 x^4 z - 14 x^3 yz + 58 x^3 z^2 - 48 x^2 y^2 z - 64 x^2 yz^2 + 10 x^2 z^3 + 108 xy^3 z - 20 xy^2 z^2 - 44 y^5 + 10 y^4 z,$$

$$H = 5 x^4 z + 10 x^3 yz - 30 x^3 z^2 + 30 x^2 y^2 z + 20 x^2 yz^2 - 40 xy^3 z + 20 y^5.$$
Hence the étale double covers $S_{\pm}$ of the complements $\mathbb{P}^2 \setminus B_{\pm}$ are conjugate but non-homeomorphic. Indeed, we have

$$T(S_{\pm}) \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix} \quad \text{and} \quad T(S_{-}) \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}. $$

**Remark**

There is another possibility

$$T(S_{\pm}) \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix} \quad \text{and} \quad T(S_{-}) \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}. $$

The verification of the fact that the first one is the case needs a careful topological calculation.
Thank you!