# Transcendental lattices of complex algebraic surfaces

Ichiro Shimada

Hiroshima University

November 25, 2009, Tohoku

Let $\mathrm{Aut}(\mathbb{C})$ be the automorphism group of the complex number field $\mathbb{C}$.

For a scheme $V \to \operatorname{Spec} \mathbb{C}$ and an element $\sigma \in \mathrm{Aut}(\mathbb{C})$, we define a scheme $V^\sigma \to \operatorname{Spec} \mathbb{C}$ by the following Cartesian diagram:

$$\begin{array}{ccc} V^\sigma & \longrightarrow & V \\ \downarrow & \square & \downarrow \\ \operatorname{Spec} \mathbb{C} & \underset{\sigma^*}{\longrightarrow} & \operatorname{Spec} \mathbb{C}. \end{array}$$

Two schemes $V$ and $V'$ over $\mathbb{C}$ are said to be *conjugate* if $V'$ is isomorphic to $V^\sigma$ over $\mathbb{C}$ for some $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Conjugate complex varieties can never be distinguished by any algebraic methods (they are isomorphic over $\mathbb{Q}$), but they can be non-homeomorphic in the classical complex topology.

The first example was given by Serre in 1964.

Other examples have been constructed by:

Abelson (1974),
Grothendieck's dessins d'enfants (1984),
Artal Bartolo, Carmona Ruber, and Cogolludo Agust (2004),
Easton and Vakil (2007), F. Charles (2009).

We will construct such examples by means of *transcendental lattices* of complex algebraic surfaces.

### Example (S.- and Arima)

Consider two smooth irreducible surfaces $S_{\pm}$ in $\mathbb{C}^3$ defined by

$$w^2(G(x,y) \pm \sqrt{5} \cdot H(x,y)) = 1, \quad \text{where}$$

$$\begin{aligned} G(x,y) &:= -9\,x^4 - 14\,x^3y + 58\,x^3 - 48\,x^2y^2 - 64\,x^2y \\ &\quad +10\,x^2 + 108\,xy^3 - 20\,xy^2 - 44\,y^5 + 10\,y^4, \\ H(x,y) &:= 5\,x^4 + 10\,x^3y - 30\,x^3 + 30\,x^2y^2 + \\ &\quad +20\,x^2y - 40\,xy^3 + 20\,y^5. \end{aligned}$$

Then $S_+$ and $S_-$ are not homeomorphic.

- § Definition of the transcendental lattice
- § The transcendental lattice is topological
- § The discriminant form is algebraic
- § Fully-rigged surfaces
- § Maximizing curves
- § Arithmetic of fully-rigged $K3$ surfaces
- § Construction of the explicit example

By a *lattice*, we mean a free $\mathbb{Z}$-module $L$ of finite rank with a non-degenerate symmetric bilinear form

$$L \times L \to \mathbb{Z}.$$

A lattice $L$ is naturally embedded into the dual lattice

$$L^\vee := \mathrm{Hom}(L, \mathbb{Z}).$$

The *discriminant group* of $L$ is the finite abelian group

$$D_L := L^\vee / L.$$

A lattice $L$ is called *unimodular* if $D_L$ is trivial.

Let $X$ be a smooth projective surface over $\mathbb{C}$. Then

$$H^2(X) := H^2(X, \mathbb{Z})/(\text{the torsion})$$

is regarded as a unimodular lattice by the cup-product.
The *Néron-Severi lattice*

$$\mathrm{NS}(X) := H^2(X) \cap H^{1,1}(X)$$

of classes of algebraic curves on $X$ is a sublattice of signature

$$\mathrm{sgn}(\mathrm{NS}(X)) = (1, \rho - 1).$$

The *transcendental lattice* of $X$ is defined to be the orthogonal complement of $\mathrm{NS}(X)$ in $H^2(X)$:

$$T(X) := \mathrm{NS}(X)^{\perp}.$$

## Proposition (Shioda)

$T(X)$ is a birational invariant of algebraic surfaces.

## Proof.

Suppose that $X$ and $X'$ are birational. There exists a smooth projective surface $X''$ with birational morphisms

$$X'' \to X \quad \text{and} \quad X'' \to X'.$$

Every birational morphism between smooth projective surfaces is a composite of blowing-ups at points.
A blowing-up at a point does not change the transcendental lattice. $\square$

Hence, for a surface $S$ (possibly singular and possibly open), the transcendental lattice $T(S)$ is well-defined.

Let $X$ be a smooth projective surface over $\mathbb{C}$,
and let $C_1, \dots, C_n \subset X$ be irreducible curves. Suppose that

$$[C_1], \dots, [C_n] \in \mathrm{NS}(X) \text{ span } \mathrm{NS}(X) \otimes \mathbb{Q} \text{ over } \mathbb{Q}.$$

We consider the open surface

$$S := X \setminus (C_1 \cup \dots \cup C_n).$$

By definition, we have $T(S) = T(X)$.

Consider the intersection pairing

$$\iota \;\;:\;\; H_2(S) \times H_2(S) \to \mathbb{Z}.$$

We put

$$H_2(S)^\perp := \{\, x \in H_2(S) \;\mid\; \iota(x, y) = 0 \;\text{ for all }\; y \in H_2(S) \,\}.$$

Then $H_2(S)/H_2(S)^\perp$ becomes a lattice.

## Proposition

The lattice $T(S) = T(X)$ is isomorphic to $H_2(S)/H_2(S)^{\perp}$.

## Proof.

We put $C := C_1 \cup \cdots \cup C_n$. Consider the diagram:

$$
\begin{array}{ccccc}
H_2(S) & \xrightarrow{j_*} & H_2(X) & & \\
\downarrow \wr & & \downarrow \wr & & \\
H^2(X, C) & \longrightarrow & H^2(X) & \xrightarrow{r_C} & H^2(C) = \bigoplus H^2(C_i),
\end{array}
$$

where $j : S \hookrightarrow X$ is the inclusion. From this, we see that $\mathrm{Im}\, j_* = T(X)$. Since $T(X)$ is non-degenerate, we have $\mathrm{Ker}\, j_* = H_2(S)^{\perp}$. $\square$

Let $\sigma$ be an element of $\mathrm{Aut}(\mathbb{C})$. Then

$$[C_1^\sigma], \dots, [C_n^\sigma] \in \mathrm{NS}(X^\sigma) \text{ span } \mathrm{NS}(X^\sigma) \otimes \mathbb{Q} \text{ over } \mathbb{Q},$$

because the intersection pairing on $\mathrm{NS}(X)$ is defined algebraically.

Since the lattice $H_2(S)/H_2(S)^\perp$ is defined topologically, we obtain the following:

**Corollary**

If $S^\sigma$ and $S$ are homeomorphic, then $T(S^\sigma) = T(X^\sigma)$ is isomorphic to $T(S) = T(X)$.

**Corollary**

If $T(X)$ and $T(X^\sigma)$ are not isomorphic, then there exists a Zariski open subset $S \subset X$ such that $S$ and $S^\sigma$ are not homeomorphic.

Let $L$ be a lattice. The $\mathbb{Z}$-valued symmetric bilinear form on $L$ extends to

$$L^\vee \times L^\vee \to \mathbb{Q}.$$

Hence, on the discriminant group $D_L := L^\vee/L$ of $L$, we have a quadratic form

$$q_L : D_L \to \mathbb{Q}/\mathbb{Z}, \;\; \bar{x} \mapsto x^2 \bmod \mathbb{Z},$$

which is called the *discriminant form* of $L$.

A lattice $L$ is said to be *even* if $x^2 \in 2\mathbb{Z}$ for any $x \in L$.
If $L$ is even, then $q_L : D_L \to \mathbb{Q}/\mathbb{Z}$ is refined to

$$q_L : D_L \to \mathbb{Q}/2\mathbb{Z}.$$

Since $H^2(X)$ is unimodular and both of $T(X)$ and $\mathrm{NS}(X)$ are primitive in $H^2(X)$, we have the following:

**Proposition**

$$(D_{T(X)}, q_{T(X)}) \cong (D_{\mathrm{NS}(X)}, -q_{\mathrm{NS}(X)}).$$

Since the Néron-Severi lattice is defined algebraically, we obtain the following:

**Corollary**

For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$(D_{T(X)}, q_{T(X)}) \cong (D_{T(X^\sigma)}, q_{T(X^\sigma)}).$$

Recall that:

Our aim is to construct conjugate open surfaces $S$ and $S^\sigma$ that are not homeomorphic.

For this, it is enough to construct conjugate smooth projective surfaces $X$ and $X^\sigma$ with non-isomorphic transcendental lattices $T(X) \not\cong T(X^\sigma)$.

But $T(X)$ and $T(X^\sigma)$ have isomorphic discriminant forms.

**Problem**

*To what extent does the discriminant form determine the lattice?*

**Proposition**

Let $L$ and $L'$ be even lattices of the same rank.
If $L$ and $L'$ have isomorphic discriminant forms and the same signature, then $L$ and $L'$ belong to the same genus.

**Theorem (Eichler)**

Suppose that $L$ and $L'$ are *indefinite*.
If $L$ and $L'$ belong to the same spinor-genus, then $L$ and $L'$ are isomorphic.

The difference between genus and spinor-genus is not big.

Hence we need to search for $X$ such that $T(X)$ is *definite*.

## Definition (Katsura)

Let $S$ be a surface with a smooth projective birational model $X$.

$$
\begin{aligned}
& S \text{ is \textit{fully-rigged}} \\
\iff \quad & \mathrm{rank}(\mathrm{NS}(X)) = h^{1,1}(X) \\
\iff \quad & \mathrm{rank}(T(S)) = 2p_g(X) \\
\iff \quad & T(S) \text{ is positive-definite.}
\end{aligned}
$$

## Remark

For abelian surfaces or $K3$ surfaces, fully-rigged surfaces are called "singular".

### Definition (Persson)

A reduced (possibly reducible) projective plane curve $B \subset \mathbb{P}^2$ of even degree $2m$ is *maximizing* if the following hold:

- $B$ has only simple singularities ($ADE$-singularities), and
- the total Milnor number of $B$ is $3m^2 - 3m + 1$.

Equivalently, $B \subset \mathbb{P}^2$ is maximizing if and only if

- the double cover $Y_B \to \mathbb{P}^2$ of $\mathbb{P}^2$ branching along $B$ has only RDPs, and
- for the minimal resolution $X_B$ of $Y_B$, the classes of the exceptional divisors span a sublattice of $\mathrm{NS}(X_B)$ with rank $h^{1,1}(X_B) - 1$.

In particular, $X_B$ is fully-rigged.

Persson (1982) found many examples of maximizing curves.

**Example**

The projective plane curve

$$B : xy(x^n + y^n + z^n)^2 - 4xy((xy)^n + (yz)^n + (zx)^n) = 0$$

has singular points of type

$$2n \times D_{n+2} \ + \ n \times A_{n-1} \ + \ A_1.$$

It is maximizing.

If $B \subset \mathbb{P}^2$ is of degree 6 and has only simple singularities, then the minimal resolution $X_B$ of the double cover of $\mathbb{P}^2$ branching along $B$ is a $K3$ surface.

In the paper

> Yang, Jin-Gen
> Sextic curves with simple singularities
> Tohoku Math. J. (2) 48 (1996), no. 2, 203–227,

Yang classified all sextic curves with only simple singularities by means of Torelli theorem for complex $K3$ surfaces.

His method also gives the transcendental lattices of the fully-rigged $K3$ surfaces $X_B$ obtained as the double plane sextics.

(We use the terminology “fully-rigged $K3$ surfaces” rather than the traditional “singular $K3$ surfaces”.)

Let $X$ be a fully-rigged $K3$ surface; that is, $X$ is a $K3$ surface with the Picard number 20. Then the transcendental lattice $T(X)$ is a positive-definite even lattice of rank 2.
The Hodge decomposition

$$T(X) \otimes \mathbb{C} = H^{2,0}(X) \oplus H^{0,2}(X)$$

induces an orientation on $T(X)$. We denote by

$$\tilde{T}(X)$$

the oriented transcendental lattice of $X$.
By Torelli theorem, we have

$$\tilde{T}(X) \cong \tilde{T}(X') \implies X \cong X'.$$

## Construction by Shioda and Inose.

Every fully-rigged $K3$ surface $X$ is obtained as a certain double cover of the Kummer surface

$$\mathrm{Km}(E \times E'),$$

where $E$ and $E'$ are elliptic curves with $CM$ by some orders of

$$\mathbb{Q}(\sqrt{-|\operatorname{disc} T(X)|}).$$

**Theorem (Shioda and Inose)**

(1) For any positive-definite oriented even lattice $\tilde{T}$ of rank 2, there exists a fully-rigged $K3$ surface $X$ such that $\tilde{T}(X) \cong \tilde{T}$.
(2) Every fully-rigged $K3$ surface is defined over a number field.

The class field theory of imaginary quadratic fields tells us how the Galois group acts on the $j$-invariants of elliptic curves with CM. Using this, S.- and Schütt (2007) proved the following:

**Theorem**

Let $X$ and $X'$ be fully-rigged $K3$ surfaces defined over $\overline{\mathbb{Q}}$.
If

$$(D_{T(X)}, q_{T(X)}) \cong (D_{T(X')}, q_{T(X')})$$

(that is, if $T(X)$ and $T(X')$ are in the same genus),
then $X$ and $X'$ are conjugate.

Therefore, if the genus contains more than one isomorphism class, then we can construct non-homeomorphic conjugate surfaces as Zariski open subsets of fully-rigged $K3$ surfaces.

From Yang's table, we know that there exists a sextic curve

$$B = L + Q,$$

where $Q$ is a quintic curve with one $A_{10}$-singular point,
and $L$ is a line intersecting $Q$ at only one smooth point of $Q$.
Hence $B$ has $A_9 + A_{10}$.

The Néron-Severi lattice $\mathrm{NS}(X_B)$ is an overlattice of

$$R_{A_9+A_{10}} \oplus \langle h \rangle$$

with index 2,
where $R_{A_9+A_{10}}$ is the negative-definite root lattice of type $A_9 + A_{10}$,
and $h$ is the vector $[\mathcal{O}_{\mathbb{P}^2}(1)]$ with $h^2 = 2$.
(The extension comes from the fact that $B$ is reducible.)

The genus of even positive-definite lattices of rank 2 corresponding to the discriminant form

$$(D_{\mathrm{NS}(X_B)}, -q_{\mathrm{NS}(X_B)}) \;\cong\; (\mathbb{Z}/55\mathbb{Z}, [2/55] \bmod 2)$$

consists of two isomorphism classes:

$$\begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}, \qquad \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}.$$

On the other hand, the maximizing sextic $B$ is defined by the normal form

$$B_{\pm} \quad : \quad z \cdot (G(x,y,z) \pm \sqrt{5} H(x,y,z)) = 0, \quad \text{where}$$

$$\begin{aligned} G &= -9\,x^4z - 14\,x^3yz + 58\,x^3z^2 - 48\,x^2y^2z - 64\,x^2yz^2 + \\ &\quad 10\,x^2z^3 + 108\,xy^3z - 20\,xy^2z^2 - 44\,y^5 + 10\,y^4z, \\ H &= 5\,x^4z + 10\,x^3yz - 30\,x^3z^2 + 30\,x^2y^2z + \\ &\quad 20\,x^2yz^2 - 40\,xy^3z + 20\,y^5. \end{aligned}$$

Hence the étale double covers $S_{\pm}$ of the complements $\mathbb{P}^2 \setminus B_{\pm}$ are conjugate but non-homeomorphic. Indeed, we have

$$T(S_+) \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix} \quad \text{and} \quad T(S_-) \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}.$$

**Remark**

There is another possibility

$$T(S_+) \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix} \quad \text{and} \quad T(S_-) \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}.$$

The verification of the fact that the first one is the case needs a careful topological calculation.

# Thank you!