On supersingular varieties

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24 September, 2010, Nagoya
Let $X$ be a smooth projective variety over $\mathbb{F}_q$. The following are equivalent:

(i) There is a polynomial $N(t) \in \mathbb{Z}[t]$ such that

$$|X(\mathbb{F}_{q^\nu})| = N(q^\nu)$$

for all $\nu \in \mathbb{Z}_{>0}$.

(ii) The eigenvalues of the $q$ th power Frobenius on the $l$-adic cohomology ring are powers of $q$ by integers.

If these are satisfied, then $b_{2i-1}(X) = 0$ and

$$N(t) = \sum_{i=0}^{\dim X} b_{2i}(X) t^i.$$ 

We say that $X$ is Frobenius supersingular if (i) and (ii) are satisfied.
If the cohomology ring of $X$ is generated by the classes of algebraic cycles over $\mathbb{F}_q$, then $X$ is Frobenius supersingular.

The converse is true if the Tate conjecture is assumed.

We have examples of Frobenius supersingular varieties of non-negative Kodaira dimension.

**Theorem**

The Fermat variety

$$X := \{x_0^{q+1} + \cdots + x_{2m+1}^{q+1} = 0\} \subset \mathbb{P}^{2m+1}$$

of dimension $2m$ and degree $q + 1$ regarded as a variety over $\mathbb{F}_{q^2}$ is Frobenius supersingular.

This follows from

$$|X(\mathbb{F}_{q^2})| = 1 + q^2 + \cdots + q^{4m} + (b_{2m}(X) - 1)q^{2m}.$$
Problems on Frobenius supersingular varieties

- Construct non-trivial examples.
- Prove (or disprove) the unirationality.
- Present explicitly algebraic cycles that generate the cohomology ring.
- Investigate the lattice given by the intersection pairing of algebraic cycles.
- Produce dense lattices by the intersection pairing in small characteristics.

We discuss these problems for the classical example of Fermat varieties of degree $q + 1$, and for the new example of Frobenius incidence varieties.
Unirationality and Supersingularity

A variety $X$ is called *(purely-inseparably) unirational* if there is a dominant (purely-inseparable) rational map

$$\mathbb{P}^n \dashrightarrow X.$$ 

**Theorem (Shioda)**

Let $S$ be a smooth projective surface defined over $k = \bar{k}$. If $S$ is unirational, then the Picard number $\rho(S)$ is equal to $b_2(S)$; that is, $S$ is *supersingular in the sense of Shioda*.

The converse is conjectured to be true for $K3$ surfaces.
Artin-Shioda conjecture

Every supersingular $K3$ surface $S$ (in the sense of Shioda) is conjectured to be (purely-inseparably) unirational.

The discriminant of the Néron-Severi lattice $\text{NS}(S)$ is $-p^{2\sigma(S)}$, where $\sigma(S)$ is a positive integer $\leq 10$, which is called the Artin invariant of $S$.

The conjecture is confirmed to be true in the following cases:

- $p$ odd and $\sigma(S) \leq 2$ (Ogus and Shioda):
- $p = 2$ (Rudakov and Shafarevich, S.-):
- $p = 3$ and $\sigma(S) \leq 6$ (Rudakov and Shafarevich, S.- and De Qi Zhang):
- $p = 5$ and $\sigma(S) \leq 3$ (S.- and Pho Duc Tai).

Method: The structure theorem for $\text{NS}(S)$ by Rudakov-Shafarevich.
Fermat variety of degree $q + 1$

Unirationality of the Fermat variety

Theorem (Shioda-Katsura, S.-)

The Fermat variety $X$ of degree $q + 1$ and dimension $n \geq 2$ in characteristic $p > 0$ is purely-inseparably unirational, where $q = p^\nu$.

Indeed, $X$ contains a linear subspace $\Lambda \subset \mathbb{P}^{n+1}$ of dimension $\lfloor n/2 \rfloor$. The unirationality is proved by the projection from the center $\Lambda$. 
Lattice

By a *quasi-lattice*, we mean a free $\mathbb{Z}$-module $L$ of finite rank with a symmetric bilinear form

$$( \ , \ ) : L \times L \to \mathbb{Z}.$$ 

If the symmetric bilinear form is non-degenerate, we say that $L$ is a *lattice*.

If $L$ is a quasi-lattice, then $L/L^\perp$ is a lattice, where

$$L^\perp := \{ x \in L \mid (x, y) = 0 \text{ for all } y \in L \}.$$
Lattices associated with the Fermat varieties

The Fermat variety

\[ X := \{ x_0^{q+1} + \cdots + x_{2m+1}^{q+1} = 0 \} \subset \mathbb{P}^{2m+1} \]

of dimension \( 2m \) and degree \( q + 1 \) contains many \( m \)-dimensional linear subspaces \( \Lambda_i \). The number is

\[
\prod_{\nu=0}^{m} (q^{2\nu+1} + 1).
\]

Each of them is defined over \( \mathbb{F}_{q^2} \).

Let \( \tilde{N}(X) \subset A^m(X) \) be the \( \mathbb{Z} \)-module generated by the rational equivalence classes of \( \Lambda_i \), where \( A(X) \) is the Chow ring.

By the intersection pairing

\[
\tilde{N}(X) \times \tilde{N}(X) \to \mathbb{Z},
\]

we can consider \( \tilde{N}(X) \) as a quasi-lattice.
Let $\mathcal{N}(X) := \tilde{\mathcal{N}}(X)/\tilde{\mathcal{N}}(X)\perp$ be the associated lattice.

**Theorem (Tate, S.-)**

1. The rank of $\mathcal{N}(X)$ is equal to $b_{2m}(X)$.
2. The discriminant of $\mathcal{N}(X)$ is a power of $p$.

**Corollary**

The cycle map induces an isomorphism $\mathcal{N}(X) \otimes \mathbb{Q}_l \cong H^{2m}(X, \mathbb{Q}_l)$.

The assertion (2) is an analogue of the result that the discriminant of the Néron-Severi lattice $\mathcal{NS}(S)$ of a supersingular $K3$ surface $S$ is a power of $p$. 
Let \( h \in \mathcal{N}(X) \) be the numerical equivalence class of a linear plane section \( X \cap \mathbb{P}^{m+1} \).

We put

\[
\mathcal{N}_{\text{prim}}(X) := \{ x \in \mathcal{N}(X) \mid (x, h) = 0 \} = \langle h \rangle^\perp.
\]

**Theorem**

The lattice \([-1]^m \mathcal{N}_{\text{prim}}(X)\) is positive-definite.

Here \([-1]^m \mathcal{N}_{\text{prim}}(X)\) is the lattice obtained from \( \mathcal{N}_{\text{prim}}(X) \) by changing the sign with \((-1)^m\).
Dense lattices

Let $L$ be a positive-definite lattice of rank $m$. The *minimal norm* of $L$ is defined by

$$N_{\min}(L) := \min\{x^2 \mid x \in L, x \neq 0\},$$

and the *normalized center density* of $L$ is defined by

$$\delta(L) := (\text{disc } L)^{-1/2} \cdot (N_{\min}(L)/4)^{m/2}. $$

Minkowski and Hlawka proved in a non-constructive way that, for each $m$, there is a positive-definite lattice $L$ of rank $m$ with

$$\delta(L) > \text{MH}(m) := \frac{\zeta(m)}{2^{m-1}V_m},$$

where $V_m$ is the volume of the $m$-dimensional unit ball.
We say that a positive-definite lattice $L$ of rank $m$ is dense if

$$\delta(L) > \text{MH}(m).$$

The intersection pairing of algebraic cycles in positive characteristic has been used to construct dense lattices.

For example, Elkies and Shioda constructed many dense lattices as Mordell-Weil lattices of elliptic surfaces in positive characteristics.
Dense lattices arising from Fermat varieties

Let $X$ be the Fermat cubic variety of dimension $2m$ in characteristic 2. Recall that $X$ contains many $m$-dimensional linear subspaces $\Lambda_i$.

We consider the positive-definite lattice

$$\langle [\Lambda_i] - [\Lambda_j] \rangle \subset [-1]^m N_{\text{prim}}(X)$$

generated by the classes $[\Lambda_i] - [\Lambda_j]$. Their properties are as follows:

<table>
<thead>
<tr>
<th>dim $X$</th>
<th>rank</th>
<th>$N_{\text{min}}$</th>
<th>$\log_2 \delta$</th>
<th>$\log_2$ MH</th>
<th>name</th>
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<td>2</td>
<td>6</td>
<td>2</td>
<td>$-3.792...$</td>
<td>$-7.344...$</td>
<td>$E_6$</td>
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<tr>
<td>4</td>
<td>22</td>
<td>4</td>
<td>$-1.792...$</td>
<td>$-13.915...$</td>
<td>$\Lambda_{22}$</td>
</tr>
<tr>
<td>6</td>
<td>86</td>
<td>8</td>
<td>$34.207...$</td>
<td>$19.320...$</td>
<td>$N_{86}$</td>
</tr>
</tbody>
</table>
Frobenius incidence variety

We fix an $n$-dimensional linear space $V$ over $\mathbb{F}_p$ with $n \geq 3$.

We denote by $G_{n,l} = G_{n-l}^n$ the Grassmannian variety of $l$-dimensional subspaces of $V$.

Let $F$ be a field of characteristic $p$, and consider an $F$-rational linear subspace $L \in G_{n,l}(F)$ of $V$.

Let $\phi$ be the $p$th power Frobenius morphism of $G_{n,l}$. For a positive integer $\nu$, we put

$$L^{(p^\nu)} := \phi^\nu(L).$$
Let $l$ and $c$ be positive integers such that $l + c < n$.

We denote by $\mathcal{I}_{n,l}^c$ the incidence subvariety of $G_{n,l} \times G_n^c$:

$$\mathcal{I}_{n,l}^c(F) = \{ (L, M) \in G_{n,l}(F) \times G_n^c(F) \mid L \subset M \}.$$ 

Let $r := p^a$ and $s := p^b$ be powers of $p$ by positive integers. We define the **Frobenius incidence variety** $X_{n,l}^c$ by

$$X_{n,l}^c := (\phi^a \times \text{id})^* \mathcal{I}_{n,l}^c \cap (\text{id} \times \phi^b)^* \mathcal{I}_{n,l}^c.$$ 

Then $X_{n,l}^c$ is defined over $\mathbb{F}_p$, and we have

$$X_{n,l}^c(F) = \{ (L, M) \in G_{n,l}(F) \times G_n^c(F) \mid L^{(r)} \subset M \text{ and } L \subset M^{(s)} \}$$

$$= \{ (L, M) \in G_{n,l}(F) \times G_n^c(F) \mid L + L^{(rs)} \subset M^{(s)} \}$$

$$= \{ (L, M) \in G_{n,l}(F) \times G_n^c(F) \mid L^{(r)} \subset M \cap M^{(rs)} \}.$$
Theorem

(1) The scheme $X_{n,l}^c$ is smooth and geometrically irreducible of dimension $(n - l - c)(l + c)$.

(2) If $X_{n,l}^c$ is regarded as a scheme over $\mathbb{F}_{rs}$, then $X_{n,l}^c$ is Frobenius supersingular.

The smoothness of $X_{n,l}^c$ is proved by computing the dimension of Zariski tangent spaces.

We prove the second assertion by counting the number of $\mathbb{F}_{(rs)\nu}$-rational points of $X_{n,l}^c$.

We put

$$q := rs.$$
The main ingredient of the proof is the finite set

\[ T_{l,d}(q, q^\nu) := \{ L \in G_{n,l}(\mathbb{F}_{q^\nu}) \mid \dim(L \cap L^{(q)}) = d \}. \]

When \( l = d \), we have \( T_{l,l}(q, q^\nu) = G_{n,l}(\mathbb{F}_q) \) for any \( \nu \).

For \( d < l \), we calculate the cardinality of the set

\[ \mathcal{P} := \{ (L, M) \in G_{n,l}(\mathbb{F}_{q^\nu}) \times G_{n,2l-d}(\mathbb{F}_{q^\nu}) \mid L + L^{(q)} \subset M \} \]

\[ = \{ (L, M) \in G_{n,l}(\mathbb{F}_{q^\nu}) \times G_{n,2l-d}(\mathbb{F}_{q^\nu}) \mid L^{(q)} \subset M \cap M^{(q)} \}, \]

in two ways using the projections \( \mathcal{P} \to G_{n,l}(\mathbb{F}_{q^\nu}) \) and \( \mathcal{P} \to G_{n,2l-d}(\mathbb{F}_{q^\nu}) \).

Then we get

\[ |\mathcal{P}| = \sum_{t=d}^{l} |T_{l,t}(q, q^\nu)| \cdot |G_{n-2l+t,t-d}(\mathbb{F}_{q^\nu})| \]

\[ = \sum_{u=l}^{2l-d} |T_{2l-d,u}(q, q^\nu)| \cdot |G_{u,l}(\mathbb{F}_{q^\nu})|. \]
By this equality, we obtain a recursive formula for $|T_{l,d}(q, q^\nu)|$.

Using the projection $X_{n,l}^c(\mathbb{F}_{q^\nu}) \to G_{n,l}(\mathbb{F}_{q^\nu})$, we obtain the following:

$$|X_{n,l}^c(\mathbb{F}_{q^\nu})| = \sum_{d=0}^{l} |T_{l,d}(q, q^\nu)| \cdot |G_{n-2l+d}^c(\mathbb{F}_{q^\nu})|.$$ 

By the recursive formula for $|T_{l,d}(q, q^\nu)|$, we prove that there is a monic polynomial $N_{n,l}^c(t)$ of degree $(l + c)(n - l - c)$ such that

$$|X_{n,l}^c(\mathbb{F}_{q^\nu})| = N_{n,l}^c(q^\nu).$$

Therefore $X_{n,l}^c$ is Frobenius supersingular.

Since $N_{n,l}^c(t)$ is monic, $X_{n,l}^c$ is geometrically irreducible. Moreover we obtain the Betti numbers of $X_{n,l}^c$. 
Example

Let \((x_1 : \cdots : x_n)\) and \((y_1 : \cdots : y_n)\) be homogeneous coordinates of \(G_{n,1} = \mathbb{P}^*(V)\) and \(G_n^1 = \mathbb{P}^*(V)\) that are dual to each other. Then \(I_{n,1}^1 = \{\sum x_i y_i = 0\}\), and hence \(X_{n,1}^1\) is defined by

\[
\begin{align*}
  x_1^r y_1 + \cdots + x_n^r y_n &= 0, \\
  x_1 y_1^s + \cdots + x_n y_n^s &= 0.
\end{align*}
\]

The Betti numbers of \(X_{n,1}^1\) are as follows:

\[
b_{2i} = b_{2(n-2)-2i} = \begin{cases} 
  i + 1 & \text{if } i < n-2, \\
  n-2 + (q^n - 1)/(q-1) & \text{if } i = n-2.
\end{cases}
\]

When \(r = s = 2\) (and hence \(q = 4\), \(X_{3,1}^1\) is the supersingular \(K3\) surface with Artin invariant 1 (Mukai’s model).
Example

The Betti numbers of $X_{7,2}^2$ are calculated as follows:

\begin{align*}
    b_0 &= b_{24} : 1 \\
    b_2 &= b_{22} : 2 \\
    b_4 &= b_{20} : 5 \\
    b_6 &= b_{18} : q^6 + q^5 + q^4 + q^3 + q^2 + q + 8 \\
    b_8 &= b_{16} : 2(q^6 + q^5 + q^4 + q^3 + q^2 + q) + 12 \\
    b_{10} &= b_{14} : 3(q^6 + q^5 + q^4 + q^3 + q^2 + q) + 14 \\
    b_{12} : q^{10} + q^9 + 2q^8 + 2q^7 + 6q^6 + \\
          + 6q^5 + 6q^4 + 5q^3 + 5q^2 + 4q + 16.
\end{align*}
Unirationality of $X_{n,l}^c$

Theorem

The Frobenius incidence variety $X_{n,l}^c$ is purely-inseparably unirational.

Idea of the proof for the case $2l + c \leq n$.

We define $\tilde{X} \subset G_{n,l} \times G_n^c$ by

$$\tilde{X}(F) = \{ (L, M) \mid L \subset M, \quad L^{(rs)} \subset M \}.$$ 

The projection $\tilde{X} \to G_{n,l}$ is dominant. Using this projection, we can show that $\tilde{X}$ is rational. The map $(L, M) \mapsto (L, M^{(s)})$ is a dominant morphism from $\tilde{X}$ to $X_{n,l}^c$. 
Algebraic cycles on $X_{n,l}$

Let $\Lambda$ be an $\mathbb{F}_{rs}$-rational linear subspace of $V$ such that $l \leq \dim \Lambda \leq n - c$. We define $\Sigma_{\Lambda} \subset G_{n,l} \times G_n^c$ by

$$\Sigma_{\Lambda}(F) := \{ (L, M) \in G_{n,l}(F) \times G_n^c(F) \mid L \subset \Lambda \text{ and } \Lambda^{(r)} \subset M \}.$$ 

It follows from $\Lambda^{(rs)} = \Lambda$ that $\Sigma_{\Lambda}$ is contained in $X_{n,l}^c$.

When $l = c$, we have $2 \dim \Sigma_{\Lambda} = \dim X_{n,l}^l$.

We can calculate the intersection numbers of these $\Sigma_{\Lambda}$ on $X_{n,l}^l$. 

We consider the case where \( l = c = 1 \):

\[
X^{1}_{n,1} \subset \mathbb{P}^{*}(V) \times \mathbb{P}^{*}(V).
\]

We put

\[
\mathcal{H} := \text{Im}( A^{n-2}(\mathbb{P}^{*}(V) \times \mathbb{P}^{*}(V)) \to A^{n-2}(X^{1}_{n,1}) ).
\]

By the intersection pairing, we can consider the submodule

\[
\tilde{\mathcal{N}}(X^{1}_{n,1}) := \mathcal{H} + \langle [\Sigma_{\Lambda}] \rangle \subset A^{n-2}(X^{1}_{n,1})
\]

as a quasi-lattice. Let

\[
\mathcal{N}(X^{1}_{n,1}) := \tilde{\mathcal{N}}(X^{1}_{n,1})/\tilde{\mathcal{N}}(X^{1}_{n,1})^\perp
\]

be the associated lattice, and put

\[
\mathcal{N}_{\text{prim}}(X^{1}_{n,1}) := \mathcal{H}^\perp \subset \mathcal{N}(X^{1}_{n,1}).
\]
Theorem

(1) The rank of $\mathcal{N}(X_{n,1}^1)$ is $b_{2(n-2)}(X_{n,1}^1)$.
(2) The discriminant of $\mathcal{N}(X_{n,1}^1)$ is a power of $p$.
(3) The lattice $[-1]^n\mathcal{N}_{\text{prim}}(X_{n,1}^1)$ is positive-definite.

Corollary

The cohomology ring of $X_{n,1}^1$ is generated by the classes of $\Sigma_\Lambda$ and the image of $A(\mathbb{P}_*(V) \times \mathbb{P}^*(V)) \to A(X_{n,1}^1)$. 
Dense lattices of rank 84 and 85

Theorem

Suppose that \( p = r = s = 2 \). Then \( \mathcal{N}_{\text{prim}}(X_{4,1}^1) \) is an even positive-definite lattice of rank 84, with discriminant \( 85 \cdot 2^{16} \), and with minimal norm 8.

In fact, \( \mathcal{N}_{\text{prim}}(X_{4,1}^1) \) is a section of a larger lattice \( \mathcal{M}_C \) of rank

\[
85 = |\mathbb{P}^3(F_4)|
\]

collected by the projective geometry over \( F_4 \) and a code over

\[
R := \mathbb{Z}/8\mathbb{Z}.
\]

We put

\[
T := \mathbb{P}^3(F_4).
\]

For \( S \subset T \), we denote by \( \nu_S \in R^T \) and \( \tilde{\nu}_S \in \mathbb{Z}^T \) the characteristic functions of \( S \).
Let $C \subset R^T$ be the submodule generated by

$$2^{2-k} (v_P - v_{P'})$$

where $P$ and $P'$ are $\mathbb{F}_4$-rational linear subspaces of $\mathbb{P}^3$ of dimension $k$ ($k = 0, 1, 2$), and let $\mathcal{M}_C$ be the pull-back of $C$ by $\mathbb{Z}^T \to R^T$.

We define a $\mathbb{Q}$-valued symmetric bilinear form on $\mathbb{Z}^T$ by

$$(\tilde{v}_{\{t\}}, \tilde{v}_{\{t'\}}) = \delta_{tt'}/4 \quad (t, t' \in T).$$

Then $\mathcal{M}_C \subset \mathbb{Z}^T$ is a lattice.

<table>
<thead>
<tr>
<th>name</th>
<th>rank</th>
<th>disc</th>
<th>$N_{\text{min}}$</th>
<th>$\log_2 \delta$</th>
<th>$\log_2 \text{MH}$</th>
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<td>$\mathcal{N}<em>{\text{prim}}(X</em>{4,1}^1)$</td>
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<td>$85 \cdot 2^{16}$</td>
<td>8</td>
<td>30.795...</td>
<td>17.546...</td>
</tr>
<tr>
<td>$\mathcal{M}_C$</td>
<td>85</td>
<td>$2^{20}$</td>
<td>8</td>
<td>32.5</td>
<td>18.429...</td>
</tr>
<tr>
<td>$\mathcal{N}_{86}$</td>
<td>86</td>
<td>$3 \cdot 2^{16}$</td>
<td>8</td>
<td>34.207...</td>
<td>19.320...</td>
</tr>
</tbody>
</table>
Thank you!