An algorithm to compute automorphism groups of $K3$ surfaces

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Let $X$ be an algebraic $K3$ surface.
We denote by $S_X$ the Néron-Severi lattice of $X$.

Suppose that $X$ is defined over $\mathbb{C}$, or is supersingular in odd characteristic. Then, thanks to the Torelli-type theorem due to Piatetski-Shapiro and Shafarevich (1971) and Ogus (1978, 1983), we can study the automorphism group

$$\text{Aut}(X)$$

of $X$ by $S_X$.

We denote by $O(S_X)$ the orthogonal group of $S_X$. Then we have a natural homomorphism

$$\text{Aut}(X) \rightarrow O(S_X).$$

It is known that this homomorphism has only a finite kernel. We present an algorithm to give a finite set of generators of the image of this homomorphism.
We concentrate on **complex** $K3$ surfaces.

With the cup-product,

\[
H := H^2(X, \mathbb{Z})
\]

is an even unimodular lattice of signature $(3,19)$. Let $T_X$ is the orthogonal complement of $S_X = H \cap H^{1,1}$ in $H$. Let $\omega_X \in T_X \otimes \mathbb{C}$ be a non-zero holomorphic 2-form, and put

\[
C_X := \{ g \in O(T_X) \mid \omega_X^g = \lambda \omega_X \text{ for some } \lambda \in \mathbb{C}^\times \}.
\]

We denote by

\[
\text{Nef}(X) := \{ x \in S_X \otimes \mathbb{R} \mid \langle x, [C] \rangle \geq 0 \text{ for any curve } C \}
\]

the nef cone of $X$, and put

\[
\text{Aut}(\text{Nef}(X)) := \{ g \in O(S_X) \mid \text{Nef}(X)^g = \text{Nef}(X) \}.
\]
Then we have:

**Theorem.** Via the natural actions of $\text{Aut}(X)$ on $S_X$ and $T_X$, the group $\text{Aut}(X)$ is identified with the subgroup of

$$\text{Aut} (\text{Nef}(X)) \times C_X$$

consisting of pairs

$$(g, h) \in \text{Aut} (\text{Nef}(X)) \times C_X$$

such that $g$ and $h$ are restrictions of an element $\gamma \in \text{O}(H)$ to $S_X$ and $T_X$, respectively.

**Remark.** There is a simple criterion for the existence of $\gamma$ by means of discriminant forms.
Hence it is important to calculate \( \text{Aut}(\text{Nef}(X)) \).

The nef cone \( \text{Nef}(X) \) is bounded by the hyperplanes \(( [C] )^\perp \) perpendicular to the classes of \((-2)\)-curves (that is, smooth rational curves) on \( X \).

The cases where \( \text{Nef}(X) \) has only finitely many walls (\( \iff \text{Aut}(X) \) is finite) were classified by Nikulin (1981, 2000) and Vinberg (2007).

The fact that \( \text{Aut}(X) \) is finitely generated was proved by Sterk (1985) and Lieblich-Maulik (2011).
Vinberg (1983) gave a set of generators of infinite $\text{Aut}(X)$ for two most algebraic $K3$ surfaces.

Using an idea of Borcherds, Kondo (1998) gave a set of generators of $\text{Aut}$ of a generic Jacobian Kummer surface.

Since then, automorphism groups of several $K3$ surfaces have been determined by this method:
Kondo-Keum (2001): Kummer surfaces of product type
Dolgachev-Keum (2002): Hessian quartics
Dolgachev-Kondo (2003): a supersingular $K3$ surface in char 2

Our method is a generalization of Borcherds-Kondo method.
Example

Let \( X \) be a complex \( K3 \) surface with Picard number 3 (that is, \( S_X \) is of rank 3).

Suppose that \( X \) admits an elliptic fibration

\[
\phi : X \to \mathbb{P}^1
\]

with a section \( \mathbb{P}^1 \to X \). Considering this section as the origin, we can consider the Mordell-Weil group \( MW_\phi \).

We assume that \( MW_\phi \) is of rank 1 (that is, \( \phi \) has no reducible fibers).

Then the group \( Aut(X) \) contains the subgroup

\[
MW_\phi \times \{\pm 1\} \cong (\mathbb{Z}/2) \ast (\mathbb{Z}/2)
\]

generated by the translations and the inversion.
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We denote by $f_\phi \in S_X$ the class of a fiber of $\phi : X \to \mathbb{P}^1$ and by $z_\phi \in S_X$ the class of the zero section of $\phi$.

Then there is $v_3 \in S_X$ such that $f_\phi, z_\phi, v_3$ form a basis of $S_X$, and that the Gram matrix of $S_X$ with respect to $f_\phi, z_\phi, v_3$ is

$$\text{Gram}_S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2k \end{bmatrix},$$

where $-2k := v_3^2$. (The number $2k$ is the discriminant of the Mordell-Weil lattice of $\phi : X \to \mathbb{P}^1$.)
We further assume that the period $\mathbb{C} \omega_X$ of $X$ is sufficiently generic. Then the natural homomorphism

$$\text{Aut}(X) \to \text{O}(S_X) \cong \{ g \in \text{GL}_3(\mathbb{Z}) \mid g \cdot \text{Gram}_S \cdot {}^t g = \text{Gram}_S \}$$

is injective. The image of $\text{MW}_\phi \times \{ \pm 1 \} \cong (\mathbb{Z}/2) \ast (\mathbb{Z}/2)$ is generated by

$$h_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad h_2 := \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & -1 \\ 2k & 0 & -1 \end{bmatrix}.$$
Suppose that $k = 11$. Then we need one more generator for $\text{Aut}(X)$:

$$h_3 := \begin{bmatrix} 20 & 9 & -3 \\ 7 & 2 & -1 \\ 154 & 66 & -23 \end{bmatrix}. $$

The set of $(-2)$-curves on $X$ is decomposed into at most two orbits under the action of $\text{Aut}(X)$. There is only one Jacobian fibration on $X$ modulo $\text{Aut}(X)$.
The fundamental domain of $\text{Aut}(X)$ for the case $k = 11$ in $\text{Nef}(X)$ in the projective disc model is as follows.
Suppose that $k = 12$. Then we need two more generators:

$$h_3 := \begin{bmatrix} 37 & 12 & -5 \\ 36 & 13 & -5 \\ 360 & 120 & -49 \end{bmatrix}, \quad h_4 := \begin{bmatrix} 97 & 48 & -14 \\ 0 & 1 & 0 \\ 672 & 336 & -97 \end{bmatrix}.$$
Generalized Borcherds-Kondo method.

Suppose that we have a primitive embedding

\[ S_X \hookrightarrow L \]

of \( S_X \) into an even \textbf{unimodular} hyperbolic lattice

\[ L := \mathbb{II}_{1,n-1} \]

of rank \( n \), where \( n = 10, 18 \) or 26. Let \( \mathcal{P}_L \subset L \otimes \mathbb{R} \) be the positive cone of \( L \) containing \( \text{Nef}(X) \). Then the hyperplanes \( (r)_{\perp} \) in \( \mathcal{P}_L \) perpendicular to the vectors \( r \in L \) with \( r^2 = -2 \) decompose \( \mathcal{P}_L \) into a union of closed chambers, which we call \textit{Conway chambers}. They are fundamental domain of the reflection group in \( O^+(L) \). Conway (1983) described the shape of Conway chambers.
The tessellation of $\mathcal{P}_L$ by the Conway chambers induces a tessellation of $\mathcal{P}_{S_X} = (S_X \otimes \mathbb{R}) \cap \mathcal{P}_L$. We call the chambers of this induced tessellation \textit{induced chambers}.

Under certain conditions on $S_X \hookrightarrow L$, we have the following:

- The decomposition by induced chambers is $O^+(S_X)$-invariant.
- The number of $O^+(S_X)$-orbits on the set of induced chambers is finite.
- The nef cone $\text{Nef}(X)$ is a union of induced chambers.
- Each induced chamber has only finitely many walls, and hence its automorphism group is finite.
Therefore we can find all $O^+(S_X)$-congruence classes of induced chambers, and hence we obtain the fundamental domain.
Remark.

Borcherds (1987) proved that, if the orthogonal complement of $S_X$ in $L$ is a root lattice, then the induced chambers are $O^+(S_X)$-congruent to each other.

Remark.

We have applied our algorithm to the Néron-Severi lattice of the complex Fermat quartic. There are too many $O^+(S_X)$-congruence classes of induced chambers.

Remark.

The preprint is available from http://arxiv.org/abs/1304.7427