THE GRAPHS OF HOFFMAN-SINGLETON, HIGMAN-SIMS, MCLAUGHLIN, AND THE HERMITIAN CURVE OF DEGREE 6 IN CHARACTERISTIC 5

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Abstract. We present algebro-geometric constructions of the graphs of Hoffman-Singleton, Higman-Sims, and McLaughlin by means of the configuration of 3150 smooth conics totally tangent to the Hermitian curve of degree 6 in characteristic 5, and the Néron–Severi lattice of the supersingular $K3$ surface in characteristic 5 with Artin invariant 1.

1. Introduction

The graphs of Hoffman-Singleton, Higman-Sims, and McLaughlin are important examples of strongly regular graphs. These three graphs are closely related. Indeed, the Higman-Sims graph is constructed from the set of 15-cocliques in the Hoffman-Singleton graph (see Hafner [10]), and the McLaughlin graph has been constructed from the Hoffman-Singleton graph by Inoue [14] recently.

The fact that the automorphism group of the Hoffman-Singleton graph contains the simple group $\text{PSU}_3(\mathbb{F}_{25})$ as a subgroup of index 2 suggests that there is a relation between these three graphs and the Hermitian curve of degree 6 over $\mathbb{F}_{25}$. In fact, Benson and Losey [2] constructed the Hoffman-Singleton graph by means of the geometry of $\mathbb{P}^2(\mathbb{F}_{25})$ equipped with a Hermitian polarity.

In this talk, we present two algebro-geometric constructions of these three graphs. The one uses the set of smooth conics totally tangent to the Hermitian curve of degree 6 in characteristic 5, and the other uses the Néron–Severi lattice of the supersingular $K3$ surface in characteristic 5 with Artin invariant 1. See [25] for the first construction, and [15] for the second construction.

2. Strongly regular graphs

Let $\Gamma = (V, E)$ be a graph, where $V$ is the set of vertices and $E \subset \binom{V}{2}$ is the set of edges. We assume that $V$ is finite. For $p \in V$, we put

$$L(p) := \{ p' \in V \mid pp' \in E \}.$$

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We say that $\Gamma$ is regular of degree $k$ if $k := |L(p)|$ does not depend on $p \in V$, and that $\Gamma$ is strongly regular with the parameter $(v, k, \lambda, \mu)$ if $\Gamma$ is regular of degree $k$ with $|V| = v$ such that, for distinct vertices $p, p' \in V$, we have

$$|L(p) \cap L(p')| = \begin{cases} \lambda & \text{if } pp' \in E, \\ \mu & \text{otherwise}. \end{cases}$$

**Definition-Example 2.1.** A triangular graph $T(m)$ is defined to be the graph $(V, E)$ such that $V = [m]$, where $[m] := \{1, 2, \ldots, m\}$, and $E$ is the set of pairs $\{i, j\}, \{i', j'\}$ such that $\{i, j\} \cap \{i', j'\} \neq \emptyset$. Then $T(m)$ is a strongly regular graph of parameters $(v, k, \lambda, \mu) = (m(m - 1)/2, 2(m - 2), m - 2, 4)$.

**Definition-Theorem 2.1.** (1) The Hoffman-Singleton graph is the unique strongly regular graph of parameters $(v, k, \lambda, \mu) = (50, 7, 0, 1)$.

(2) The Higman-Sims graph is the unique strongly regular graph of parameters $(v, k, \lambda, \mu) = (100, 22, 0, 6)$.

(3) The McLaughlin graph is the unique strongly regular graph of parameters $(v, k, \lambda, \mu) = (275, 112, 30, 56)$.

**Theorem 2.1.** (1) The automorphism group of the Hoffman-Singleton graph contains $\text{PSU}_3(F_{25})$ as a subgroup of index $2$.

(2) The automorphism group of the Higman-Sims graph contains the Higman-Sims group as a subgroup of index $2$.

(3) The automorphism group of the McLaughlin graph contains the McLaughlin group as a subgroup of index $2$.

See [9], [11], [13], and [17]. See also [4] for constructions for these graphs.

**Remark 2.2.** Constructions of these graphs by the Leech lattice are known. Below is a part of Table 10.4 of Conway-Sloane’s book [7]. See also Borcherds’ paper [3].

<table>
<thead>
<tr>
<th>Name</th>
<th>Order</th>
<th>Structure</th>
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</thead>
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<tr>
<td>.533</td>
<td>$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$</td>
<td>PSU$<em>3(F</em>{25})$</td>
</tr>
<tr>
<td>.7</td>
<td>$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$</td>
<td>HS</td>
</tr>
<tr>
<td>.1033</td>
<td>$2^{10} \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$</td>
<td>HS.2</td>
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<tr>
<td>.332</td>
<td>$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$</td>
<td>HS</td>
</tr>
<tr>
<td>.5</td>
<td>$2^8 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$</td>
<td>McL.2</td>
</tr>
<tr>
<td>.832</td>
<td>$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$</td>
<td>McL</td>
</tr>
<tr>
<td>.322</td>
<td>$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$</td>
<td>McL</td>
</tr>
<tr>
<td>.522</td>
<td>$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$</td>
<td>McL.2</td>
</tr>
</tbody>
</table>
3. Hermitian varieties

In this and the next sections, we fix a power \( q := p^r \) of a prime integer \( p \). Let \( k \) denote an algebraic closure of the finite field \( \mathbb{F}_{q^2} \). Every algebraic variety will be defined over \( k \).

Let \( n \) be an integer \( \geq 2 \). We define the Hermitian variety \( X \) to be the hypersurface of \( \mathbb{P}^n \) defined by
\[
x_0^{q+1} + \cdots + x_n^{q+1} = 0.
\]
The automorphism group \( \text{Aut}(X) \subset \text{Aut}(\mathbb{P}^n) = \text{PGL}_{n+1}(k) \) of this hypersurface \( X \) is equal to \( \text{PGU}_{n+1}(\mathbb{F}_{q^2}) \).

We say that a point \( P \) of \( X \) is a special point if \( P \) satisfies the following equivalent conditions. Let \( \mathcal{T}_P \) be the hyperplane tangent to \( X \) at \( P \).

(i) \( P \) is an \( \mathbb{F}_{q^2} \)-rational point of \( X \).

(ii) \( \mathcal{T}_P \cap X \) is a cone.

We denote by \( \mathcal{P}_X \) the set of special points of \( X \). Then we have
\[
|\mathcal{P}_X| = \frac{1}{q} \left( \frac{q^{2(n+1)} - 1}{q^2 - 1} + \frac{(-q)^{n+1} - 1}{q + 1} \right),
\]
and \( \text{Aut}(X) = \text{PGU}_{n+1}(\mathbb{F}_{q^2}) \) acts on \( \mathcal{P}_X \) transitively. See [12, Chapter 23] or [23], for example.

A curve \( C \subset \mathbb{P}^n \) is said to be a rational normal curve if \( C \) is projectively equivalent to the image of the morphism \( \mathbb{P}^1 \to \mathbb{P}^n \) given by
\[
[x : y] \mapsto [x^{n+1} : x^n y : \cdots : xy^n : y^{n+1}].
\]
It is known that a curve \( C \subset \mathbb{P}^n \) is a rational normal curve if and only if \( C \) is non-degenerate (that is, there exist no hyperplanes of \( \mathbb{P}^n \) containing \( C \)), and \( \text{deg}(C) = n + 1 \).

We say that a rational normal curve \( C \) is totally tangent to the Hermitian variety \( X \) if \( C \) is tangent to \( X \) at distinct \( q + 1 \) points and the intersection multiplicity at each intersection point is \( n \).

A subset \( S \) of a rational normal curve \( C \) is a Baer subset if there exists a coordinate \( t : C \to \mathbb{P}^1 \) on \( C \) such that \( S \) is the inverse image by \( t \) of the set \( \mathbb{P}^1(\mathbb{F}_q) = \mathbb{F}_q \cup \{ \infty \} \) of \( \mathbb{F}_q \)-rational points of \( \mathbb{P}^1 \).

**Theorem 3.1** ([24]). Suppose that \( n \not\equiv 0 \pmod{p} \) and \( 2n \leq q \). Let \( \mathcal{Q}_X \) denote the set of rational normal curves totally tangent to \( X \).

(1) The set \( \mathcal{Q}_X \) is non-empty, and \( \text{Aut}(X) \) acts on \( \mathcal{Q}_X \) transitively with the stabilizer subgroup isomorphic to \( \text{PGL}_2(\mathbb{F}_q) \). In particular, we have
\[
|\mathcal{Q}_X| = |\text{PGU}_{n+1}(\mathbb{F}_{q^2})|/|\text{PGL}_2(\mathbb{F}_q)|.
\]
(2) For any $C \in \mathcal{Q}_X$, the points in $C \cap X$ form a Baer subset of $C$.
(3) Every $C \in \mathcal{Q}_X$ is defined over $\mathbb{F}_{q^2}$, and we have $C \cap X \subset \mathcal{P}_X$.

Remark 3.2. B. Segre obtained Theorem 3.1 for the case $n = 2$ in [22, n. 81].

4. HERMITIAN CURVES

In this section, we put $n = 2$ and consider the Hermitian curve

$$x^{q+1} + y^{q+1} + z^{q+1} = 0$$

of degree $q + 1$ in characteristic $p$. Then the condition (ii) above for $P \in X$ to be a special point of $X$ is equivalent to $T_P \cap X = \{P\}$, and, by [8] and [16], it is further equivalent to the condition

(iii) $P$ is a Weierstrass point of the curve $X$.

The number of special points of $X$ is equal to $q^3 + 1$, and $\text{Aut}(X)$ acts on $\mathcal{P}_X$ double-transitively.

A line $L \subset \mathbb{P}^2$ is a special secant line of $X$ if $L$ contains distinct two points of $\mathcal{P}_X$. If $L$ is a special secant line, then $L$ intersects $X$ transversely, and we have $L \cap X \subset \mathcal{P}_X$. Let $\mathcal{S}_X$ denote the set of special secant lines of $X$. We have

$$|\mathcal{S}_X| = q^4 - q^3 + q^2.$$  

Suppose that $p$ is odd and $q \geq 5$. Then we have $|Q_X| = q^2(q^3 + 1)$. Let $Q \in Q_X$ be a conic totally tangent to $X$. A special secant line $L$ of $X$ is said to be a special secant line of $Q$ if $L$ passes through two distinct points of $Q \cap X$. We denote by $\mathcal{S}(Q)$ the set of special secant lines of $Q$. Since $|Q \cap X| = q + 1$, we obviously have $|\mathcal{S}(Q)| = q(q + 1)/2$.

5. GEOMETRIC CONSTRUCTION BY THE HERMITIAN CURVE

In this section, we consider the Hermitian curve

$$X : x^6 + y^6 + z^6 = 0$$

of degree 6 in characteristic 5. We have

$$|\text{Aut}(X)| = 378000, \quad |\mathcal{P}_X| = 126, \quad |Q_X| = 3150, \quad |\mathcal{S}_X| = 525,$$

and for $Q \in Q_X$, we have $|Q \cap X| = 6$ and $|\mathcal{S}(Q)| = 15$.

Our construction proceeds as follows.

Proposition 5.1. Let $G$ be the graph whose set of vertices is $Q_X$ and whose set of edges is the set of pairs $\{Q, Q'\}$ of distinct conics in $Q_X$ such that $Q$ and $Q'$ intersect transversely (that is, $|Q \cap Q'| = 4$) and $|\mathcal{S}(Q) \cap \mathcal{S}(Q')| = 3$. Then $G$ has exactly 150 connected components, and each connected component is isomorphic to the triangular graph $T(7)$.
Let $\mathcal{D}$ denote the set of connected components of the graph $G$.

**Proposition 5.2.** Let $D \in \mathcal{D}$ be a connected component of the graph $G$. Then $Q \cap Q' \cap X = \emptyset$ for any distinct conics $Q, Q'$ in $D$. Since $|D| \times |Q \cap X| = |\mathcal{P}_X|$, each $D \in \mathcal{D}$ gives rise to a decomposition of $\mathcal{P}_X$ into a disjoint union of $21$ sets $Q \cap X$ of six points, where $Q$ runs through $D$.

**Proposition 5.3.** Suppose that $Q \in \mathcal{Q}_X$ and $D' \in \mathcal{D}$ satisfy $Q \notin D'$. Then one of the following holds:

\[
\begin{align*}
\alpha) \quad |Q \cap Q' \cap X| &= \begin{cases} 
2 & \text{for } 3 \text{ conics } Q' \in D', \\
0 & \text{for } 18 \text{ conics } Q' \in D'.
\end{cases} \\
\beta) \quad |Q \cap Q' \cap X| &= \begin{cases} 
2 & \text{for } 1 \text{ conic } Q' \in D', \\
1 & \text{for } 4 \text{ conics } Q' \in D', \\
0 & \text{for } 16 \text{ conics } Q' \in D'.
\end{cases} \\
\gamma) \quad |Q \cap Q' \cap X| &= \begin{cases} 
1 & \text{for } 6 \text{ conics } Q' \in D', \\
0 & \text{for } 15 \text{ conics } Q' \in D'.
\end{cases}
\end{align*}
\]

For $Q \in \mathcal{Q}_X$ and $D' \in \mathcal{D}$ satisfying $Q \notin D'$, we define $t(Q, D')$ to be $\alpha, \beta$ or $\gamma$ according to the cases in Proposition 5.3.

**Proposition 5.4.** Suppose that $D, D' \in \mathcal{D}$ are distinct, and hence disjoint as subsets of $\mathcal{Q}_X$. Then one of the following holds:

\[
\begin{align*}
\beta^{21} \quad t(Q, D') &= \beta \quad \text{for all } Q \in D. \\
\gamma^{21} \quad t(Q, D') &= \gamma \quad \text{for all } Q \in D. \\
\alpha^{15}\gamma^6 \quad t(Q, D') &= \begin{cases} 
\alpha & \text{for } 15 \text{ conics } Q \in D, \\
\gamma & \text{for } 6 \text{ conics } Q \in D.
\end{cases} \\
\alpha^3\gamma^{18} \quad t(Q, D') &= \begin{cases} 
\alpha & \text{for } 3 \text{ conics } Q \in D, \\
\gamma & \text{for } 18 \text{ conics } Q \in D.
\end{cases}
\end{align*}
\]

For distinct $D, D' \in \mathcal{D}$, we define $T(D, D')$ to be $\beta^{21}, \gamma^{21}, \alpha^{15}\gamma^6$ or $\alpha^3\gamma^{18}$ according to the cases in Proposition 5.4.

Our main results are as follows.

**Theorem 5.5.** Let $H$ be the graph whose set of vertices is $\mathcal{D}$, and whose set of edges is the set of pairs $\{D, D'\}$ such that $D \neq D'$ and $T(D, D') = \alpha^{15}\gamma^6$. Then $H$ has exactly three connected components, and each connected component is the Hoffman-Singleton graph.

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We denote by $C_1, C_2, C_3$ the set of vertices of the connected components of $H$. The orbit of an element $D \in \mathcal{D}$ by the subgroup $\text{PSU}_3(\mathbb{F}_{25}) \subset \text{Aut}(X)$ of index 3 is one of the connected component $C_i$ of $H$.

**Proposition 5.6.** If $D$ and $D'$ are in the same connected component of $H$, then $T(D, D')$ is either $\gamma^{21}$ or $\alpha^{15}\gamma^6$. If $D$ and $D'$ are in different connected components of $H$, then $T(D, D')$ is either $\beta^{21}$ or $\alpha^3\gamma^{18}$.

**Theorem 5.7.** Let $H'$ be the graph whose set of vertices is $\mathcal{D}$, and whose set of edges is the set of pairs $\{D, D'\}$ such that $D \neq D'$ and $T(D, D')$ is either $\beta^{21}$ or $\alpha^{15}\gamma^6$. For any $i$ and $j$ with $i \neq j$, the restriction $H'|\{C_i \cup C_j\}$ of $H'$ to $C_i \cup C_j$ is the Higman-Sims graph.

Using our results, we can recast the construction of the McLaughlin graph by Inoue [14] into a simpler form.

Let $E_1$ denote the set of edges of the Hoffman-Singleton graph $H|C_1$; that is,

$$E_1 := \{ \{D_1, D_2\} \mid D_1, D_2 \in C_1, \ T(D_1, D_2) = \alpha^{15}\gamma^6 \}.$$  

We define a symmetric relation $\sim$ on $E_1$ by setting $\{D_1, D_2\} \sim \{D'_1, D'_2\}$ if and only if $\{D_1, D_2\}$ and $\{D'_1, D'_2\}$ are disjoint and there exists an edge $\{D''_1, D''_2\} \in E_1$ that has a common vertex with each of the edges $\{D_1, D_2\}$ and $\{D'_1, D'_2\}$.

**Theorem 5.8.** Let $H''$ be the graph whose set of vertices is $E_1 \cup C_2 \cup C_3$, and whose set of edges consists of

- $\{E, E'\}$, where $E, E' \in E_1$ are distinct and satisfy $E \sim E'$,
- $\{E, D\}$, where $E = \{D_1, D_2\} \in E_1$, $D \in C_2 \cup C_3$, and both of $T(D_1, D)$ and $T(D_2, D)$ are $\alpha^3\gamma^{18}$, and
- $\{D, D'\}$, where $D, D' \in C_2 \cup C_3$ are distinct and satisfy and $T(D, D') = \alpha^{15}\gamma^6$ or $\alpha^3\gamma^{18}$.

Then $H''$ is the McLaughlin graph.

**Proof of Theorems.** We make the list of defining equations of the conics in $Q_X$, and calculate the adjacency matrices of $G$, $H$, $H'$ and $H''$. We then show that $H|C_i$ is strongly regular of parameters $(50, 7, 0, 1)$, $H'|\{C_i \cup C_j\}$ is strongly regular of parameters $(100, 22, 0, 6)$, and $H''$ is strongly regular of parameters $(275, 112, 30, 56)$.

**Remark 5.9.** There are many other ways to define the edges of $H$ and $H'$. For example, the classical 15-coclique construction of the Higman-Sims graph from the Hoffman-Singleton graph can be rephrased neatly in terms of the geometry of $Q_X$. 

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6. GROUP THEORETIC INTERPRETATION

The above construction can be expressed in terms of the structure of subgroups of $\text{Aut}(X) = \text{PGU}_3(\mathbb{F}_{25})$.

For an element $a$ of a set $A$ on which $\text{PGU}_3(\mathbb{F}_{25})$ acts, we denote by $\text{stab}(a)$ the stabilizer subgroup in $\text{PGU}_3(\mathbb{F}_{25})$ of $a$. By $\mathfrak{S}_m$ and $\mathfrak{A}_m$, we denote the symmetric group and the alternating group of degree $m$, respectively.

Let $Q$ be an element of $Q_X$. Then $\text{stab}(Q)$ is isomorphic to $\text{PGL}_2(\mathbb{F}_5) \cong \mathfrak{S}_5$.

**Theorem 6.1.** Let $Q$ and $Q'$ be distinct elements of $Q_X$. Then $Q$ and $Q'$ are adjacent in the graph $G$ if and only if $\text{stab}(Q) \cap \text{stab}(Q')$ is isomorphic to $\mathfrak{A}_4$. Moreover, $Q$ and $Q'$ are in the same connected component of $G$ if and only if the subgroup $\langle \text{stab}(Q), \text{stab}(Q') \rangle$ of $\text{PGU}_3(\mathbb{F}_{25})$ is isomorphic to $\mathfrak{S}_5$.

**Proposition 6.2.** For each $D \in D$, the action of $\text{stab}(D)$ on the triangular graph $D = T(7)$ identifies $\text{stab}(D)$ with the subgroup $\mathfrak{A}_7$ of $\text{Aut}(T(7)) \cong \mathfrak{S}_7$.

**Theorem 6.3.** Let $D$ and $D'$ be distinct elements of $D$. We identify $\text{stab}(D)$ with $\mathfrak{A}_7$ by Proposition 6.2. Then $T(D, D')$ is

\[
\begin{aligned}
\beta^{21} & \text{ if and only if } \text{stab}(D) \cap \text{stab}(D') \cong \text{PSL}_2(\mathbb{F}_7), \\
\gamma^{21} & \text{ if and only if } \text{stab}(D) \cap \text{stab}(D') \cong \mathfrak{A}_5, \\
\alpha^{15} \gamma^6 & \text{ if and only if } \text{stab}(D) \cap \text{stab}(D') \cong \mathfrak{A}_6, \\
\alpha^3 \gamma^{18} & \text{ if and only if } \text{stab}(D) \cap \text{stab}(D') \cong (\mathfrak{A}_4 \times 3) : 2.
\end{aligned}
\]

**Remark 6.4.** By ATLAS [6], we see that the maximal subgroups of $\mathfrak{A}_7$ are $\mathfrak{A}_6$, $\text{PSL}_2(\mathbb{F}_7)$, $\text{PSL}_2(\mathbb{F}_7)$, $\mathfrak{S}_5$, $(\mathfrak{A}_4 \times 3) : 2$.

7. SUPERSINGULAR K3 SURFACE

First we recall the definition of the Néron–Severi lattice of a smooth projective surface $Y$ defined over an algebraically closed field. A divisor $D$ on $Y$ is **numerically equivalent to zero** if

\[ D \cdot C = 0 \quad \text{for any curve } C \text{ on } Y, \]

where $D \cdot C$ is the intersection number of $D$ and $C$ on $Y$. Let $S_Y$ be the $\mathbb{Z}$-module of numerical equivalence classes of divisors on $Y$. Then $S_Y$ with the symmetric bilinear form $\langle \cdot, \cdot \rangle$ induced by the intersection pairing becomes a lattice, which is called the Néron–Severi lattice of $Y$.

A $K3$ surface $Y$ is said to be **supersingular** if the rank of $S_Y$ attains the possible maximum 22. Supersingular $K3$ surfaces exist only in positive characteristics. Suppose that $Y$ is a supersingular $K3$ surface in characteristic $p > 0$. 
Let \( S'_Y := \text{Hom}(S_Y, \mathbb{Z}) \) denote the dual lattice of \( S_Y \). Artin [1] proved that \( S'_Y / S_Y \) is a \( p \)-elementary abelian group of rank \( 2\sigma \), where \( \sigma \) is an integer such that \( 1 \leq \sigma \leq 10 \). This integer \( \sigma \) is called the Artin invariant of \( Y \). It is known that the isomorphism class of the lattice \( S_Y \) depends only on \( p \) and \( \sigma \) (Rudakov and Shafarevich [21]), and that a supersingular \( K3 \) surface with Artin invariant 1 in characteristic \( p \) exists and is unique up to isomorphisms (Ogus [19, 20], Rudakov and Shafarevich [21]).

We work over an algebraically closed field of characteristic 5, and consider the smooth surface \( Y \) defined by
\[
w^2 = x^6 + y^6 + z^6
\]
in the weighted projective space \( \mathbb{P}(3, 1, 1, 1) \). Then \( Y \) is a double cover of \( \mathbb{P}^2 \) branched along the Hermitian curve \( X \). Proposition 7.1. The surface \( Y \) is a supersingular \( K3 \) surface with Artin invariant 1. In particular, its Néron–Severi lattice \( S_Y \) is isomorphic to the unique lattice characterized by the following properties:

- \( S_Y \) is even and of signature \((1, 21)\),
- \( S'_Y / S_Y \cong (\mathbb{Z}/5\mathbb{Z})^2 \).

In fact, we can give a basis of \( S_Y \) explicitly. Let \( P \) be a special point of \( X \). Then the tangent line \( T_P \) to \( X \) at \( P \) intersects \( X \) at \( P \) with multiplicity 6. Hence the pullback of \( T_P \) by the double covering \( Y \to \mathbb{P}^2 \) splits into two smooth rational curves meeting at one point with multiplicity 3. Since the number of \( \mathbb{F}_{25} \)-rational points of \( X \) is 126, we obtain 252 smooth rational curves on \( Y \). There exist 22 curves among these 252 curves such that their numerical equivalence classes form a lattice of rank 22 and discriminant \(-25\). Therefore they generate \( S_Y \).

The class of the pull-back of a line of \( \mathbb{P}^2 \) is denoted by \( h_0 \in S_Y \). We have \( h_0^2 = 2 \). Then the automorphism group
\[
\text{Aut}(Y, h_0) := \{ g \in \text{Aut}(Y) \mid h_0^2 = h_0 \}
\]
of the polarized \( K3 \) surface \((Y, h_0)\) is isomorphic to \( \text{PGU}_3(\mathbb{F}_{25}) : 2 \) of order 756000, where the extra involution comes from \( \text{Gal}(Y/\mathbb{P}^2) \).

8. CONSTRUCTION BY THE NÉRON–SEVERI LATTICE

This construction stems from [15]. In an attempt to calculate the full automorphism group \( \text{Aut}(Y) \) by Borcherds method [3], we embedded \( S_Y \) into an even unimodular lattice \( L_{26} \) of signature \((1, 25)\). Note that the lattice \( L_{26} \) is unique up to isomorphisms. From the lattice data \((S_Y, h_0)\), the Hoffman-Singleton graph and Higman-Sims graph can be constructed.
Let $U$ be the hyperbolic plane
\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix},
\]
and let $\Lambda$ be the negative definite Leech lattice. As $L_{26}$, we use $U \oplus \Lambda$. Vectors of $L_{26}$ are written as $(a, b, \lambda)$, where $a, b \in \mathbb{Z}$, $(a, b) \in U$ and $\lambda \in \Lambda$. Let $\mathcal{P}(L_{26})$ be the connected component of $\{v \in L_{26} \otimes \mathbb{R} \mid v^2 > 0\}$ that contains $w_0 := (1, 0, 0)$ on its boundary. Each vector $r \in L_{26}$ with $r^2 = -2$ defines a reflection
\[
s_r : x \mapsto x + \langle x, r \rangle r.
\]
Let $W(L_{26})$ denote the subgroup of $O(L_{26})$ generated by these $s_r$. Then $W(L_{26})$ acts on $\mathcal{P}(L_{26})$. We put
\[
\mathcal{R}_0 := \{ r \in L_{26} \mid r^2 = -2, \langle r, w_0 \rangle = 1 \},
\]
\[
\mathcal{D}_0 := \{ x \in \mathcal{P}(L_{26}) \mid \langle x, r \rangle \geq 0 \text{ for any } r \in \mathcal{R}_0 \}.
\]
The map
\[
\lambda \mapsto r_\lambda := (-1 - \lambda^2/2, 1, \lambda)
\]
gives a bijection from $\Lambda$ to $\mathcal{R}_0$, and the group $\text{Aut}(\mathcal{D}_0) := \{ g \in O(L_{26}) \mid \mathcal{D}_0^g = \mathcal{D}_0 \}$ is isomorphic to the Conway group $\text{Co}_\infty$. Conway [5] proved the following:

**Theorem 8.1.** The domain $\mathcal{D}_0$ is a standard fundamental domain of the action of $W(L_{26})$ on $\mathcal{P}(L_{26})$.

By Nikulin [18], we see that there exists a primitive embedding $S_Y \hookrightarrow L_{26}$ unique up to $O(L_{26})$. The orthogonal complement $R$ of $S_Y$ in $L_{26}$ has a Gram matrix
\[
\begin{bmatrix}
-2 & -1 & 0 & 1 \\
-1 & -2 & -1 & 0 \\
0 & -1 & -4 & -2 \\
1 & 0 & -2 & -4 \\
\end{bmatrix}.
\]
We denote by
\[
\text{pr}_S : L_{26} \rightarrow S_Y, \quad \text{pr}_R : L_{26} \rightarrow R,
\]
the orthogonal projections to $S_Y$ and $R$, respectively.

**Theorem 8.2** ([15]). There exists a primitive embedding $S_Y \hookrightarrow L_{26}$ such that $\text{pr}_S(w_0) = h_0$. 

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In the following, we use this primitive embedding. The set
\[ V := \{ r_\lambda \in \mathcal{R}_0 \mid \langle \text{pr}_S(r_\lambda), h_0 \rangle = 1, \quad \langle \text{pr}_S(r_\lambda), \text{pr}_S(r_\lambda) \rangle = -8/5 \} \]
consists of 300 elements. For each \( r_\lambda \in V \), there exists a unique \( r'_\lambda \in V \) such that \( \langle r_\lambda, r'_\lambda \rangle = 3 \), and for any vector \( r_\mu \in V \) other than \( r_\lambda, r'_\lambda \), we have that \( \langle r_\lambda, r_\mu \rangle \) is 0 or 1.

**Definition 8.3.** Let \( F \) be the graph whose set of vertices is \( V \) and whose set of edges is the set of pairs \( \{r_\lambda, r_\mu\} \) such that \( \langle r_\lambda, r_\mu \rangle = 1 \).

The subset \( \text{pr}_R(V) \) of \( R \) consists of six elements \( \rho_1, \ldots, \rho_6 \). Their inner-products are given by
\[
\begin{bmatrix}
-2 & -1 & -1 & 1 & 1 & 2 \\
-1 & -2 & 1 & -1 & 2 & 1 \\
-1 & 1 & -2 & 2 & -1 & 1 \\
1 & -1 & 2 & -2 & 1 & -1 \\
1 & 2 & -1 & 1 & -2 & -1 \\
2 & 1 & 1 & -1 & -1 & -2
\end{bmatrix}
\]

We put
\[ V_i := \text{pr}_R^{-1}(\rho_i) \cap V. \]
If \( r_\lambda \in V_i \), then the unique vector \( r'_\lambda \in V \) with \( \langle r_\lambda, r'_\lambda \rangle = 3 \) belongs to \( V_i' \), where \( \langle \rho_i, \rho_i' \rangle = 2/5 \).

**Theorem 8.4.** For each \( i \), \( F|V_i \) is the Hoffman-Singleton graph.
If \( \langle \rho_i, \rho_i' \rangle = -1/5 \), then \( F|(V_i \cup V_i') \) is the Higman-Sims graph.

**References**


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