Connected components of the moduli of elliptic $K3$ surfaces

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We work over the complex number field. All $K3$ surfaces in this talk are algebraic.

Thanks to the Torelli theorem for $K3$ surfaces, we can study the moduli of $K3$ surfaces by lattice theory.

We study connected components of the moduli of elliptic $K3$ surfaces with a fixed combinatorial data. For this, it is necessary to calculate all the isomorphism classes of lattices in a given genus.

I determine the connected components of elliptic $K3$ surfaces with a fixed combinatorial data, by means of **Miranda-Morrison theory**.
An **elliptic K3 surface** is a triple \((X, f, s)\), where \(X\) is a K3 surface, \(f : X \rightarrow \mathbb{P}^1\) is a fibration whose general fiber is a curve of genus 1, and \(s : \mathbb{P}^1 \rightarrow X\) is a section of \(f\).

Let \((X, f, s)\) be an elliptic K3 surface. It is well-known that the set of sections of \(f\) has a natural structure of the finitely-generated abelian group with the zero element \(s\), which is called the **Mordell-Weil group**. We put

\[
A_f := \text{the torsion part of the Mordell-Weil group of } (X, f, s).
\]

If an irreducible curve \(C\) on \(X\) is contained in a singular fiber of \(f\) and is disjoint from the zero section \(s\), then \(C\) is a smooth rational curve. These curves form an **ADE-configuration**.

\[
\Phi_f := \text{the ADE-type of the set } \mathcal{R}_f \text{ of these curves}.
\]
The *combinatorial type* of \((X, f, s)\) is defined to be \((\Phi_f, A_f)\). The combinatorial type determines a lattice polarization of \(X\).

**Theorem (S.- 1999)**

*There exist exactly 3693 combinatorial types that can be realized as combinatorial types of elliptic K3 surfaces.*

<table>
<thead>
<tr>
<th>No.</th>
<th>Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(A_1)</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>…</td>
<td></td>
</tr>
<tr>
<td>3692</td>
<td>(2A_4 + 2A_3 + 2A_2)</td>
<td>0</td>
</tr>
<tr>
<td>3693</td>
<td>(6A_3)</td>
<td>(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})</td>
</tr>
</tbody>
</table>

**The problem**

Determine the connected components of the moduli of elliptic K3 surfaces with a fixed combinatorial data \((\Phi, A)\).
This work is motivated by the following two works:


In [AD], the connected components of the equisingular families of irreducible sextic plane curves with fixed type of $ADE$-singularities are calculated. In [G], the same calculation was done for non-special singular quartic surfaces with only $ADE$-singularities.

In both of [AD] and [G], the Miranda-Morrison theory was applied. I developed an algorithm to calculate a spinor norm of an isometry of a $p$-adic lattice, and made the method fully-automated.

I hope this algorithm is applicable for the moduli of lattice-polarized $K3$ surfaces in general.
Two elliptic K3 surfaces \((X, f, s)\) and \((X', f', s')\) are isomorphic if there exists a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\sim} & X' \\
f \downarrow & & \downarrow f' \\
\mathbb{P}^1 & \xrightarrow{\sim} & \mathbb{P}^1 \\
\end{array}
\]

that is compatible with \(s\) and \(s'\). A connected family of elliptic K3 surfaces of type \((\Phi, A)\) is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{F} & \mathbb{P}^1_B \\
\pi \downarrow & & \pi_P \leftarrow \\
B & & \\
\end{array}
\]

with a section \(S: \mathbb{P}^1_B \to \mathcal{X}\) of \(F\), where \(B\) is a connected analytic variety, \(\pi: \mathcal{X} \to B\) is a family of K3 surfaces, \(\pi_P: \mathbb{P}^1_B \to B\) is a \(\mathbb{P}^1\)-fibration, and for any point \(t \in B\), the pullback \((X_t, f_t, s_t)\) of \((\mathcal{X}, F, S)\) by \(\{t\} \hookrightarrow B\) is an elliptic K3 surface of type \((\Phi, A)\).
We say \((X, f, s)\) and \((X', f', s')\) are *connected* if there exists a connected family \((X, F, S)/B\) with two fibers isomorphic to \((X, f, s)\) and \((X', f', s')\).

We define a *connected component of the moduli of elliptic K3 surfaces of type* \((\Phi, A)\) to be an equivalence class of the relation of connectedness.

**Main result**

I determined the connected components of the moduli of elliptic K3 surfaces of a fixed type for each of the realizable 3693 combinatorial types.

Recall that \(R_f\) is the set of smooth rational curves contained in fibers of \(f\) and disjoint from \(s\). We say that \((X, f, s)\) is *extremal* if the cardinality of \(R_f\) attains the possible maximum 18 (in other words, the sum of the indices of ADE-symbols in \(\Phi_f\) is 18).
List of combinatorial types \((\Phi, A)\) with non-connected moduli.

Extremal elliptic \(K3\) surfaces

<table>
<thead>
<tr>
<th>no.</th>
<th>(\Phi)</th>
<th>(A)</th>
<th>(T)</th>
<th>([r, c])</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(E_8 + A_9 + A_1)</td>
<td>0</td>
<td>[2, 0, 10]</td>
<td>[2, 0]</td>
</tr>
<tr>
<td>2</td>
<td>(E_8 + A_6 + A_3 + A_1)</td>
<td>0</td>
<td>[6, 2, 10]</td>
<td>[0, 2]</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>89</td>
<td>(2A_5 + 4A_2)</td>
<td>(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z})</td>
<td>[6, 0, 6]</td>
<td>[0, 2]</td>
</tr>
</tbody>
</table>

Non-extremal elliptic \(K3\) surfaces

<table>
<thead>
<tr>
<th>no.</th>
<th>(r)</th>
<th>(\Phi)</th>
<th>(A)</th>
<th>([c_1, \ldots, c_k])</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17</td>
<td>(E_7 + D_6 + A_3 + A_1)</td>
<td>(\mathbb{Z}/2\mathbb{Z})</td>
<td>[1, 1]</td>
</tr>
<tr>
<td>2</td>
<td>17</td>
<td>(E_7 + 2A_5)</td>
<td>0</td>
<td>[2]</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>107</td>
<td>11</td>
<td>(A_3 + 8A_1)</td>
<td>(\mathbb{Z}/2\mathbb{Z})</td>
<td>[1, 1]</td>
</tr>
</tbody>
</table>

The non-connectedness of the moduli comes from three different reasons; one is algebraic, and the other two are transcendental.
For a $K3$ surface $X$, let

$$S_X := H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$$

denote the Néron-Severi lattice of $X$ (the $\mathbb{Z}$-module of topological classes of divisors on $X$ with the cup product), and

$$T_X := (S_X \hookrightarrow H^2(X, \mathbb{Z}))^\perp$$

the transcendental lattice of $S_X$.

For an elliptic $K3$ surface $(X, f, s)$, let

$$L_f = \langle \{[C] \mid C \in \mathcal{R}_f \} \rangle \subset S_X$$

denote the submodule generated by the set of classes $[C]$ of smooth rational curves $C \in \mathcal{R}_f$.

Note that $(X, f, s)$ is extremal if and only if the rank of $L_f$ attains the possible maximum $18$. 
(1) The lattice $L_f$ is a root lattice, and its $ADE$-type is $\Phi_f$.

(2) The Mordell-Weil group of $(X, f, s)$ is isomorphic to $S_X/(U_f \oplus L_f)$, where $U_f$ is the sublattice generated by the classes of a fiber of $f$ and the zero section $s$. We put $$M_f \coloneqq \text{the primitive closure of } L_f \text{ in } S_X,$$
so that $A_f \cong M_f/L_f$.

(3) The Hodge structure of $H^2(X)$ defines a canonical positive-sign structure on the transcendental lattice $T_X$ (a choice of one of the two connected components of the manifold parametrizing oriented 2-dimensional positive-definite subspace of $T_X \otimes \mathbb{R}$). The complex conjugation switches the positive-sign structures.
Let \((X, f, s)\) and \((X', f', s')\) be general members in the connected components \(C\) and \(C'\), respectively. The term “general” means

\[
S_X = U_f \oplus M_f, \quad S_X' = U_{f'} \oplus M_{f'}.
\]

The dimension of \(C\) is

\[
20 - \text{rank } S_X = 18 - \text{rank } M_f = 18 - \text{rank } L_f.
\]

(a) If there exists no isomorphism \(R_f \sim \rightarrow R_{f'}\) that induces \(M_f \sim \rightarrow M_{f'}\), then \(C \neq C'\). If there exists such an isomorphism \(R_f \sim \rightarrow R_{f'}\), we say that \(C\) and \(C'\) are algebraically equivalent.

(b) Even if \(C\) and \(C'\) are algebraically equivalent, the primitive embeddings \(M_f \hookrightarrow H^2(X, \mathbb{Z})\) and \(M_{f'} \hookrightarrow H^2(X', \mathbb{Z})\) may not be isomorphic under any isomorphism \(R_f \sim \rightarrow R_{f'}\) and \(H^2(X, \mathbb{Z}) \cong H^2(X', \mathbb{Z})\). In this case, we have \(C \neq C'\). In particular, if \(T_X\) and \(T_{X'}\) are not isomorphic, we have \(C \neq C'\).
(c) Even if the embeddings are isomorphic, if there exists no isomorphism of the embeddings that is compatible with the positive-sign structures of $T_X$ and $T_{X'}$, we have $C \neq C'$. In this case, we say that $C$ and $C'$ are complex conjugate.

From the list of non-connected moduli, we obtain the following.

**Theorem**

The moduli of non-extremal elliptic K3 surfaces of type $(\Phi, A)$ has more than one connected component that are algebraically equivalent if and only if $A$ is trivial and $\Phi$ is one of the following:

- $E_7 + 2A_5$, $E_6 + A_{11}$, $E_6 + A_6 + A_5$, $E_6 + 2A_5 + A_1$,
- $D_5 + 2A_6$, $D_4 + 2A_6 + A_1$, $A_{11} + A_5 + A_1$, $A_7 + 2A_5$,
- $2A_6 + A_3 + 2A_1$, $A_6 + 2A_5 + A_1$, $E_6 + 2A_5$, $3A_5 + A_1$.

For each of these types, the moduli has exactly two connected components, and they are complex conjugate to each other.
Corollary

The isomorphism class of $T_X$ of a general member $(X, f, s)$ of a connected component of non-extremal elliptic K3 surfaces of type $(\Phi, A)$ is determined by the algebraically equivalence class.

This corollary is rather unfortunate, because it shows that there are no phenomena of arithmetic Zariski pair type in non-extremal elliptic K3 surfaces (that is, with positive dimensional moduli).
Examples of non-extremal elliptic $K3$ surfaces

We investigate the combinatorial type

$$(\Phi, A) = (2D_6 + 4A_1, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}).$$

We have $r = 16$, and hence the moduli is of dimension 2.

The moduli has two connected components I and II with non-isomorphic $M_f$ (that is, they are not algebraically equivalent).

We say that a section $\tau: \mathbb{P}^1 \to X$ of an elliptic $K3$ surface $(X, f, s)$ is narrow at $P \in \mathbb{P}^1$ if $\tau$ and $s$ intersect the same irreducible component of $f^{-1}(P)$. 
In the class I, the three non-trivial torsion sections are as follows;

<table>
<thead>
<tr>
<th>$D_6, D_6$</th>
<th>$A_1, A_1, A_1, A_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>not narrow</td>
<td>narrow at 2 points</td>
</tr>
<tr>
<td>not narrow</td>
<td>narrow at other 2 points</td>
</tr>
<tr>
<td>not narrow</td>
<td>not narrow at all 4 points</td>
</tr>
</tbody>
</table>

In the class II, the three non-trivial torsion sections are as follows;

<table>
<thead>
<tr>
<th>$D_6, D_6$</th>
<th>$A_1, A_1, A_1, A_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>not narrow</td>
<td>narrow at 1 point</td>
</tr>
<tr>
<td>not narrow</td>
<td>narrow at 2 points</td>
</tr>
<tr>
<td>not narrow</td>
<td>narrow at 1 point</td>
</tr>
</tbody>
</table>

The fact that the two $M_f$ are non-isomorphic can be shown directly. For this, we need the notion of the discriminant form of an even lattice.
Let $L$ be an even lattice; that is, $L$ is a free $\mathbb{Z}$-module of finite rank with a non-degenerate symmetric bilinear form

$$\langle \cdot, \cdot \rangle: L \times L \to \mathbb{Z}$$

such that $\langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in L$. Then we have a canonical finite quadratic form

$$q_L: D_L := \text{Hom}(L, \mathbb{Z})/L \to \mathbb{Q}/2\mathbb{Z}$$

which is called the *discriminant form* of $L$.

For I, the $M_f$ has discriminant form $q: (\mathbb{Z}/2\mathbb{Z})^4 \to \mathbb{Q}/2\mathbb{Z}$ such that $q(x) \in \{0, 1\}$ for all $x \in (\mathbb{Z}/2\mathbb{Z})^4$.

For II, the $M_f$ has discriminant form $q: (\mathbb{Z}/2\mathbb{Z})^4 \to \mathbb{Q}/2\mathbb{Z}$ such that $q(x) = 1/2$ for some $x \in (\mathbb{Z}/2\mathbb{Z})^4$. 

Examples of extremal elliptic $K3$ surfaces

A $K3$ surface is *singular* if the rank of $S_X$ attains the possible maximum 20.
In particular, an extremal elliptic $K3$ surface is singular. The moduli of extremal elliptic $K3$ surfaces is of dimension 0.

If $X$ is singular, then $T_X$ is a positive-definite even lattice of rank 2 with a canonical orientation.

Theorem (Shioda-Inose)

*The isomorphism class of a singular $K3$ surface $X$ is determined by the isomorphism class of the transcendental lattice $T_X$ with the canonical orientation.*
Consider the combinatorial type

$$(\Phi, A) = (E_7 + A_{10} + A_1, 0).$$

The moduli has 3 connected components. They have isomorphic $M_f$. One has the transcendental lattice

$$T_f \cong \begin{bmatrix} 2 & 0 \\ 0 & 22 \end{bmatrix}.$$

The other two have the transcendental lattice

$$T_f \cong \begin{bmatrix} 6 & 2 \\ 2 & 8 \end{bmatrix},$$

and these two are complex conjugate.

The non-connected moduli whose connected components cannot be distinguished by the algebraic data $M_f$ corresponds to *arithmetical Zariski pairs*. 
Example of an arithmetic Zariski pair of plane curves of degree 6

For singular $K3$ surfaces, we have the following:

Theorem (Shioda-Inose)

*A singular $K3$ surface is defined over $\overline{\mathbb{Q}}$.*

Theorem (Schütt-S.)

*Let $X$ and $X'$ be a pair of singular $K3$ surfaces with $S_X \cong S_{X'}$. Then $X$ and $X'$ are conjugate under $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.***
Consider the plane curves of degree 6 whose singularities are of type $A_{10} + A_9$. Since the Milnor number 19 is maximal, the moduli is of dimension 0. There are four connected components, that is, there exists four isomorphism classes of such plane curves. Two of them are irreducible, while the other two are line plus irreducible quintic.

The reducible two curves $C_\pm \subset \mathbb{P}^2$ are defined by

$$z \cdot (G(x, y, z) \pm \sqrt{5} \cdot H(x, y, z)) = 0,$$

where

$$G(x, y, z) = -9x^4z - 14x^3yz + 58x^3z^2 - 48x^2y^2z - 64x^2yz^2 + 10x^2z^3 + 108xy^3z - 20xy^2z^2 - 44y^5 + 10y^4z,$$

$$H(x, y, z) = 5x^4z + 10x^3yz - 30x^3z^2 + 30x^2y^2z + 20x^2yz^2 - 40xy^3z + 20y^5.$$
Hence $C_+$ and $C_-$ are $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugate. For $C_\pm$, let $X_\pm$ be the singular $K3$ surface obtained as the minimal resolution of the double cover of $\mathbb{P}^2$ branched along $C_\pm$. The transcendental lattice of $X_+$ is

$$
\begin{pmatrix}
2 & 1 \\
1 & 28
\end{pmatrix},
$$

while the transcendental lattice of $X_-$ is

$$
\begin{pmatrix}
8 & 3 \\
3 & 8
\end{pmatrix}.
$$

In particular, the embeddings $C_+ \hookrightarrow \mathbb{P}^2$ and $C_- \hookrightarrow \mathbb{P}^2$ are not homeomorphic; that is, $C_+$ and $C_-$ form an arithmetic Zariski pair.
To analyze the *transcendental* part of the problem in non-extremal cases, we need a refinement of Miranda-Morrison theory.

Rick Miranda and David R. Morrison. *Embeddings of integral quadratic forms.*


We say that two even lattices $L$ and $L'$ are *in the same genus* if $L \otimes \mathbb{R}$ and $L' \otimes \mathbb{R}$ have the same signature and their discriminant forms are isomorphic.

Let $G$ be a genus of isomorphism classes of even indefinite lattices of rank $\geq 3$ determined by a signature $(s_+, s_-)$ and a finite quadratic form $q: D \to \mathbb{Q}/2\mathbb{Z}$.

Let $L$ be a member of $G$. We have a natural homomorphism $O(L) \to O(q)$. 

**Miranda-Morrison theory**

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Let $L$ be a member of $G$. We have a natural homomorphism $O(L) \to O(q)$.
Miranda and Morrison defined a finite abelian group $\mathcal{M}$ that fits in an exact sequence

$$0 \rightarrow \text{Coker}(\mathcal{O}(L) \rightarrow \mathcal{O}(q)) \rightarrow \mathcal{M} \rightarrow G \rightarrow 0,$$

and showed how to calculate $\mathcal{M}$.

(1) Their result depends on the strong approximation theorem for the spin group of indefinite lattices of rank $\geq 3$.
(2) The group $\mathcal{M}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\ell$ for some $\ell$.
(3) In order to calculate $\mathcal{M}$, we need not to know $L$. It is enough to know the $\mathbb{Z}_p$-lattices $L \otimes \mathbb{Z}_p$ for each $p|2\text{disc}(L)$. Since we have a complete classification of $\mathbb{Z}_p$-lattices, the $\mathbb{Z}_p$-lattices $L \otimes \mathbb{Z}_p$ can be calculated from the discriminant from $q$. 
We fix the algebraic data $M$, which is negative-definite of rank $r$, and compute the connected components such that $M \cong M_f$.

By Nikulin’s theorem, if $M_f \cong M$, then the genus $\mathcal{G}_T$ of the $T_f$ is determined by the signature $(2, 18 - r)$ and the discriminant form

$$q_{T_f} \cong -q_M.$$

The embeddings of $U \oplus M$ into the $K3$ lattice (the even unimodular lattice of signature $(3, 19)$) is in one-to-one correspondence with the Miranda-Morrison group $\mathcal{M}$ for $\mathcal{G}_T$. 
We need to refine the Miranda-Morrison theory:

- We have to take the positive-sign structures of $T_f$ into account. We enlarge $\mathcal{M}$ to $\widetilde{\mathcal{M}} \subset \mathcal{M} \times \{\pm 1\}$.

- We have to divide $\widetilde{\mathcal{M}}$ by the automorphisms coming from the permutation of the root system $\mathcal{R}_f$ of smooth rational curves.

For the second task, we write the following algorithm.
The input is
- a finite quadratic form

\[ q: D \to \mathbb{Q}_p/2\mathbb{Z}_p \]

on a finite abelian \( p \)-group \( D \), and,
- an automorphism \( g \) of \( q \).

1. Calculate the Gram matrix of an even \( \mathbb{Z}_p \)-lattice \( L_p \) whose discriminant form is \( q \) and with minimal rank.
2. Find a lift \( \tilde{g} \in \text{O}(L_p) \) of \( g \).
3. Calculate the spinor norm of \( \tilde{g} \).

Since \( |\mathbb{Z}_p| \) is uncountable, we have to use approximation in the \( p \)-adic topology. The estimate of the approximation error is necessary.
The preprint is available from: arXiv:1610.04706

Thank you for your attention