平成 23 年度卒業論文
低次元のベキ零リー環の微分環

広島大学理学部数学科
B085763 藤丸拓也
指導教員 田丸博士 准教授

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1 はじめに

この論文はリー環の微分環がテーマである．リー環の微分環を求めるとそのリー環の自己同型群がわかる．リー環の自己同型群がわかると幾何的な応用ができることが知られている．以下 $\mathfrak{g}$ をリー環とする．

定義 1.1 $D : \mathfrak{g} \to \mathfrak{g}$ が *derivation* であるとは次の 2 つを満たすこと:

1. $[D(X), Y] + [X, D(Y)] = D[X, Y] \quad (\forall X, Y \in \mathfrak{g})$,
2. D は線型写像.

定義 1.2 $\mathrm{Der}(\mathfrak{g}) := \{D : \mathfrak{g} \to \mathfrak{g} \mid D$ は *derivation*$\}$ を微分環という．

この論文では低次元のベキ零リー環の微分環を計算した.

定義 1.3 $\mathfrak{g}_0 = \mathfrak{g}$ とおき, $\mathfrak{g}_i = [\mathfrak{g}_{i-1}, \mathfrak{g}]$ $(i = 1, 2, 3, \ldots)$ と帰納的に定義する．$\mathfrak{g}$ がベキ零リー環であるとは次が成り立つこと: $\exists k \in \mathbb{N}$ s.t. $\mathfrak{g}_k = 0$.

可換でないベキ零リー環は, 3 次元ベキ零リー環 1 つ, 4 次元ベキ零リー環 2 つ, 5 次元ベキ零リー環 8 つに分類されている ([2]). この論文ではそれぞれの微分環を求めた．3 次元直交リー環, 3 次元特殊線型リー環の微分環も求めている．

結果を求めるにあたって, それぞれのリー環の基底を求め, 括弧積を定義し, 最後に表現行列の形で表した.

命題 1.4 $\mathfrak{g}$ を 3 次元ベキ零リー環とする．括弧積が $[e_1, e_2] = e_3$, その他が 0 となるような $\mathfrak{g}$ の基底 $\{e_1, e_2, e_3\}$ をとると,

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{11} + a_{22} \end{pmatrix} \middle| a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, a_{32} \in \mathbb{R} \right\}.$$

2　準備

2.1　ステップ数の定義

定義 2.1 $\mathfrak{g}$ をベキ零リー環とする. 定義 1.3 と同様に $\mathfrak{g}_i$ をおく. $\mathfrak{g}$ が k -ステップベキ零リー環であるとは次を満たすこと: $\mathfrak{g}_k = 0$, $\mathfrak{g}_{k-1} \neq 0$

2.2　3 次元ベキ零リー環の分類

定理 2.2 ([2]) $\mathfrak{g}$ を可換でない 3 次元ベキ零リー環とするならば, $\mathfrak{g}$ は基底 $\{e_1, e_2, e_3\}$ は以下を満たすリー環と同型である.

1. $[e_1, e_2] = e_3$, その他が 0.

続いて, 3 次元ベキ零リー環のステップ数を示す.

命題 2.3 定理 2.2 と同様に $\mathfrak{g}$ をおく. $\mathfrak{g}$ は 2 -ステップベキ零リー環である.

(証明開始)
$\mathfrak{g}$ を 3 次元リー環としてその基底を $\{e_1,\ e_2,\ e_3\}$ とする.
$\mathfrak{g}_0 = \mathfrak{g}$ なので, $\mathfrak{g}_0 = \mathrm{span}\{e_1,\ e_2,\ e_3\}$ となる.
(claim, $\mathfrak{g}_1$=span$\{e_3\}$.)
$\forall \mathbf{a},\ \mathbf{b} \in \mathfrak{g}_0$をとる.
$\exists a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3 \in \mathbb{R}$ s.t $\mathbf{a} = a_1e_1 + a_2e_2 + a_3e_3,\ \mathbf{b} = b_1e_1 + b_2e_2 + b_3e_3.$

$$\begin{aligned}
[\mathbf{a}, \mathbf{b}] &= [a_1e_1 + a_2e_2 + a_3e_3, b_1e_1 + b_2e_2 + b_3e_3] \\
&= [a_1e_1 + a_2e_2 + a_3e_3, b_1e_1] + [a_1e_1 + a_2e_2 + a_3e_3, b_2e_2] + [a_1e_1 + a_2e_2 + a_3e_3, b_3e_3] \\
&= [a_1e_1, b_1e_1] + [a_2e_2, b_1e_1] + [a_3e_3, b_1e_1] \\
&\quad + [a_1e_1, b_2e_2] + [a_2e_2, b_2e_2] + [a_3e_3, b_2e_2] \\
&\quad + [a_1e_1, b_3e_3] + [a_2e_2, b_3e_3] + [a_3e_3, b_3e_3] \\
&= a_1b_1[e_1, e_1] + a_2b_1[e_2, e_1] + a_3b_1[e_3, e_1] \\
&\quad + a_1b_2[e_1, e_2] + a_2b_2[e_2, e_2] + a_3b_2[e_3, e_2] \\
&\quad + a_1b_3[e_1, e_3] + a_2b_3[e_2, e_3] + a_3b_3[e_3, e_3] \\
&= 0 - a_2b_1e_3 + 0 + a_1b_2e_3 + 0 + 0 + 0 + 0 + 0 \\
&= (a_1b_2 - a_2b_1)e_3.
\end{aligned}$$

$\mathfrak{g}_1 = [\mathfrak{g}_0, \mathfrak{g}_0]$ なので, $\mathfrak{g}_1 = \mathrm{span}\{e_3\}$ となる.
(claim, $\mathfrak{g}_2 = 0$.)
$\forall \mathbf{c} \in \mathfrak{g}_0$, $\forall \mathbf{d} \in \mathfrak{g}_1$, をとる.
$\exists c_1$, c_2, c_3, $d_3 \in \mathbb{R}$ s.t $\mathbf{c} = c_1 e_1 + c_2 e_2 + c_3 e_3$, $\mathbf{d} = d_3 e_3$.

$$\begin{aligned}
[\mathbf{d}, \mathbf{c}] &= [d_3 e_3, c_1 e_1 + c_2 e_2 + c_3 e_3] \\
&= [d_3 e_3, c_1 e_1] + [d_3 e_3, c_2 e_2] + [d_3 e_3, c_3 e_3] \\
&= d_3 c_1 [e_3, e_1] + d_3 c_2 [e_3, e_2] + d_3 c_3 [e_3, e_3] \\
&= 0.
\end{aligned}$$

$\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_0]$ なので, $\mathfrak{g}_2 = 0$.
(証明終了)

2.3 4 次元ベキ零リー環の分類

定理 2.4 ([2]) $\mathfrak{g}$ を可換でない 4 次元ベキ零リー環とするならば, $\mathfrak{g}$ は基底 $\{e_1$, e_2, e_3, $e_4\}$ は以下のいずれかを満たすリー環と同型である.

1. $[e_1, e_2] = e_4$, その他が 0 の場合,
2. $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, その他が 0 の場合.

続いて, 4 次元ベキ零リー環のステップ数を示す. 証明は命題 2.3 と同様に行えばよいので省略する.

命題 2.5 $\mathfrak{g}$ を 4 次元リー環とする. $\mathfrak{g}$ の括弧積が定理 2.4 の 1 の場合, $\mathfrak{g}$ は 2 -ステップベキ零リー環である.

命題 2.6 $\mathfrak{g}$ を 4 次元リー環とする. $\mathfrak{g}$ の括弧積が定理 2.4 の 2 の場合, $\mathfrak{g}$ は 3 -ステップベキ零リー環である.

2.4 5 次元ベキ零リー環の分類

定理 2.7 ([2]) $\mathfrak{g}$ を可換でない 5 次元ベキ零リー環とするならば, $\mathfrak{g}$ は基底 $\{e_1$, e_2, e_3, e_4, $e_5\}$ は以下のいずれかを満たすリー環と同型である.

1. $[e_1, e_2] = e_5$, その他が 0,

2. $[e_1, e_3] = e_5, [e_2, e_4] = e_5$, その他が 0,
3. $[e_1, e_2] = e_4, [e_1, e_3] = e_5$, その他が 0,
4. $[e_1, e_2] = e_4, [e_1, e_4] = e_5$, その他が 0,
5. $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5$, その他が 0,
6. $[e_1, e_2] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5$, その他が 0,
7. $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5$, その他が 0,
8. $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5$, その他が 0.

続いて, 5 次元ベキ零リー環のステップ数を示す. 証明は命題 2.3 と同様に行えばよいので省略する.

命題 2.8 $\mathfrak{g}$ を 5 次元リー環とする. $\mathfrak{g}$ の括弧積が定理 2.7 の 1, 2, 3 の場合, $\mathfrak{g}$ は 2 -ステップベキ零リー環である.

命題 2.9 $\mathfrak{g}$ を 5 次元リー環とする. $\mathfrak{g}$ の括弧積が定理 2.7 の 4, 5, 6 の場合, $\mathfrak{g}$ は 3 -ステップベキ零リー環である.

命題 2.10 $\mathfrak{g}$ を 5 次元リー環とする. $\mathfrak{g}$ の括弧積が定理 2.7 の 7, 8 の場合, $\mathfrak{g}$ は 4 -ステップベキ零リー環である.

3　3 次元ベキ零リー環の微分環

命題 3.1 $\mathfrak{g}$ を 3 次元ベキ零リー環とする．括弧積が $[e_1, e_2] = e_3$, その他が 0 となるような $\mathfrak{g}$ の基底 $\{e_1, e_2, e_3\}$ をとると,

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{11} + a_{22} \end{pmatrix} | a_{ij} \in \mathbb{R} \right\}.$$

となり $\mathrm{Der}(\mathfrak{g})$ は 6 次元である．

(証明開始)

$D(e_1) = a_{11}e_1 + a_{21}e_2 + a_{31}e_3$,

$D(e_2) = a_{12}e_1 + a_{22}e_2 + a_{32}e_3$,

$D(e_3) = a_{13}e_1 + a_{23}e_2 + a_{33}e_3$ とおく．$(a_{ij} \in \mathbb{R})$

D が $\mathrm{Der}(\mathfrak{g})$ に入るための必要十分条件は,

$[D(e_1), e_2] + [e_1, D(e_2)] = D[e_1, e_2]$,

$[D(e_1), e_3] + [e_1, D(e_3)] = D[e_1, e_3]$,

$[D(e_2), e_3] + [e_2, D(e_3)] = D[e_2, e_3]$ の 3 式を満たすこと．

$$\begin{aligned} [D(e_1), e_2] &= [a_{11}e_1 + a_{21}e_2 + a_{31}e_3, e_2] \\ &= [a_{11}e_1, e_2] + [a_{21}e_2, e_2] + [a_{31}e_3, e_2] \\ &= a_{11}[e_1, e_2] + a_{21}[e_2, e_2] + a_{31}[e_3, e_2] \\ &= a_{11}e_3 + O + O. \\ [e_1, D(e_2)] &= [e_1, a_{12}e_1 + a_{22}e_2 + a_{32}e_3] \\ &= [e_1, a_{12}e_1] + [e_1, a_{22}e_2] + [a_{32}e_1, e_3] \\ &= a_{12}[e_1, e_1] + a_{22}[e_1, e_2] + a_{32}[e_1, e_3] \\ &= O + a_{22}e_3 + O. \\ D[e_1, e_2] &= D(e_3) \\ &= a_{13}e_1 + a_{23}e_2 + a_{33}e_3. \end{aligned}$$

$$
\begin{aligned}
[D(e_1), e_3] &= [a_{11}e_1 + a_{21}e_2 + a_{31}e_3, e_3] \\
&= [a_{11}e_1, e_3] + [a_{21}e_2, e_3] + [a_{31}e_3, e_3] \\
&= a_{11}[e_1, e_3] + a_{21}[e_2, e_3] + a_{31}[e_3, e_3] \\
&= +O + O + O. \\
[e_1, D(e_3)] &= [e_1, a_{13}e_1 + a_{23}e_2 + a_{33}e_3] \\
&= [e_1, a_{13}e_1] + [e_1, a_{23}e_2] + [a_{33}e_1, e_3] \\
&= a_{13}[e_1, e_1] + a_{23}[e_1, e_2] + a_{33}[e_1, e_3] \\
&= O + a_{23}e_3 + O. \\
D[e_1, e_3] &= D(O) \\
&= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_2), e_3] &= [a_{12}e_1 + a_{22}e_2 + a_{32}e_3, e_3] \\
&= [a_{11}e_1, e_2] + [a_{21}e_2, e_2] + [a_{31}e_3, e_2] \\
&= a_{12}[e_1, e_3] + a_{22}[e_2, e_3] + a_{32}[e_3, e_3] \\
&= O + O + O. \\
[e_2, D(e_3)] &= [e_2, a_{13}e_1 + a_{23}e_2 + a_{33}e_3] \\
&= [e_2, a_{13}e_1] + [e_2, a_{23}e_2] + [e_2, a_{33}e_3] \\
&= a_{13}[e_2, e_1] + a_{23}[e_2, e_2] + a_{33}[e_2, e_3] \\
&= -a_{13}e_3 + O + O. \\
D[e_2, e_3] &= D(O) \\
&= O.
\end{aligned}
$$

(claim, (左辺) $\subset$ (右辺).)

$[D(e_1), e_2] + [e_1, D(e_2)] = D[e_1, e_2]$ ならば

$$(a_{11} + a_{22})e_3 = a_{13}e_1 + a_{23}e_2 + a_{33}e_3.$$

$\{\ e_1,\ e_2,\ e_3\ \}$ は基底なので $a_{13} = 0$ かつ $a_{23} = 0$ かつ $a_{11} + a_{22} = a_{33}$.

$[D(e_1), e_3] + [e_1, D(e_3)] = D[e_1, e_3]$ ならば

$$a_{23}e_3 = O.$$

$\{\ e_1,\, e_2,\, e_3\ \}$ は基底なので $a_{23} = 0$.

$[D(e_2), e_3] + [e_2, D(e_3)] = D[e_2, e_3]$ ならば

$$-a_{13}e_3 = O.$$

$\{\ e_1,\, e_2,\, e_3\ \}$ は基底なので $a_{13} = 0$.

(claim, (左辺) $\supset$ (右辺).)

$a_{13} = 0$ かつ $a_{23} = 0$ かつ $a_{11} + a_{22} = a_{33}$ ならば
$[D(e_1), e_2] + [e_1, D(e_2)] = D[e_1, e_2]$ が成立する.

$a_{23} = 0$ ならば $[D(e_1), e_3] + [e_1, D(e_3)] = D[e_1, e_3]$ が成立する.

$a_{13} = 0$ ならば $[D(e_2), e_3] + [e_2, D(e_3)] = D[e_2, e_3]$ が成立する.

双線型より, $\forall X, Y \in \mathfrak{g}$ に対し, $[D(X), Y] + [X, D(Y)] = D[X, Y]$ が成立する.

(証明終了)

4　4 次元ベキ零リー環の微分環

4.1　2 -ステップ 4 次元ベキ零リー環の微分環

命題 4.1 $\mathfrak{g}$ を 4 次元ベキ零リー環とする．括弧積が $[e_1, e_2] = e_4$, その他が 0 となるような $\mathfrak{g}$ の基底 $\{e_1, e_2, e_3, e_4\}$ をとると,

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 2a_{11} + a_{22} \end{pmatrix} | a_{ij} \in \mathbb{R} \right\}$$

となり $\mathrm{Der}(\mathfrak{g})$ は 10 次元である.

(証明開始)

$D(e_1) = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4$

$D(e_2) = a_{12}e_1 + a_{22}e_2 + a_{32}e_3 + a_{42}e_4$

$D(e_3) = a_{13}e_1 + a_{23}e_2 + a_{33}e_3 + a_{43}e_4$

$D(e_4) = a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4$ とおく．$(a_{ij} \in \mathbb{R})$

$$\begin{aligned} [D(e_1), e_2] &= a_{11}e_4. \\ [e_1, D(e_2)] &= a_{22}e_4. \\ D[e_1, e_2] &= a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_3] &= O. \\ [e_1, D(e_3)] &= a_{23}e_4. \\ D[e_1, e_3] &= O. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_4] &= O. \\ [e_1, D(e_4)] &= a_{24}e_4. \\ D[e_1, e_4] &= O. \end{aligned}$$

$$[D(e_2), e_3] = O.$$
$$[e_2, D(e_3)] = -a_{13}e_4.$$
$$D[e_2, e_3] = O.$$

$$[D(e_2), e_4] = O.$$
$$[e_2, D(e_4)] = -a_{14}e_4.$$
$$D[e_2, e_4] = O.$$

$$[D(e_3), e_4] = O.$$
$$[e_3, D(e_4)] = O.$$
$$D[e_3, e_4] = O.$$

証明は同様なので省略する.

(証明終了)

4.2 3 -ステップ 4 次元ベキ零リー環の微分環

命題 4.2 $\mathfrak{g}$ を 4 次元ベキ零リー環とする. 括弧積が $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, その他が 0 となるような $\mathfrak{g}$ の基底 $\{e_1, e_2, e_3, e_4\}$ をとると,

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{11}+a_{22} & 0 \\ a_{41} & a_{42} & a_{32} & 2a_{11}+a_{22} \end{pmatrix} | a_{ij} \in \mathbb{R} \right\}$$

となり $\mathrm{Der}(\mathfrak{g})$ は 7 次元である.

(証明開始)

$D(e_1) = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4$

$D(e_2) = a_{12}e_1 + a_{22}e_2 + a_{32}e_3 + a_{42}e_4$

$D(e_3) = a_{13}e_1 + a_{23}e_2 + a_{33}e_3 + a_{43}e_4$

$D(e_4) = a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4$ とおく. $(a_{ij} \in \mathbb{R})$

D が $\mathrm{Der}(\mathfrak{g})$ に入るための必要十分条件は,

$$
\begin{aligned}
[D(e_1), e_2] &= a_{11}e_3.\\
[e_1, D(e_2)] &= a_{22}e_3 + a_{32}e_4.\\
D[e_1, e_2] &= a_{13}e_1 + a_{23}e_2 + a_{33}e_3 + a_{43}e_4.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_1), e_3] &= a_{11}e_4.\\
[e_1, D(e_3)] &= a_{23}e_3 + a_{33}e_4.\\
D[e_1, e_3] &= a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_1), e_4] &= O.\\
[e_1, D(e_4)] &= a_{24}e_3 + a_{34}e_4.\\
D[e_1, e_4] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_2), e_3] &= a_{12}e_4.\\
[e_2, D(e_3)] &= -a_{13}e_3.\\
D[e_2, e_3] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_2), e_4] &= O.\\
[e_2, D(e_4)] &= -a_{14}e_3.\\
D[e_2, e_4] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_3), e_4] &= O.\\
[e_3, D(e_4)] &= -a_{14}e_4.\\
D[e_3, e_4] &= O.
\end{aligned}
$$

証明は同様なので省略する.

(証明終了)

5　5 次元ベキ零リー環の微分環

5.1　2 -ステップ 5 次元ベキ零リー環の微分環

命題 5.1 $\mathfrak{g}$ を 5 次元ベキ零リー環とする．括弧積が $[e_1, e_2] = e_5$, その他が 0 となるような $\mathfrak{g}$ の基底 $\{e_1, e_2, e_3, e_4, e_5\}$ をとると，

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{11}+a_{22} \end{pmatrix} \mid a_{ij} \in \mathbb{R} \right\}$$

となり $\mathrm{Der}(\mathfrak{g})$ は 16 次元である．

(証明開始)

$D(e_1) = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4 + a_{51}e_5$
$D(e_2) = a_{12}e_1 + a_{22}e_2 + a_{32}e_3 + a_{42}e_4 + a_{52}e_5$
$D(e_3) = a_{13}e_1 + a_{23}e_2 + a_{33}e_3 + a_{43}e_4 + a_{53}e_5$
$D(e_4) = a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4 + a_{54}e_5$
$D(e_5) = a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5$ とおく．$(a_{ij} \in \mathbb{R})$

$$\begin{aligned} [D(e_1), e_2] &= a_{11}e_5. \\ [e_1, D(e_2)] &= a_{22}e_5. \\ D[e_1, e_2] &= a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_3] &= O. \\ [e_1, D(e_3)] &= a_{23}e_5. \\ D[e_1, e_3] &= O. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_4] &= O. \\ [e_1, D(e_4)] &= a_{24}e_5. \\ D[e_1, e_4] &= O. \end{aligned}$$

$$[D(e_1), e_5] = O.$$
$$[e_1, D(e_5)] = a_{25}e_5.$$
$$D[e_1, e_5] = O.$$

$$[D(e_2), e_3] = O.$$
$$[e_2, D(e_3)] = -a_{13}e_5.$$
$$D[e_2, e_3] = O.$$

$$[D(e_2), e_4] = O.$$
$$[e_2, D(e_4)] = -a_{14}e_5.$$
$$D[e_2, e_4] = O.$$

$$[D(e_2), e_5] = O.$$
$$[e_2, D(e_5)] = -a_{15}e_5.$$
$$D[e_2, e_5] = O.$$

$$[D(e_3), e_4] = O.$$
$$[e_3, D(e_4)] = O.$$
$$D[e_3, e_4] = O.$$

$$[D(e_3), e_5] = O.$$
$$[e_3, D(e_5)] = O.$$
$$D[e_3, e_5] = O.$$

$$[D(e_4), e_5] = O.$$
$$[e_4, D(e_5)] = O.$$
$$D[e_4, e_5] = O.$$

証明は同様なので省略する.

(証明終了)

命題 5.2 $\mathfrak{g}$ を 5 次元ベキ零リー環とする. 括弧積が $[e_1, e_3] = e_5$, $[e_2, e_4] = e_5$, その他が 0 となるような $\mathfrak{g}$ の基底 $\{e_1, e_2, e_3, e_4, e_5\}$ をとると,

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{23} & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 \\ a_{31} & a_{41} & a_{33} & -a_{21} & 0 \\ a_{41} & a_{42} & a_{12} & a_{11} - a_{22} + a_{33} & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{11} + a_{33} \end{pmatrix} | a_{ij} \in \mathbb{R} \right\}$$

となり $\mathrm{Der}(\mathfrak{g})$ は 15 次元である.

(証明開始)

$D(e_1) = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4 + a_{51}e_5$

$D(e_2) = a_{12}e_1 + a_{22}e_2 + a_{32}e_3 + a_{42}e_4 + a_{52}e_5$

$D(e_3) = a_{13}e_1 + a_{23}e_2 + a_{33}e_3 + a_{43}e_4 + a_{53}e_5$

$D(e_4) = a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4 + a_{54}e_5$

$D(e_5) = a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5$ とおく. $(a_{ij} \in \mathbb{R})$

$$\begin{aligned} [D(e_1), e_2] &= -a_{41}e_5. \\ [e_1, D(e_2)] &= a_{32}e_5. \\ D[e_1, e_2] &= O. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_3] &= a_{11}e_5. \\ [e_1, D(e_3)] &= a_{33}e_5. \\ D[e_1, e_3] &= a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_4] &= a_{21}e_5. \\ [e_1, D(e_4)] &= a_{34}e_5. \\ D[e_1, e_4] &= O. \end{aligned}$$

$$
\begin{aligned}
[D(e_1), e_5] &= O. \\
[e_1, D(e_5)] &= a_{35}e_5. \\
D[e_1, e_5] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_2), e_3] &= a_{12}e_5. \\
[e_2, D(e_3)] &= a_{43}e_5. \\
D[e_2, e_3] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_2), e_4] &= a_{22}e_5. \\
[e_2, D(e_4)] &= a_{44}e_5. \\
D[e_2, e_4] &= a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_2), e_5] &= O. \\
[e_2, D(e_5)] &= a_{45}e_5. \\
D[e_2, e_5] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_3), e_4] &= a_{23}e_5. \\
[e_3, D(e_4)] &= -a_{14}e_5. \\
D[e_3, e_4] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_3), e_5] &= O. \\
[e_3, D(e_5)] &= -a_{15}e_5. \\
D[e_3, e_5] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_4), e_5] &= O. \\
[e_4, D(e_5)] &= -a_{25}e_5. \\
D[e_4, e_5] &= O.
\end{aligned}
$$

証明は同様なので省略する.

(証明終了)

命題 5.3 $\mathfrak{g}$ を 5 次元ベキ零リー環とする. 括弧積が $[e_1, e_2] = e_4$, $[e_1, e_3] = e_5$, その他が 0 となるような $\mathfrak{g}$ の基底 $\{e_1, e_2, e_3, e_4, e_5\}$ をとると,

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{11}+a_{22} & a_{23} \\ a_{51} & a_{52} & a_{53} & a_{32} & a_{11}+a_{33} \end{pmatrix} | a_{ij} \in \mathbb{R} \right\}$$

となり $\mathrm{Der}(\mathfrak{g})$ は 13 次元である.

(証明開始)

$D(e_1) = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4 + a_{51}e_5$

$D(e_2) = a_{12}e_1 + a_{22}e_2 + a_{32}e_3 + a_{42}e_4 + a_{52}e_5$

$D(e_3) = a_{13}e_1 + a_{23}e_2 + a_{33}e_3 + a_{43}e_4 + a_{53}e_5$

$D(e_4) = a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4 + a_{54}e_5$

$D(e_5) = a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5$ とおく. $(a_{ij} \in \mathbb{R})$

$$\begin{aligned} [D(e_1), e_2] &= a_{11}e_4. \\ [e_1, D(e_2)] &= a_{22}e_4 + a_{32}e_5. \\ D[e_1, e_2] &= a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4 + a_{54}e_5. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_3] &= a_{11}e_5. \\ [e_1, D(e_3)] &= a_{23}e_4 + a_{33}e_5. \\ D[e_1, e_3] &= a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_4] &= O. \\ [e_1, D(e_4)] &= a_{24}e_4 + a_{34}e_5. \\ D[e_1, e_4] &= O. \end{aligned}$$

$$
\begin{aligned}
[D(e_1), e_5] &= O. \\
[e_1, D(e_5)] &= a_{25}e_4 + a_{35}e_5. \\
D[e_1, e_5] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_2), e_3] &= a_{12}e_5. \\
[e_2, D(e_3)] &= -a_{13}e_4. \\
D[e_2, e_3] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_2), e_4] &= O. \\
[e_2, D(e_4)] &= -a_{14}e_4. \\
D[e_2, e_4] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_2), e_5] &= O. \\
[e_2, D(e_5)] &= -a_{15}e_4. \\
D[e_2, e_5] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_3), e_4] &= O. \\
[e_3, D(e_4)] &= -a_{14}e_5. \\
D[e_3, e_4] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_3), e_5] &= O. \\
[e_3, D(e_5)] &= -a_{15}e_5. \\
D[e_3, e_5] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_4), e_5] &= O. \\
[e_4, D(e_5)] &= O. \\
D[e_4, e_5] &= O.
\end{aligned}
$$

証明は同様なので省略する.

(証明終了)

5.2 3 -ステップ 5 次元ベキ零リー環の微分環

命題 5.4 $\mathfrak{g}$ を 5 次元ベキ零リー環とする．括弧積が $[e_1, e_2] = e_4$, $[e_1, e_4] = e_5$, その他が 0 となるような $\mathfrak{g}$ の基底 $\{e_1, e_2, e_3, e_4, e_5\}$ をとると，

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & 0 & a_{11} + a_{22} & 0 \\ a_{51} & a_{52} & a_{53} & a_{42} & 2a_{11} + a_{22} \end{pmatrix} \mid a_{ij} \in \mathbb{R} \right\}$$

となり $\mathrm{Der}(\mathfrak{g})$ は 11 次元である．

(証明開始)

$D(e_1) = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4 + a_{51}e_5$

$D(e_2) = a_{12}e_1 + a_{22}e_2 + a_{32}e_3 + a_{42}e_4 + a_{52}e_5$

$D(e_3) = a_{13}e_1 + a_{23}e_2 + a_{33}e_3 + a_{43}e_4 + a_{53}e_5$

$D(e_4) = a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4 + a_{54}e_5$

$D(e_5) = a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5$ とおく．($a_{ij} \in \mathbb{R}$)

$$\begin{aligned} [D(e_1), e_2] &= a_{11}e_4. \\ [e_1, D(e_2)] &= a_{22}e_4 + a_{32}e_5. \\ D[e_1, e_2] &= a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4 + a_{54}e_5. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_3] &= O. \\ [e_1, D(e_3)] &= a_{23}e_4 + a_{43}e_5. \\ D[e_1, e_3] &= O. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_4] &= a_{11}e_5. \\ [e_1, D(e_4)] &= a_{24}e_4 + a_{44}e_5. \\ D[e_1, e_4] &= a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5. \end{aligned}$$

$$[D(e_1), e_5] = O.$$
$$[e_1, D(e_5)] = a_{25}e_4 + a_{45}e_5.$$
$$D[e_1, e_5] = O.$$

$$[D(e_2), e_3] = O.$$
$$[e_2, D(e_3)] = -a_{13}e_4.$$
$$D[e_2, e_3] = O.$$

$$[D(e_2), e_4] = a_{12}e_5.$$
$$[e_2, D(e_4)] = -a_{14}e_4.$$
$$D[e_2, e_4] = O.$$

$$[D(e_2), e_5] = O.$$
$$[e_2, D(e_5)] = -a_{15}e_4.$$
$$D[e_2, e_5] = O.$$

$$[D(e_3), e_4] = a_{13}e_5.$$
$$[e_3, D(e_4)] = O.$$
$$D[e_3, e_4] = O.$$

$$[D(e_3), e_5] = O.$$
$$[e_3, D(e_5)] = O.$$
$$D[e_3, e_5] = O.$$

$$[D(e_4), e_5] = O.$$
$$[e_4, D(e_5)] = -a_{15}e_5.$$
$$D[e_4, e_5] = O.$$

証明は同様なので省略する.

(証明終了)

命題 5.5 $\mathfrak{g}$ を 5 次元リー環としてその基底を $\{e_1,\ e_2,\ e_3,\ e_4,\ e_5\}$ とする. 括弧積が $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_2, e_3] = e_5$, その他が 0 の場合

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}+a_{22} & 0 & 0 \\ a_{41} & a_{42} & a_{32} & 2a_{11}+a_{22} & a_{12} \\ a_{51} & a_{52} & -a_{31} & a_{21} & a_{11}+2a_{22} \end{pmatrix} | a_{ij} \in \mathbb{R} \right\}$$

となり $\mathrm{Der}(\mathfrak{g})$ は 10 次元である.

(証明開始)

$D(e_1) = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4 + a_{51}e_5$

$D(e_2) = a_{12}e_1 + a_{22}e_2 + a_{32}e_3 + a_{42}e_4 + a_{52}e_5$

$D(e_3) = a_{13}e_1 + a_{23}e_2 + a_{33}e_3 + a_{43}e_4 + a_{53}e_5$

$D(e_4) = a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4 + a_{54}e_5$

$D(e_5) = a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5$とおく.$(a_{ij} \in \mathbb{R})$

$$\begin{aligned} [D(e_1), e_2] &= a_{11}e_3 + O - a_{31}e_5. \\ [e_1, D(e_2)] &= a_{22}e_3 + a_{32}e_4. \\ D[e_1, e_2] &= a_{13}e_1 + a_{23}e_2 + a_{33}e_3 + a_{43}e_4 + a_{53}e_5. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_3] &= a_{11}e_4 + a_{21}e_5. \\ [e_1, D(e_3)] &= a_{23}e_3 + a_{33}e_4. \\ D[e_1, e_3] &= a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4 + a_{54}e_5. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_4] &= O. \\ [e_1, D(e_4)] &= a_{24}e_3 + a_{34}e_4. \\ D[e_1, e_4] &= O. \end{aligned}$$

$$
\begin{aligned}
[D(e_1), e_5] &= O. \\
[e_1, D(e_5)] &= a_{25}e_3 + a_{35}e_4. \\
D[e_1, e_5] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_2), e_3] &= a_{12}e_4 + a_{22}e_5. \\
[e_2, D(e_3)] &= -a_{13}e_3 + O + a_{33}e_5. \\
D[e_2, e_3] &= a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_2), e_4] &= O. \\
[e_2, D(e_4)] &= -a_{14}e_3 + a_{34}e_5. \\
D[e_2, e_4] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_2), e_5] &= O. \\
[e_2, D(e_5)] &= -a_{15}e_3 + a_{35}e_5. \\
D[e_2, e_5] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_3), e_4] &= O. \\
[e_3, D(e_4)] &= -a_{14}e_4 - a_{24}e_5. \\
D[e_3, e_4] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_3), e_5] &= O. \\
[e_3, D(e_5)] &= -a_{15}e_4 - a_{25}e_5. \\
D[e_3, e_5] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_4), e_5] &= O. \\
[e_4, D(e_5)] &= O. \\
D[e_4, e_5] &= O.
\end{aligned}
$$

証明は同様なので省略する.

(証明終了)

命題 5.6 $\mathfrak{g}$ を 5 次元リー環としてその基底を $\{e_1,\ e_2,\ e_3,\ e_4,\ e_5\}$ とする. 括弧積が $[e_1, e_2] = e_4$, $[e_1, e_4] = e_5$, $[e_2, e_3] = e_5$, その他が 0 の場合

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & 2a_{11} & 0 & 0 \\ a_{41} & a_{42} & -a_{21} & a_{11} + a_{22} & 0 \\ a_{51} & a_{52} & a_{53} & -a_{31} + a_{42} & 2a_{11} + a_{22} \end{pmatrix} | a_{ij} \in \mathbb{R} \right\}$$

となり $\mathrm{Der}(\mathfrak{g})$ は 10 次元である.

(証明開始)

$D(e_1) = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4 + a_{51}e_5$

$D(e_2) = a_{12}e_1 + a_{22}e_2 + a_{32}e_3 + a_{42}e_4 + a_{52}e_5$

$D(e_3) = a_{13}e_1 + a_{23}e_2 + a_{33}e_3 + a_{43}e_4 + a_{53}e_5$

$D(e_4) = a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4 + a_{54}e_5$

$D(e_5) = a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5$とおく.$(a_{ij} \in \mathbb{R})$

$$\begin{aligned} [D(e_1), e_2] &= a_{11}e_4 - a_{31}e_5. \\ [e_1, D(e_2)] &= a_{22}e_4 + a_{42}e_5. \\ D[e_1, e_2] &= a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4 + a_{54}e_5. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_3] &= a_{21}e_5. \\ [e_1, D(e_3)] &= a_{23}e_4 + a_{43}e_5. \\ D[e_1, e_3] &= O. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_4] &= a_{11}e_5. \\ [e_1, D(e_4)] &= a_{24}e_4 + a_{44}e_5. \\ D[e_1, e_4] &= a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5. \end{aligned}$$

$$
\begin{aligned}
[D(e_1), e_5] &= O. \\
[e_1, D(e_5)] &= a_{25}e_4 + a_{45}e_5. \\
D[e_1, e_5] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_2), e_3] &= a_{22}e_5. \\
[e_2, D(e_3)] &= -a_{13}e_4 + a_{33}e_5. \\
D[e_2, e_3] &= a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_2), e_4] &= a_{12}e_5. \\
[e_2, D(e_4)] &= -a_{14}e_4 + a_{34}e_5. \\
D[e_2, e_4] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_2), e_5] &= O. \\
[e_2, D(e_5)] &= -a_{15}e_4 + a_{35}e_5. \\
D[e_2, e_5] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_3), e_4] &= a_{13}e_5. \\
[e_3, D(e_4)] &= -a_{24}e_5. \\
D[e_3, e_4] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_3), e_5] &= O. \\
[e_3, D(e_5)] &= -a_{25}e_5. \\
D[e_3, e_5] &= O.
\end{aligned}
$$

$$
\begin{aligned}
[D(e_4), e_5] &= O. \\
[e_4, D(e_5)] &= -a_{15}e_5. \\
D[e_4, e_5] &= O.
\end{aligned}
$$

証明は同様なので省略する.

(証明終了)

5.3　4 -ステップ 5 次元ベキ零リー環の微分環

命題 5.7 $\mathfrak{g}$ を 5 次元リー環としてその基底を $\{e_1,\ e_2,\ e_3,\ e_4,\ e_5\}$ とする. 括弧積が $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_1, e_4] = e_5$, その他が 0 の場合

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}+a_{22} & 0 & 0 \\ a_{41} & a_{42} & a_{32} & 2a_{11}+a_{22} & 0 \\ a_{51} & a_{52} & a_{42} & a_{32} & 3a_{11}+a_{22} \end{pmatrix} | a_{ij} \in \mathbb{R} \right\}$$

となり $\mathrm{Der}(\mathfrak{g})$ は 9 次元である.

(証明開始)

$D(e_1) = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4 + a_{51}e_5$

$D(e_2) = a_{12}e_1 + a_{22}e_2 + a_{32}e_3 + a_{42}e_4 + a_{52}e_5$

$D(e_3) = a_{13}e_1 + a_{23}e_2 + a_{33}e_3 + a_{43}e_4 + a_{53}e_5$

$D(e_4) = a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4 + a_{54}e_5$

$D(e_5) = a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5$とおく.($a_{ij} \in \mathbb{R}$)

$$\begin{aligned} [D(e_1), e_2] &= a_{11}e_3. \\ [e_1, D(e_2)] &= a_{22}e_3 + a_{32}e_4 + a_{42}e_5. \\ D[e_1, e_2] &= a_{13}e_1 + a_{23}e_2 + a_{33}e_3 + a_{43}e_4 + a_{53}e_5. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_3] &= a_{11}e_4. \\ [e_1, D(e_3)] &= a_{23}e_3 + a_{33}e_4 + a_{43}e_5. \\ D[e_1, e_3] &= a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4 + a_{54}e_5. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_4] &= a_{11}e_5. \\ [e_1, D(e_4)] &= a_{24}e_3 + a_{34}e_4 + a_{44}e_5. \\ D[e_1, e_4] &= a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5. \end{aligned}$$

$$[D(e_1), e_5] = O.$$
$$[e_1, D(e_5)] = a_{25}e_3 + a_{35}e_4 + a_{45}e_5.$$
$$D[e_1, e_5] = O.$$

$$[D(e_2), e_3] = a_{12}e_4.$$
$$[e_2, D(e_3)] = -a_{13}e_3.$$
$$D[e_2, e_3] = O.$$

$$[D(e_2), e_4] = a_{12}e_5.$$
$$[e_2, D(e_4)] = -a_{14}e_3.$$
$$D[e_2, e_4] = O.$$

$$[D(e_2), e_5] = O.$$
$$[e_2, D(e_5)] = -a_{15}e_3.$$
$$D[e_2, e_5] = O.$$

$$[D(e_3), e_4] = a_{13}e_5.$$
$$[e_3, D(e_4)] = -a_{14}e_4.$$
$$D[e_3, e_4] = O.$$

$$[D(e_3), e_5] = O.$$
$$[e_3, D(e_5)] = -a_{15}e_4.$$
$$D[e_3, e_5] = O.$$

$$[D(e_4), e_5] = O.$$
$$[e_4, D(e_5)] = -a_{15}e_5.$$
$$D[e_4, e_5] = O.$$

証明は同様なので省略する.

(証明終了)

命題 5.8 $\mathfrak{g}$ を 5 次元リー環としてその基底を $\{e_1,\ e_2,\ e_3,\ e_4,\ e_5\}$ とする. 括弧積が $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_1, e_4] = e_5$, $[e_2, e_3] = e_5$, その他が 0 の場合

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & 2a_{11} & 0 & 0 & 0 \\ a_{31} & a_{32} & 3a_{11} & 0 & 0 \\ a_{41} & a_{42} & a_{32} & 4a_{11} & 0 \\ a_{51} & a_{52} & -a_{31} + a_{42} & a_{21} + a_{32} & 5a_{11} \end{pmatrix} | a_{ij} \in \mathbb{R} \right\}$$

となり $\mathrm{Der}(\mathfrak{g})$ は 8 次元である.

(証明開始)

$D(e_1) = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4 + a_{51}e_5$

$D(e_2) = a_{12}e_1 + a_{22}e_2 + a_{32}e_3 + a_{42}e_4 + a_{52}e_5$

$D(e_3) = a_{13}e_1 + a_{23}e_2 + a_{33}e_3 + a_{43}e_4 + a_{53}e_5$

$D(e_4) = a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4 + a_{54}e_5$

$D(e_5) = a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5$ とおく.$(a_{ij} \in \mathbb{R})$

$$\begin{aligned} [D(e_1), e_2] &= a_{11}e_3 - a_{31}e_5. \\ [e_1, D(e_2)] &= a_{22}e_3 + a_{32}e_4 + a_{42}e_5. \\ D[e_1, e_2] &= a_{13}e_1 + a_{23}e_2 + a_{33}e_3 + a_{43}e_4 + a_{53}e_5. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_3] &= a_{11}e_4 + a_{21}e_5. \\ [e_1, D(e_3)] &= a_{23}e_3 + a_{33}e_4 + a_{43}e_5. \\ D[e_1, e_3] &= a_{14}e_1 + a_{24}e_2 + a_{34}e_3 + a_{44}e_4 + a_{54}e_5. \end{aligned}$$

$$\begin{aligned} [D(e_1), e_4] &= a_{11}e_5. \\ [e_1, D(e_4)] &= a_{24}e_3 + a_{34}e_4 + a_{44}e_5. \\ D[e_1, e_4] &= a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5. \end{aligned}$$

$$\begin{aligned}[D(e_1), e_5] &= O.\\ [e_1, D(e_5)] &= a_{25}e_3 + a_{35}e_4 + a_{45}e_5.\\ D[e_1, e_5] &= O.\end{aligned}$$

$$\begin{aligned}[D(e_2), e_3] &= a_{12}e_4 + a_{22}e_5.\\ [e_2, D(e_3)] &= -a_{13}e_3 + a_{33}e_5.\\ D[e_2, e_3] &= a_{15}e_1 + a_{25}e_2 + a_{35}e_3 + a_{45}e_4 + a_{55}e_5.\end{aligned}$$

$$\begin{aligned}[D(e_2), e_4] &= a_{12}e_5.\\ [e_2, D(e_4)] &= -a_{14}e_3 + a_{34}e_5.\\ D[e_2, e_4] &= O.\end{aligned}$$

$$\begin{aligned}[D(e_2), e_5] &= O.\\ [e_2, D(e_5)] &= -a_{15}e_3 + a_{35}e_5.\\ D[e_2, e_5] &= O.\end{aligned}$$

$$\begin{aligned}[D(e_3), e_4] &= a_{13}e_5.\\ [e_3, D(e_4)] &= -a_{14}e_4 - a_{24}e_5.\\ D[e_3, e_4] &= O.\end{aligned}$$

$$\begin{aligned}[D(e_3), e_5] &= O.\\ [e_3, D(e_5)] &= -a_{15}e_4 - a_{25}e_5.\\ D[e_3, e_5] &= O.\end{aligned}$$

$$\begin{aligned}[D(e_4), e_5] &= O.\\ [e_4, D(e_5)] &= -a_{15}e_5.\\ D[e_4, e_5] &= O.\end{aligned}$$

証明は同様なので省略する.

(証明終了)

6　3 次元単純リー環の微分環

定義 6.1 $\mathfrak{o}(3) := \{X \in \mathfrak{gl}_3(\mathbb{R}) \mid {}^t X + X = 0\}$ を 3 次元直交リー環と呼ぶ.

命題 6.2 $\mathfrak{o}(3)$ の基底を $X = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ とすると

$$\mathrm{Der}(\mathfrak{o}(3)) = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix} \mid a_{ij} \in \mathbb{R} \right\}.$$

となり $\mathrm{Der}(\mathfrak{o}(3))$ は 3 次元である.

(証明開始)

$D(X) = a_{11}X + a_{21}Y + a_{31}Z,$

$D(Y) = a_{12}X + a_{22}Y + a_{32}Z,$

$D(Z) = a_{13}X + a_{23}Y + a_{33}Z$ とおく $(a_{ij} \in \mathbb{R})$.

$$\begin{aligned} [D(X), Y] &= -a_{31}X + a_{11}Z. \\ [X, D(Y)] &= -a_{32}Y + a_{22}Z. \\ D[X, Y] &= a_{13}X + a_{23}Y + a_{33}Z. \end{aligned}$$

$$\begin{aligned} [D(X), Z] &= a_{21}X - a_{11}Y. \\ [X, D(Z)] &= -a_{33}Y + a_{23}Z. \\ D[X, Z] &= -a_{12}X - a_{22}Y - a_{32}Z. \end{aligned}$$

$$\begin{aligned} [D(Y), Z] &= a_{22}X - a_{12}Y. \\ [X, D(Z)] &= a_{33}X - a_{13}Z. \\ D[X, Z] &= a_{11}X + a_{21}Y + a_{31}Z. \end{aligned}$$

証明は同様なので省略する.
(証明終了)

定義 6.3 $\mathfrak{sl}_2(\mathbb{R}) := \{X \in \mathfrak{gl}_2(\mathbb{R}) \mid tr(X) = 0\}$ を 3 次元特殊線型リー環と呼ぶ.

命題 6.4 $\mathfrak{sl}_2(\mathbb{R})$ の基底を $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ とすると

$$\mathrm{Der}(\mathfrak{sl}_2(\mathbb{R})) = \left\{ \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & -a_{11} & a_{23} \\ -2a_{23} & -2a_{13} & 0 \end{pmatrix} | a_{ij} \in \mathbb{R} \right\}$$

となり $\mathrm{Der}(\mathfrak{sl}_2(\mathbb{R}))$ は 3 次元である.

(証明開始)
$D(X) = a_{11}X + a_{21}Y + a_{31}Z,$
$D(Y) = a_{12}X + a_{22}Y + a_{32}Z,$
$D(Z) = a_{13}X + a_{23}Y + a_{33}Z$ とおく $(a_{ij} \in \mathbb{R})$.

$$\begin{aligned} [D(X), Y] &= -a_{31}Y + 2a_{11}Z. \\ [X, D(Y)] &= -a_{32}X + 2a_{22}Z. \\ D[X, Y] &= 2a_{13}X + 2a_{23}Y + 2a_{33}Z. \end{aligned}$$

$$\begin{aligned} [D(X), Z] &= -a_{11}X + a_{21}Y. \\ [X, D(Z)] &= -a_{33}X + 2a_{23}Z. \\ D[X, Z] &= -a_{11}X - a_{21}Y - a_{31}Z. \end{aligned}$$

$$\begin{aligned} [D(Y), Z] &= -a_{12}X + a_{22}Y. \\ [X, D(Z)] &= a_{33}Y - 2a_{13}Z. \\ D[X, Z] &= a_{12}X + a_{22}Y + a_{32}Z. \end{aligned}$$

証明は同様なので省略する.
(証明終了)

7　終わりに

最後になりましたが, この場を借りて, 本論文を作成するにあたり, 指導教員の田丸博士先生をはじめ, 田丸ゼミに参加していた方々に感謝を申し上げます.

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