Introduction to Riemannian symmetric spaces and R-spaces

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0 Preface

0.1 Topic of this lecture

- **Riemannian symmetric spaces:**
  - **an important class of Riemannian manifolds,**
  - **a beautiful theory (a beautiful application of Lie groups),**
  - **many applications (for example, to submanifold geometry).**

0.2 Plan of this lecture

- **Part 1. Homogeneous spaces.**
  - **Ref.: Foundations of Differentiable Manifolds and Lie Groups (by Warner)**
- **Part 2. Symmetric spaces.**
  - **Ref.: Differential geometry, Lie groups, and symmetric spaces (by Helgason)**
- **Part 3. R-spaces.**
  - **Ref.: I do not know good textbooks...**

1 Homogeneous spaces

- **Summary of Part 1:**
  - M : a homogeneous manifold (i.e., $\exists$ a transitive action)

    $\Leftrightarrow$ there is an expression $M = G/K$.

- **Contents:**
  - Group actions & transitive actions
  - G/K as a set
  - G/K as a manifold & main result
  - Some applications (homogeneous submanifolds)

1.1 Group actions (1/3)

- **Def.:**
  - G : a group, M : a set, $\mathrm{Aut}(M) := \{f : M \to M \mid \text{bijective}\}$.
  - A group homomorphism $\varphi : G \to \mathrm{Aut}(M)$ is called an action.
  - Notations: $G \curvearrowright M$, $g.p := \varphi_g(p)$, ...

- **Example:**
  - $\mathrm{GL}_n(\mathbb{R}) \curvearrowright \mathbb{R}^n$,
  - $\mathrm{GL}_n(\mathbb{R}) \curvearrowright G_k(\mathbb{R}^n) := \{V \subset \mathbb{R}^n \mid \dim(V) = k\}$ (Grassmann),
  - $\mathrm{GL}_n(\mathbb{R}) \curvearrowright M_n(\mathbb{R})$ by conjugation ($g.X := gXg^{-1}$).
  - $\mathrm{SL}_2(\mathbb{R}) \curvearrowright \mathbb{R}\mathrm{H}^2 := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ (upper half plane) by the linear fractional transformation.

1.2 Group actions (2/3)

- **Lem.:** If $G \curvearrowright M$, then
  - $G \supset H$: subgroup $\Longrightarrow$ $H \curvearrowright M$,
  - $M \supset M'$: preserved by G $\Longrightarrow$ $G \curvearrowright M'$.

- **Example:**
  - $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SO}(n) \curvearrowright \mathbb{R}^n$,
  - $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SO}(n) \curvearrowright G_k(\mathbb{R}^n)$,
  - $\mathrm{O}(n)$, $\mathrm{SO}(n) \curvearrowright S^{n-1}$ $(\subset \mathbb{R}^n)$,
  - $\mathrm{O}(n)$, $\mathrm{SO}(n) \curvearrowright \mathrm{sym}_n^0(\mathbb{R})$ by conjugation,

    where $\mathrm{sym}_n^0(\mathbb{R}) := \{X \in M_n(\mathbb{R}) \mid {}^tX = X,\ \mathrm{tr}(X) = 0\}$.

1.3 Group actions (3/3)

- **Def.:**
  - $G \curvearrowright M$: **transitive**

    $:\Leftrightarrow \forall p, q \in M, \ \exists g \in G \ : \ g.p = q$

    $(\Leftrightarrow \exists o \in M \ : \ \forall p \in M, \ \exists g \in G \ : \ g.p = o)$.

- **Example: The following actions are transitive:**
  - $\mathrm{GL}_n(\mathbb{R}), \mathrm{SL}_n(\mathbb{R}), \mathrm{O}(n), \mathrm{SO}(n) \curvearrowright G_k(\mathbb{R}^n)$,
  - $\mathrm{O}(n), \mathrm{SO}(n) \curvearrowright S^{n-1} \ (\subset \mathbb{R}^n)$,
  - $\mathrm{SL}_2(\mathbb{R}) \curvearrowright \mathbb{R}\mathrm{H}^2$.

1.4 Homogeneous sets (1/3)

[Main Thm.: M : homogeneous set $\Leftrightarrow$ $M = G/K$.]

▼ **Def.:**

- M is called a **homogeneous set** with respect to G

  $:\Leftrightarrow$ $\exists$ a transitive action $G \curvearrowright M$.

▼ **Def.:**

- Let G : a group, K : a subgroup.
- Define $g \sim h :\Leftrightarrow g^{-1}h \in K$ (an equivalent relation).
- The quotient space $G/K := G/\sim$ is called the **coset space**.

  (Note: $[g] = gK$)

1.5 Homogeneous sets (2/3)

[Main Thm.: M : homogeneous set $\Leftrightarrow$ $M = G/K$.]

▼ **Thm.:**

- M : **a homogeneous set with respect to** G

  $\Rightarrow$ $M = G/G_p$ **(bijective),**

  where $G_p := \{g \in G \mid g.p = p\}$: **the isotropy subgroup at** p**.**

- G/K **is a homogeneous set with respect to** G**.**

▼ **Example:**

- $G_k(\mathbb{R}^n) = \mathrm{GL}_n(\mathbb{R})/(*)$,
- $S^{n-1} = \mathrm{O}(n)/\mathrm{O}(n-1)$,
- $\mathbb{R}\mathrm{H}^2 = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$.

1.6 Homogeneous sets (3/3)

[Main Thm.: M : homogeneous set $\Leftrightarrow M = G/K$.]

- **Lem.:**

  - $G \supset H$: a subgroup, $G, H \curvearrowright M$: transitive.

    $\Rightarrow M = G/G_p = H/(G_p \cap H)$.

- **Example:**

  - $$G_k(\mathbb{R}^n) = \mathrm{SL}_n(\mathbb{R})/(*)$$
    $$= \mathrm{O}(n)/\mathrm{O}(k) \times \mathrm{O}(n-k)$$
    $$= \mathrm{SO}(n)/\mathrm{S}(\mathrm{O}(k) \times \mathrm{O}(n-k)).$$

    (Notation: $\mathrm{S}(\mathrm{O}(k) \times \mathrm{O}(n-k)) := \mathrm{SO}(n) \cap (\mathrm{O}(k) \times \mathrm{O}(n-k))$.)

  - $S^{n-1} = \mathrm{O}(n)/\mathrm{O}(n-1) = \mathrm{SO}(n)/\mathrm{SO}(n-1)$.

1.7 Homogeneous manifolds (1/3)

[Aim: M : homogeneous mfd $\Leftrightarrow M = G/K$.]

▼ **Def.:**

- **An action of a Lie group $G \curvearrowright M$: smooth**

  $:\Leftrightarrow \varphi_g : M \to M$ is smooth ($\forall g \in G$).

▼ **Example:**

- **All actions we mentioned before are smooth.**

▼ **Def.:**

- **M is called a homogeneous manifold with respect to G**

  $:\Leftrightarrow \exists$ a transitive smooth action $G \curvearrowright M$.

1.8 Homogeneous manifolds (2/3)

[Main Thm.: M : homogeneous mfd $\Leftrightarrow$ $M = G/K$.]

- **Prop.:**
  - G : a Lie group, K : a closed subgroup.

    $\Rightarrow$ $\exists 1$ the manifold structure on G/K s.t. $G \curvearrowright M$: smooth.

- **Thm. (homogeneous $\Leftrightarrow$ $M = G/K$):**
  - M : a homogeneous manifold with respect to G

    $\Rightarrow$ $M = G/G_p$ (diffeo.), where $G_p := \{g \in G \mid g.p = p\}$.
  - G/K is a homogeneous manifold with respect to G.

1.9 Homogeneous manifolds (3/3)

[Main Thm.: M : homogeneous mfd $\Leftrightarrow$ $M = G/K$.]

- **How to define a mfd structure on G/K:**
  - **We use**
    - — $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ **(a direct sum decomposition),**
    - — $\exp : \mathfrak{g} \to G$ **(the exponential map),**
    - — $\pi : G \to G/K$ **(the natural projection).**
  - $\pi \circ \exp|_{\mathfrak{m}} : \mathfrak{m} \to G/K$ **defines a local chart around $[e]$.**
- **Cor.:**
  - $\mathfrak{m} \cong T_o(G/K)$ **(a natural identification).**

1.10 Some applications

- **Def.:**
  - **Let $G \curvearrowright M$, $p \in M$.**
  - **$G.p := \{g.p \in M \mid g \in G\}$ is called the orbit.**
- **Fact:**
  - **$M \supset G.p$: a submanifold.**
- **Example:**
  - **Orbits of $SO(3) \curvearrowright \mathrm{sym}_3^0(\mathbb{R}) \cong \mathbb{R}^5$**

    (Veronese surface, Cartan hypersurface)

2 Symmetric spaces

- **Summary of Part 2:**
  - M : a Riemannian symmetric space

    $\leftrightarrow$ (G, K, σ) : a symmetric pair $\leftrightarrow$ $(\mathfrak{g}, \mathfrak{k}, \theta)$: a symmetric pair

- **Contents:**
  - Definition
  - (G, K, σ) : Symmetric pairs of Lie groups
  - $(\mathfrak{g}, \mathfrak{k}, \theta)$: Symmetric pairs of Lie algebras
  - Algebraic structures - Cartan decompositions, Rank, Roots
  - Some applications - Iwasawa decomposition

2.1 Definition (1/2)

[Aim: define symmetric spaces]

▼ **Setting:**

- $M = (M, g)$: a connected Riemannian manifold.

▼ **Def.:**

- An isometry $s_p : M \to M$ is called a **symmetry** at p

  $:\Leftrightarrow$ $s_p^2 = \mathrm{id}$, $p \in \mathrm{Fix}(s_p, M)$ is isolated.

▼ **Def.:**

- (M, g) is called a **symmetric space**

  $:\Leftrightarrow$ $\forall p \in M$, $\exists s_p$: symmetry at p.

2.2 Definition (2/2)

[Aim: define symmetric spaces]

▼ **Example:**

The following spaces are symmetric:

- $\mathbb{R}^n$ $(s_p(x) := 2p - x)$,
- S^n $(s_p(x) := -x + 2\langle x, p\rangle p)$,
- $G_k(\mathbb{R}^n)$,
- G : **any compact Lie group with a bi-invariant metric** $(s_p(x) := px^{-1}p)$.

2.3 Symmetric pairs of Lie groups (1/3)

[Aim: symmetric spaces ↔ symmetric pairs]

▼ **Def.:**

- (G, K, σ) : a **Riemannian symmetric pair**

  :⇔
  - G : a connected Lie group,
  - K : a compact subgroup of G,
  - σ : an involutive (i.e., $\sigma^2 = \mathrm{id}$) automorphism of G,
  - $\mathrm{Fix}(\sigma, G)^0 \subset K \subset \mathrm{Fix}(\sigma, G)$.

▼ **Remark:**

- K : compact $\longleftrightarrow$ $\exists$ invariant Riemannian metric.
- $\mathrm{Fix}(\sigma, G)^0$: the connected component containing $e \in G$.

2.4 Symmetric pairs of Lie groups (2/3)

[Aim: symmetric spaces ↔ symmetric pairs]

▼ **Example: The following (G, K, σ) are symmetric pairs:**

- $(\mathrm{SL}_n(\mathbb{R}), \mathrm{SO}(n), \sigma)$, $\quad \sigma(g) := {}^t g^{-1}$.
- $S^n = (\mathrm{SO}(n+1), \mathrm{SO}(n), \sigma)$, $\quad \sigma(g) := I_{1,n} g I_{1,n}$.
- $\mathbb{R}\mathrm{P}^n = (\mathrm{SO}(n+1), \mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(n)), \sigma)$, $\quad \sigma(g) := I_{1,n} g I_{1,n}$.
- $G_a(\mathbb{R}^{a+b}) = (\mathrm{SO}(a+b), \mathrm{S}(\mathrm{O}(a) \times \mathrm{O}(b)), \sigma)$, $\quad \sigma(g) := I_{a,b} g I_{a,b}$.
- $G_a(\mathbb{C}^{a+b}) = (\mathrm{SU}(a+b), \mathrm{S}(\mathrm{U}(a) \times \mathrm{U}(b)), \sigma)$, $\quad \sigma(g) := I_{a,b} g I_{a,b}$.

▼ **Remark:**

- $G = \mathrm{SO}(n)$, $\sigma(g) := I_{1,n} g I_{1,n}$,

  $\Rightarrow \ \mathrm{Fix}(\sigma, G)^0 = \mathrm{SO}(n) \subsetneq \mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(n)) = \mathrm{Fix}(\sigma, G)$.

2.5 Symmetric pairs of Lie groups (3/3)

[Aim: symmetric spaces ↔ symmetric pairs]

- **Thm.:**
  - (M, g) : a symmetric space

    $\Rightarrow (\mathrm{Isom}(M, g)^0, \mathrm{Isom}(M, g)^0_x, s_x)$: a symmetric pair.
  - (G, K, σ) : a symmetric pair

    $\Rightarrow M := G/K$ becomes a symmetric space s.t. $s_o = \widetilde{\sigma}$.
- **Note:**
  - $\widetilde{\sigma} : G/K \to G/K : [g] \mapsto [\sigma(g)]$.

2.6 Symmetric pairs of Lie algebras (1/3)

[Aim: $(G, K, \sigma) \leftrightarrow$ symmetric pairs of Lie algebras $(\mathfrak{g}, \mathfrak{k}, \theta)$]

▼ **Def.:**

- $(\mathfrak{g}, \mathfrak{k}, \theta)$: a **Riemannian symmetric pair**

  $:\Leftrightarrow$
  - $\mathfrak{g}$: a Lie algebra,
  - $\mathfrak{k}$: a compact Lie subalgebra of $\mathfrak{g}$,
  - θ : an involutive (i.e., $\theta^2 = \mathrm{id}$) automorphism of $\mathfrak{g}$,
  - $\mathrm{Fix}(\theta, \mathfrak{g}) = \mathfrak{k}$.

▼ **Note:**

- $\mathfrak{k}$: compact $:\Leftrightarrow$ $\exists K$: a compact Lie group s.t. $\mathrm{Lie}(K) = \mathfrak{k}$.

2.7 Symmetric pairs of Lie algebras (2/3)

[Aim: $(G, K, \sigma) \leftrightarrow$ symmetric pairs of Lie algebras $(\mathfrak{g}, \mathfrak{k}, \theta)$]

- **Example:** The following $(\mathfrak{g}, \mathfrak{k}, \theta)$ are symmetric pairs:
  - $(\mathfrak{sl}_n(\mathbb{R}), \mathfrak{o}(n), \theta)$, $\quad \theta(X) := -{}^tX$.
  - $(\mathfrak{o}(n+1), \mathfrak{o}(n), \theta)$, $\quad \theta(X) := I_{1,n} X I_{1,n}$.

2.8 Symmetric pairs of Lie algebras (3/3)

[Aim: $(G, K, \sigma) \leftrightarrow$ symmetric pairs of Lie algebras $(\mathfrak{g}, \mathfrak{k}, \theta)$]

- **Thm.:**
  - **(G, K, σ) : a symmetric pair**

    $\Rightarrow$ $(\mathrm{Lie}(G), \mathrm{Lie}(K), d\sigma)$: a symmetric pair.
  - **All symmetric pairs $(\mathfrak{g}, \mathfrak{k}, \theta)$ can be constructed in this way.**

- **Remark:**
  - **$(G, K, \sigma) \leftrightarrow (\mathfrak{g}, \mathfrak{k}, \theta)$ is not one-to-one.**

    (e.g., $(\mathrm{SO}(n+1), \mathrm{SO}(n), \sigma)$, $(\mathrm{SO}(n+1), \mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(n)), \sigma)$ give the same $(\mathfrak{o}(n+1), \mathfrak{o}(n), \theta)$)

2.9 Algebraic structures (1/4) - Cartan decomposition

- **Def.:**
  - For $(\mathfrak{g}, \mathfrak{k}, \theta)$, let $\mathfrak{p} := \{X \in \mathfrak{g} \mid \theta(X) = -X\}$.
  - Then, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the **Cartan decomposition** (or the **canonical decomposition**).
- **Remark:**
  - $T_o(G/K) \cong \mathfrak{p}$ (the natural identification).
- **Example:**
  - $\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{o}(n) \oplus \mathrm{sym}_n^0(\mathbb{R})$.
  - $\mathfrak{o}(n+1) = \mathfrak{o}(n) \oplus \mathfrak{p}, \quad \mathfrak{p} \cong \mathbb{R}^n$.
  - $\mathfrak{o}(a+b) = (\mathfrak{o}(a) \oplus \mathfrak{o}(b)) \oplus \mathfrak{p}, \quad \mathfrak{p} \cong M_{a,b}(\mathbb{R})$.

2.10 Algebraic structures (2/4) - Rank

- **Thm.:**
  - Any maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ are conjugate.
- **Def.:**
  - $\mathrm{rank}\,(M) := \dim \mathfrak{a}$, which is called the **rank**.
- **Example:**
  - $\mathrm{rank}\,(\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)) = n - 1.$
  - $\mathrm{rank}\,(S^n) = 1.$
  - $\mathrm{rank}\,(G_a(\mathbb{R}^{a+b})) = \min\{a, b\}.$

2.11 Algebraic structures (3/4) - Duality

- **Prop.:**
  - $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: the Cartan decomposition of $(\mathfrak{g}, \mathfrak{k}, \theta)$

    $\Rightarrow$ • $\mathfrak{g}^d := \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}$ is a Lie algebra,

    • $(\mathfrak{g}^d, \mathfrak{k}, \theta)$: a symmetric pair.

- **Def.:**
  - $(\mathfrak{g}^d, \mathfrak{k}, \theta)$ is called the dual of $(\mathfrak{g}, \mathfrak{k}, \theta)$.

- **Example:**
  - S^n (or $\mathbb{R}P^n$) and $\mathbb{R}H^n$ are dual.
  - $\mathbb{C}P^n$ and $\mathbb{C}H^n$ are dual.

2.12 Algebraic structures (4/4) - Root systems

- **Def.:**
  - Let $\mathfrak{p} \supset \mathfrak{a}$: a maximal abelian subspace.
  - $\alpha : \mathfrak{a} \to \mathbb{R}$ $(\alpha \neq 0)$ is called a **root**

    $:\Leftrightarrow$ $0 \neq \mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \ (\forall H \in \mathfrak{a})\}$.
  - $\Delta := \{\alpha : \text{root}\}$ is called the **root system**.
- **Example:**
  - Roots for $(\mathfrak{sl}_n(\mathbb{R}), \mathfrak{o}(n), \theta)$ ...
- **Thm.:**
  - If it is "noncompact type", then $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$.

2.13 Some applications - Iwasawa decomposition (1/2)

- **Def.:**
  - $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$: a set of **simple roots**

    $:\Leftrightarrow \ \forall \alpha \in \Delta, \ \exists c_1, \ldots, c_r \in \mathbb{Z}_{\geq 0} \ : \ \alpha = \pm(c_1\alpha_1 + \cdots + c_r\alpha_r).$

- **Def.:**
  - $\alpha \in \Delta$ is **positive** with respect to $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$

    $:\Leftrightarrow \ \exists c_1, \ldots, c_r \in \mathbb{Z}_{\geq 0} \ : \ \alpha = c_1\alpha_1 + \cdots + c_r\alpha_r.$

- **Example:**
  - $SL_n(\mathbb{R})/SO(n)$ is easy to describe...

2.14 Some applications - Iwasawa decomposition (2/2)

- **Thm.:**

  If it is noncompact type, then

  - $\mathfrak{n} := \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$ **: a (nilpotent) Lie algebra.**
  - $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ **(as vector space), called the Iwasawa decomposition.**

- **Example:**

  - $SL_n(\mathbb{R})/SO(n)$ **is easy to describe...**

- **Thm.:**

  - $AN \curvearrowright M = G/K$ **: transitive.**
  - $M \cong AN$ **(called the solvable model).**

3 R-spaces

- **Summary of Part 3:**
  - $M = G/K$ **: a symmetric space of rank 2**

    $\longrightarrow$ **homogeneous hypersurfaces in spheres**

- **Contents:**
  - **Examples**
  - **Isotropy representations**
  - **Orbit type**
  - **Homogeneous hypersurfaces in spheres**
  - **Appendix: parabolic subgroups**

3.1 Examples

- **Recall:**
  - $\mathrm{SO}(3) \curvearrowright \mathrm{sym}^0_3(\mathbb{R}) = \mathbb{R}^5$ **by conjugation**

    $\Rightarrow$ **orbit** $= \mathbb{R}\mathrm{P}^2, \mathrm{SO}(3)/\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(1) \times \mathrm{O}(1)) \subset S^4.$

- **Recall:**
  - $\mathfrak{sl}_3(\mathbb{R}) = \mathfrak{o}(3) \oplus \mathrm{sym}^0_3(\mathbb{R})$

    is the Cartan decomposition of $\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$.

- **Comment:**
  - **This is a typical example of R-spaces.**

3.2 Isotropy representations (1/3)

- **Def.:**
  - Let $M = G/K$: a symmetric space,

    $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: the Cartan decomposition.
  - Then, $K \curvearrowright \mathfrak{p}$ by Ad is called the isotropy representation.
- **Note:**
  - $\forall g \in G,\ \mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$: the adjoint representation.
  - $G \subset \mathrm{GL}_n(\mathbb{R}) \implies \mathrm{Ad}_g(X) = gXg^{-1}$.
  - $g \in K \implies \mathrm{Ad}_g(\mathfrak{k}) = \mathfrak{k},\ \mathrm{Ad}_g(\mathfrak{p}) = \mathfrak{p}$.

3.3 Isotropy representations (2/3)

- **Example:**
  - $SL_n(\mathbb{R})/SO(n)$.
  - $SO(n+1)/SO(n)$.

- **Prop.:**
  - $(\mathfrak{g}', \mathfrak{k})$: the dual of $(\mathfrak{g}, \mathfrak{k})$

    $\Rightarrow$ their isotropy representations are equivariant,

    i.e., the natural map $\mathfrak{p} \to \sqrt{-1}\mathfrak{p}$ commutes with the actions.

- **Def.:**
  - R-spaces

    $:=$ orbits of the isotropy representations of symmetric spaces.

3.4 Isotropy representations (3/3)

- **Def. (in general):**
  - **Let $M = G/K$: a homogeneous space.**
  - $K \curvearrowright T_oM$ **by** $(dg)_o : T_oM \to T_oM$ $(g \in K)$

    is called the isotropy representation.

- **Prop.:**
  - $M = G/K$ **: symmetric,**

    $\Rightarrow$ **these two isotropy representations are equivariant.**

    (Recall: $\mathfrak{p} \cong T_o(G/K)$ **: the natural identification.)**

3.5 Orbit type (1/3)

- **Example:**
  - **$SO(3) \curvearrowright \mathrm{sym}_3^0(\mathbb{R}) = \mathbb{R}^5$ by conjugation**

    $\Rightarrow$ every orbit through a diagonal matrix.

  ($\because$) Any symmetric matrix is diagonalizable. $\square$

- **Prop.:**
  - **$\mathrm{Ad} : K \curvearrowright \mathfrak{p}$: an isotropy representation,**
  - **$\mathfrak{p} \supset \mathfrak{a}$: a maximal abelian subspace**

    $\Rightarrow$ every orbit through $\exists H \in \mathfrak{a}$.

    (I.e., we have only to consider the orbits through $H \in \mathfrak{a}$.)

3.6 Orbit type (2/3)

- **Prop.:**
  - Ad : $K \curvearrowright \mathfrak{p}$: an isotropy representation, $H \in \mathfrak{a}$

    $\Rightarrow$ $\mathrm{Lie}(K_H) = \mathfrak{k} \cap (\mathfrak{g}_0 \oplus \bigoplus_{\alpha(H)=0} \mathfrak{g}_\alpha)$.

- **Proof:**
  - (Step 1) $\mathrm{Lie}(K_H) = \{X \in \mathfrak{k} \mid [X, H] = 0\}$.
  - (Step 2) $[X, H] = 0 \Leftrightarrow X \in \mathfrak{g}_0 \oplus \bigoplus_{\alpha(H)=0} \mathfrak{g}_\alpha$. $\square$

3.7 Orbit type (3/3)

- **Example:**
  - **SO(3) $\curvearrowright$ $\mathrm{sym}_3^0(\mathbb{R}) = \mathbb{R}^5$ by conjugation**

    $\Rightarrow$ ... (see Whiteboard)

3.8 Homogeneous hypersurfaces in spheres (1/3)

- **Prop.:**
  - rank $(G/K) = r$

    $\Rightarrow$ $\dim \mathfrak{p} - \dim K.H \geq r$ ("=" $\Leftrightarrow$ H is regular).

- **Example:**
  - $SO(3) \curvearrowright \mathrm{sym}_3^0(\mathbb{R}) = \mathbb{R}^5$ by conjugation

    $\Rightarrow$ ... (see Whiteboard)

- **Proof of Prop.:**
  - $\dim K.H = \dim K - \dim K_H$.
  - $\mathrm{Lie}(K_H) = \mathfrak{k} \cap (\mathfrak{g}_0 \oplus \bigoplus_{\alpha(H)=0} \mathfrak{g}_\alpha) \supset \mathfrak{k} \cap \mathfrak{g}_0$.

3.9 Homogeneous hypersurfaces in spheres (2/3)

- **Prop.:**

  - R-spaces $\subset S^n$ (the unit sphere in $\mathfrak{p}$).

  ($\because$) K preserves the inner product defined by the Killing form. $\square$

- **Prop.:**

  - rank $(G/K) = 1$

    $\Rightarrow$ $K \curvearrowright \mathfrak{p}$: transitive on the unit sphere.

- **Thm.:**

  - rank $(G/K) = 2$

    $\Rightarrow$ $\exists H \in \mathfrak{a}$ s.t. $K.H$ is a hypersurface in the unit sphere in $\mathfrak{p}$.

3.10 Homogeneous hypersurfaces in spheres (3/3)

- **Remark:**
  - $M = G/K$ = **two-plane Grassmann (over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$)**

    $\Rightarrow$ **homogeneous hypersurfaces in spheres**

    (with four distinct principal curvatures).

- **Thm. (Hsiang - Lawson):**
  - **Every homogeneous hypersurface can be obtained in this way**

    (i.e., the isotropy representations of symmetric spaces of rank 2).

3.11 Appendix: parabolic subgroups (1/2)

Def.:

- **Let G/K : noncompact type, $\Lambda \supsetneq \Phi$: a subset.**
- $\mathfrak{q}_{\Phi} := \mathfrak{g}_0 \oplus \bigoplus_{\alpha > 0 \text{ or } \alpha \in \langle \Phi \rangle} \mathfrak{g}_{\alpha}$ **is called a parabolic subalgebra.**

Example:

- **Parabolic subalgebras in $\mathfrak{sl}_3(\mathbb{R})$,**
- **Parabolic subalgebras in $\mathfrak{sl}_n(\mathbb{R})$**

  $\leftrightarrow$ **block decompositions of matrices.**

3.12 Appendix: parabolic subgroups (2/2)

- **Thm.:**

  R-spaces = G/Q_{Φ}, i.e.,

  - $K.H = K/K_H$: an R-space

    $\Rightarrow\ \exists\Phi \subset \Lambda$ s.t. $K.H = G/Q_{\Phi}$.

  - $G \supset Q_{\Phi}$: parabolic

    $\Rightarrow\ \exists H \in \mathfrak{a}$ s.t. $G/Q_{\Phi} = K.H$.

- **Comment:**

  - This is an another definition of R-spaces.

  - Q_{Φ} are important for studying noncompact homogeneous spaces.