The space of left-invariant metrics
— on a generalization of Milnor frames

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0 Abstract

- **We are studying:**
  - **the existence and nonexistence problems of distinguished left-invariant metrics on Lie groups,**
  - **by a generalization of Milnor frames (Milnor-type theorems).**

- **Contents:**

1 Introduction

§1.1 Contents

- **Recall that we are studying:**
  - **the existence and nonexistence problems of distinguished left-invariant metrics on Lie groups,**
  - **by a generalization of Milnor frames (Milnor-type theorems).**

- **Contents of this section:**

  §1.2: Distinguished left-invariant metrics

  §1.3: The existence and nonexistence problem

  §1.4: (Classical) Milnor frames

  §1.5: What are Milnor-type theorems?

§1.2 Distinguished left-invariant metrics (1/2)

- **Our theme:**
  - Left-invariant Riemannian metrics g on Lie groups G.
- **They can be studied in terms of:**
  - $(\mathfrak{g}, \langle , \rangle)$: the corresponding metric Lie algebras.
  - The Levi-Civita connection $\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfies

$$2\langle \nabla_X Y, Z \rangle = \langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle.$$

- **They provide examples of:**
  - Einstein :$\Leftrightarrow$ $\mathrm{Ric} = c \cdot \mathrm{id}$ $(\exists c \in \mathbb{R})$,
  - algebraic Ricci soliton

    :$\Leftrightarrow$ $\mathrm{Ric} = c \cdot \mathrm{id} + D$ $(\exists c \in \mathbb{R},\ \exists D \in \mathrm{Der}(\mathfrak{g}))$,
  - Ricci soliton :$\Leftrightarrow$ $\mathrm{ric} = cg + \mathfrak{L}_X g$ $(\exists c \in \mathbb{R},\ \exists X \in \mathfrak{X}(G))$.

(Fact: Einstein $\Rightarrow$ algebraic Ricci soliton $\Rightarrow$ Ricci soliton)

§1.2 Distinguished left-invariant metrics (2/2)

▼ **Example (Heisenberg Lie algebra):**

- $\mathfrak{h}^3 := \mathrm{span}\{e_1, e_2, e_3\}$ with $[e_1, e_2] = e_3$, $[e_1, e_3] = [e_2, e_3] = 0$.
- $\langle , \rangle$: canonical inner product ($\{e_1, e_2, e_3\}$: o.n.b.)

$\Rightarrow$ ∘ $\mathrm{Ric} = \dfrac{1}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

- $\mathrm{Der}(\mathfrak{h}^3) = \left\{ \left(\begin{array}{cc|c} a & * & 0 \\ * & b & 0 \\ \hline * & * & a+b \end{array} \right) \mid a, b \in \mathbb{R} \right\}$.
- Hence, $(\mathfrak{h}^3, \langle , \rangle)$ is an algebraic Ricci soliton ($\mathrm{Ric} = c \cdot \mathrm{id} + D$).

$$(\because)\ \mathrm{Ric} = -\frac{3}{2} \cdot \mathrm{id} + \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}.$$

§1.3 The existence and nonexistence problem (1/2)

- **Problem:**
  - For a given Lie group G, examine whether G admits a **distinguished** left-invariant metric.
    (e.g., Einstein, (algebraic) Ricci soliton)

- **Known results:**
  - Classification for low dimensional ones.
    (Einstein when dim ≤ 5, alg. Ricci soliton when dim ≤ 4, ...)
  - Some higher-dimensional examples.
  - Far from the complete understanding...

§1.3 The existence and nonexistence problem (2/2)

- **Recall the problem:**
  - **Examine whether G admits a distinguished left-invariant metric.**
- **Note:**
  - **In general, this problem is difficult.**
  - **One of the reasons: there are so many left-invariant metrics...**

$$\widetilde{\mathfrak{M}} := \{\text{left-invariant metrics on } G\}$$
$$\cong \{\text{inner products on } \mathfrak{g}\}$$
$$\cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n).$$

(Note that $n := \dim G$, and $g.\langle \cdot, \cdot \rangle := \langle g^{-1}(\cdot), g^{-1}(\cdot) \rangle$)

§1.4 (Classical) Milnor frames (1/2)

- **Thm. (Milnor 1976):**
  - $\mathfrak{g}$: 3-dimensional unimodular
  - $\langle , \rangle$: any inner product on $\mathfrak{g}$

  $\Rightarrow \ \exists \{x_1, x_2, x_3\}$: o.n.b. w.r.t. $\langle , \rangle$, $\exists \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$:

  $$[x_1, x_2] = \lambda_3 x_3, \quad [x_2, x_3] = \lambda_1 x_1, \quad [x_3, x_1] = \lambda_2 x_2.$$

- **Remark:**
  - $\{x_1, x_2, x_3\}$ is called the Milnor frame.
  - For each $\mathfrak{g}$, possible values of $\lambda_1, \lambda_2, \lambda_3$ are determined.
  - All inner products can be studied by using up to 3 parameters. (Note that $\widetilde{\mathfrak{M}} = GL_3(\mathbb{R})/O(3)$ has dimension 6.)

§1.4 (Classical) Milnor frames (2/2)

- **Example:**
  - $\langle , \rangle$: any inner product on $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{R})$

    $\Rightarrow$ ◦ $\exists \{x_1, x_2, x_3\}$: o.n.b. wrt $\langle , \rangle$, $\exists \lambda_1 > 0, \lambda_2 > 0, \lambda_3 < 0$:

    $$[x_1, x_2] = \lambda_3 x_3, \ [x_2, x_3] = \lambda_1 x_1, \ [x_3, x_1] = \lambda_2 x_2.$$

- **Cor.:**
  - The eigenvalues of $\mathrm{Ric}_{\langle , \rangle}$ are either $(+, -, -)$ or $(0, 0, -)$.
  - $\mathrm{SL}_2(\mathbb{R})$ does not admit left-invariant Einstein metrics.

- **Comment:**
  - The Milnor frames are quite useful.
  - But, the proof strongly depends on dimension 3...

§1.5 What are Milnor-type theorems? (1/2)

- **Today's topic is:**
  - **Milnor-type theorems,**
    a generalization of Milnor frames to higher-dim. Lie groups.

- **A format of Milnor-type theorems:**
  - $\mathfrak{g}$ **: a Lie algebra,**
  - $\langle , \rangle$ **: any inner product on** $\mathfrak{g}$

    $\Rightarrow \ \exists k > 0, \ \exists \{x_1, \ldots, x_n\}$ **: o.n.b. w.r.t.** $k\langle , \rangle$ **:**

    the bracket relations contain l **parameters.**

- **Comment:**
  - **If** $l << \dim \widetilde{\mathfrak{M}}$, **then such Milnor-type theorem is useful.**

§1.5 What are Milnor-type theorems? (2/2)

- **Thm. (Hashinaga-T.-Terada, preprint):**
  - **For ANY Lie algebra $\mathfrak{g}$, there is a procedure to obtain a Milnor-type theorem for $\mathfrak{g}$.**
- **Applications:**
  - **Milnor-type theorems for several Lie groups, and**
  - **(non)existence results of distinguished left-invariant metrics on some Lie groups.**

2 Results (for Riemannian case)

§2.1 Contents

- **We have:**
  - **a procedure to obtain Milnor-type theorems,**
  - **and some applications.**

- **Contents of this section:**

§2.2 A procedure to obtain Milnor-type theorems (1/4)

- **Def.:**
  - $\mathfrak{PM} := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}$ (the orbit space)

    is called the moduli space of left-invariant metrics.

- **Recall:**
  - $\widetilde{\mathfrak{M}} := \{\langle , \rangle : \text{an inner product on } \mathfrak{g}\} \cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$.
  - $\mathbb{R}^\times := \{c \cdot \mathrm{id} : \mathfrak{g} \to \mathfrak{g} \mid c \neq 0\}$.
  - $\mathrm{Aut}(\mathfrak{g}) := \{\varphi : \mathfrak{g} \to \mathfrak{g} : \text{an automorphism}\}$.
  - $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ by $g.\langle \cdot, \cdot \rangle := \langle g^{-1}(\cdot), g^{-1}(\cdot) \rangle$.

- **Story:**
  - An expression of $\mathfrak{PM} = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}$

    $\Rightarrow$ A "Milnor-type theorem" for $\mathfrak{g}$.

§2.2 A procedure to obtain Milnor-type theorems (2/4)

Story:

- An expression of $\mathfrak{PM} = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}$

  $\Rightarrow$ A "Milnor-type theorem" for $\mathfrak{g}$.

Note:

- $\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$: a noncompact Riemannian symmetric space,
- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$: an isometric action.
- There are some general theory on the orbit spaces...

Fact (Berndt-Brück 2002):

- M : a Hadamard manifold (e.g., $M = \widetilde{\mathfrak{M}}$)
- $H \curvearrowright M$: of cohomogeneity one (H : connected)

  $\Rightarrow$ $H \backslash M \cong \mathbb{R}$ or $[0, +\infty)$.

§2.2 A procedure to obtain Milnor-type theorems (3/4)

- **Thm. (Hashinaga-T.-Terada, preprint):**
  - $\{e_1, \ldots, e_n\}$: o.n.b. of $\mathfrak{g}$ w.r.t. $\langle , \rangle_0$
  - $\mathrm{GL}_n(\mathbb{R}) \supset \mathfrak{U}$: a set of representatives of $\mathfrak{PM}$

    (i.e., $\{g.\langle , \rangle_0 \mid g \in \mathfrak{U}\}$ intersects all orbits)
  - $\langle , \rangle$: an arbitrary inner product

    $\Rightarrow \ \exists k > 0, \ \exists \varphi \in \mathrm{Aut}(\mathfrak{g}), \ \exists g \in \mathfrak{U}$:

    $\{\varphi g e_1, \ldots, \varphi g e_n\}$ is orthonormal w.r.t. $k\langle , \rangle$.

- **A sketch of the proof:**
  - By assumption, $\exists g \in \mathfrak{U} : \langle , \rangle \in \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).(g.\langle , \rangle_0)$.
  - By definition, $\exists c\varphi \in \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) : \langle , \rangle = (c\varphi g).\langle , \rangle_0$.

§2.2 A procedure to obtain Milnor-type theorems (4/4)

- **Recall:**
  - $\mathrm{GL}_n(\mathbb{R}) \supset \mathfrak{U}$: a set of representatives of $\mathfrak{PM}$
  - $\langle , \rangle$: an arbitrary inner product

    $\Rightarrow\ \exists k > 0,\ \exists \varphi \in \mathrm{Aut}(\mathfrak{g}),\ \exists g \in \mathfrak{U}$:

    $\{\varphi g e_1, \ldots, \varphi g e_n\}$ is orthonormal w.r.t. $k\langle , \rangle$.

- **Comment:**
  - $\{\varphi g e_1, \ldots, \varphi g e_n\}$ is a generalized Milnor frame.
  - Note: φ preserves a bracket product.
  - If $\mathfrak{U}$ can be expressed by l variables ($l \fallingdotseq \dim \mathfrak{U}$)

    $\Rightarrow$ the bracket relations among them contain only l variables.

§2.3 Three-dimensional case (1/3)

- **Result of [Hashinaga-T.]:**
  - **We have obtained Milnor-type theorems for all 3-dimensional solvable Lie algebras.**
- **In this talk, we only consider:**
  - $\mathfrak{r}'_{3,a} := \mathrm{span}\{e_1, e_2, e_3\}$ $(a \geq 0)$ **where** $[e_1, e_2] = ae_2 - e_3$, $[e_1, e_3] = e_2 + ae_3$.
  - **Note:** $\mathfrak{r}'_{3,a}$ **is solvable.**
  - **Note:** $\mathfrak{r}'_{3,a}$ **is unimodular** $\Leftrightarrow$ $a = 0$.

§2.3 Three-dimensional case (2/3)

- **Prop. (Hashinaga-T., preprint):**
  - $\langle , \rangle$: an (arbitrary) inner product on $\mathfrak{r}'_{3,a}$

    $\Rightarrow \exists k > 0, \ \exists \lambda \geq 1, \ \exists \{x_1, x_2, x_3\}$: o.n.b. w.r.t. $k\langle , \rangle$:

    $$[x_1, x_2] = ax_2 - \lambda x_3, \quad [x_1, x_3] = (1/\lambda)x_2 + ax_3.$$

- **Proof:**
  - $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$: of cohomogeneity one.
  - A set of representatives of $\mathfrak{PM}$ can be given by

    $$\mathfrak{U} := \{\mathrm{diag}(1, 1, 1/\lambda) \mid \lambda \geq 1\} \cong [1, +\infty).$$

- **Cor. (previously known):**
  - $\mathfrak{r}'_{3,a}$ admits a left-invariant Einstein metric.

    (It corresponds to the case $\lambda = 1$.)

§2.3 Three-dimensional case (3/3)

- **Recall (when $a = 0$):**
  - $\langle , \rangle$: an (arbitrary) inner product on $\mathfrak{r}'_{3,0}$

    $\Rightarrow\ \exists k > 0,\ \exists \lambda \geq 1,\ \exists \{x_1, x_2, x_3\}$: o.n.b. w.r.t. $k\langle , \rangle$:

    $$[x_1, x_2] = -\lambda x_3, \quad [x_1, x_3] = (1/\lambda) x_2.$$

- **Recall (Milnor's Theorem):**
  - $\langle , \rangle$: an (arbitrary) inner product on $\mathfrak{r}'_{3,0}$

    $\Rightarrow\ \exists \lambda_1, \lambda_2 > 0,\ \exists \{y_1, y_2, y_3\}$: o.n.b. w.r.t. $\langle , \rangle$:

    $$[y_1, y_2] = 0, \quad [y_2, y_3] = \lambda_1 y_1, \quad [y_3, y_1] = \lambda_2 y_2.$$

- **Our theorem recovers Milnor's theorem for $\mathfrak{r}'_{3,0}$:**
  - It is enough to put (and suitable change of indices)

    $$y_i := k^{-1/2} x_i,\ \lambda_1 := 0,\ \lambda_2 := \lambda^{-1} k^{-1/2},\ \lambda_3 := \lambda k^{-1/2}.$$

§2.4 Higher-dimensional examples (1/3)

- **Comment:**
  - In general, it is not easy to express $\mathfrak{PM}$.
  - If $\dim \mathfrak{PM} = 1$ (i.e., cohomogeneity one), then it is easy to handle.

- **Recall:**
  - M : a Hadamard manifold (e.g., $M = \widetilde{\mathfrak{M}}$)
  - $H \curvearrowright M$: of cohomogeneity one (H : connected)

    $\Rightarrow$ $H \backslash M \cong \mathbb{R}$ or $[0, +\infty)$.

§2.4 Higher-dimensional examples (2/3)

- **Consider:**
  - $\mathfrak{g}^n_{1,1} := \mathrm{span}\{e_1, \ldots, e_n\}$,

    where $[e_i, e_n] = e_i$ (for $i = 1, \ldots, n-2$),

    $[e_{n-1}, e_n] = e_1 + e_{n-1}$.

- **Thm. (Taketomi-T., in preparation):**
  - $\langle , \rangle$: an arbitrary inner product on $\mathfrak{g}^n_{1,1}$

    $\Rightarrow$ $\exists k > 0$, $\exists \lambda > 0$, $\exists\{x_1, \ldots, x_n\}$: o.n.b. w.r.t. $k\langle , \rangle$:

    $[x_i, x_n] = x_i$ (for $i = 1, \ldots, n-2$),

    $[x_{n-1}, x_n] = \lambda x_1 + x_{n-1}$.

- **Cor.:**
  - The above $\mathfrak{g}^n_{1,1}$ does not admit left-invariant Ricci solitons.

§2.4 Higher-dimensional examples (3/3)

- **Prop. (Taketomi-T., in preparation):**
  - $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}^{n}_{1,1}) \curvearrowright \widetilde{\mathfrak{M}}$ satisfies:
    - The orbit space $\cong \mathbb{R}_{>0}$.
    - All orbits $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}^{n}_{1,1}).\langle , \rangle$ are hypersurfaces in $\widetilde{\mathfrak{M}}$.
    - More strongly, all orbits are congruent to each other.

      (Looks like a horosphere foliation.)

- **Comment:**
  - Our expectation: $\langle , \rangle$ is special $\Leftrightarrow$ $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ is special?
  - The above observation certificates this expectation.

    ($\nexists$ Ricci soliton, $\nexists$ special orbits)

3 Results (for pseudo-Riemannian case)

§3.1 Contents

- **Comment:**
  - Our method can also be applied to left-invariant pseudo-Riemannian metrics.

- **Contents of this section:**

  §3.2: Pseudo-Riemannian Milnor-type theorems

  §3.3: An example

§3.2 Pseudo-Riemannian Milnor-type theorems (1/1)

- **Def:**
  - $\mathfrak{g}$: $(p+q)$-dim. Lie algebra
  - $\widetilde{\mathfrak{M}}_{(p,q)} := \{\langle , \rangle$: an inner product on $\mathfrak{g}$ with signature $(p,q)\}$

    $\cong \mathrm{GL}_{p+q}(\mathbb{R})/\mathrm{O}(p,q)$.
  - $\mathfrak{PM}_{(p,q)} := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}_{(p,q)}$: the moduli space.

- **Thm. (Kubo-Onda-Taketomi-T., in preparation):**
  - A set of representatives of $\mathfrak{PM}_{(p,q)}$

    $\Rightarrow$ a pseudo-Riemannian version Milnor-type theorem

    (by the same procedure)

§3.3 An example (1/3)

- **Def.:**
  - $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n} := \mathrm{span}\{e_1, \ldots, e_n\}$ with $[e_1, e_j] = e_j$ $(j \geq 2)$ is called the **Lie algebra of $\mathbb{R}\mathrm{H}^n$**.

- **Thm. (Kubo-Onda-Taketomi-T.):**
  - $\langle , \rangle$: an arbitrary inner product on $\mathfrak{g}_{\mathbb{R}\mathrm{H}^{p+q}}$ with signature (p, q)
    $\Rightarrow$ $\exists k > 0$, $\exists \lambda \in \{0, 1, 2\}$,
    $\exists \{x_1, \ldots, x_n\}$: pseudo-o.n.b. w.r.t. $k\langle , \rangle$:
    $[x_1, x_j] = x_j$, $[x_1, x_n] = -\lambda x_1 + x_n$, $[x_j, x_n] = -\lambda x_j$
    (for $j \geq 2$)

- **Idea of Proof:**
  - $\mathfrak{PM}_{(p,q)} := \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}}_{(p,q)}$ consists of 3 points.

§3.3 An example (2/3)

- **Recall:**
  - $\#\mathfrak{PM}_{(p,q)} = 3$ for $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$.
- **Cor.:**
  - $\mathfrak{g} := \mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$
    $\Rightarrow$ $\forall \langle , \rangle$: pseudo-Riemannian, it has a constant curvature c.
    (c can take any signature, $c > 0$, $c = 0$, $c < 0$)
- **Comment:**
  - In Riemannian case, always $c < 0$ (Milnor 1976).
  - Lorentz version of this corollary has been known (Nomizu 1979).
  - Our Milnor-type theorem simplifies the proof,
    and extends it to an arbitrary signature case.

§3.3 An example (3/3)

- **Question:**
  - Why $\#\mathfrak{PM}_{(p,q)} = 3$ for $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}H^n}$?
- **In general:**
  - $\mathfrak{PM}_{(p,q)} := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}_{(p,q)} \cong O(p,q) \backslash (GL_{p+q}(\mathbb{R}) / \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))$.
- **For $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}H^n}$:**
  - $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is parabolic so that $GL_{p+q}(\mathbb{R}) / \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \cong \mathbb{R}P^{p+q-1}$.
  - The above 3 orbits correspond to $\{[v] \in \mathbb{R}P^{p+q-1}\}$ with v : timelike, lightlike, spacelike.
- **Comment:**
  - This is a very special case...
  - For further studies, we need to know actions on $GL_{p+q}(\mathbb{R}) / O(p,q)$.

4 Summary and Problems

Summary

- **Story:**
  - **The space of left-invariant metrics (both Riemannian and pseudo-Riemannian settings)**
    $\Rightarrow$ the moduli space (= the orbit space)
    $\Rightarrow$ Milnor-type theorems
    $\Rightarrow$ one can examine ALL left-invariant metrics.
  - **This can be applied to the existence and nonexistence problem of distinguished (e.g., Einstein, Ricci soliton) metrics.**

- **Point:**
  - **Actions on symmetric spaces play important roles.**

Problems

- **Problem 1:**
  - **A continuation of this study, i.e.,**
    - **Get more Milnor-type theorems,**
    - **Study isometric actions on symmetric spaces (both Riemannian and pseudo-Riemannian cases).**
- **Problem 2:**
  - **Apply our method to other geometric structures, e.g.,**
    - **left-invariant complex structures,**
    - **left-invariant symplectic structures, ...**