

Left-invariant metrics on Lie groups and submanifold geometry

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1 Introduction

1.1 Abstract

We propose a framework to study

- geometry of left-invariant metrics on Lie groups,
from the view point of submanifold geometry.

This talk is organized as follows:

§1: Introduction

§2: Preliminaries on submanifold geometry

§3: Three-dimensional case

§4: Higher-dimensional examples

§5: A pseudo-Riemannian version

§6: Summary and Problems

1.2 Theme : Left-invariant metrics (1/2)

Our theme:

- Left-invariant Riemannian metrics g on Lie groups G .

They can be studied in terms of:

- $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$: the corresponding metric Lie algebras.
- The Levi-Civita connection $\nabla : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfies

$$2\langle \nabla_X Y, Z \rangle = \langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle.$$

They provide examples of:

- Einstein $:\Leftrightarrow \text{Ric} = c \cdot \text{id} \quad (\exists c \in \mathbb{R}),$
- algebraic Ricci soliton
 $:\Leftrightarrow \text{Ric} = c \cdot \text{id} + D \quad (\exists c \in \mathbb{R}, \exists D \in \text{Der}(\mathfrak{g})),$
- Ricci soliton $:\Leftrightarrow \text{ric} = cg + \mathfrak{L}_X g \quad (\exists c \in \mathbb{R}, \exists X \in \mathfrak{X}(G)).$

1.3 Theme : Left-invariant metrics (2/2)

Fact:

- **Einstein \Rightarrow algebraic Ricci soliton \Rightarrow Ricci soliton.**

Example (Heisenberg Lie algebra):

- $\mathfrak{h}^3 := \text{span}\{e_1, e_2, e_3\}$ with $[e_1, e_2] = e_3$, $[e_1, e_3] = [e_2, e_3] = 0$.
- \langle, \rangle : canonical inner product ($\{e_1, e_2, e_3\}$: o.n.b.)

$$\Rightarrow \circ \text{ Ric} = \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\circ \text{ Der}(\mathfrak{h}^3) = \left\{ \left(\begin{array}{cc|c} a & * & 0 \\ * & b & 0 \\ \hline * & * & a+b \end{array} \right) \mid a, b \in \mathbb{R} \right\}.$$

- **Hence, $(\mathfrak{h}^3, \langle, \rangle)$ is an algebraic Ricci soliton ($\text{Ric} = c \cdot \text{id} + D$).**

1.4 Problem : The existence of distinguished metrics (1/2)

Problem:

- For a given Lie group G , examine whether G admits a “distinguished” left-invariant metric.
(e.g., Einstein, (algebraic) Ricci soliton)

Note:

- In general, this problem is difficult.
- Reason: there are so many left-invariant metrics...

$\tilde{\mathfrak{M}} := \{\text{left-invariant metrics on } G\}$

$\cong \{\text{inner products on } \mathfrak{g}\}$

$\cong \text{GL}_n(\mathbb{R})/\text{O}(n).$

(Note that $n := \dim G$, and $g.\langle \cdot, \cdot \rangle := \langle g^{-1}\cdot, g^{-1}\cdot \rangle$)

1.5 Problem : The existence of distinguished metrics (2/2)

Problem:

- **Examine whether G admits a distinguished left-invariant metric.**

Philosophy:

- **It would be related to the algebraic structure of the Lie group.**
- **(Alekseevskii conjecture) G : noncompact Einstein \Rightarrow solvable.**

Related studies:

- **(Lauret) Approach from GIT (alg. Ricci soliton for solvable case).**
- **Classification: Einstein (G, g) when $\dim \leq 5$.**
- **Classification: alg. Ricci soliton (G, g) when $\dim \leq 4$.**
- **Classification: Ricci soliton (G, g) when $\dim \leq 3$.**
- **Some higher-dimensional examples have been known.**

1.6 Strategy : Approach from submanifold geometry (1/2)

Recall:

- **Examine whether G admits a distinguished left-invariant metric.**
- **It would be related to the algebraic structure of G .**

Opportunistic hope:

- **Are there any invariants (based on the algebraic structure) which determine the existence of distinguished metrics?**

Observation:

- **$\text{Aut}(\mathfrak{g})$ is an (algebraic) invariant of a Lie algebra \mathfrak{g} .**
(This is a strong invariant, but not so useful...)
- **$\tilde{\mathfrak{M}} := \{\text{left-invariant metrics on } G\} \cong \text{GL}_n(\mathbb{R})/\text{O}(n)$.**
- **$\text{Aut}(\mathfrak{g}) \curvearrowright \tilde{\mathfrak{M}}$**
gives an isometry (hence preserves curvatures).

1.7 Strategy : Approach from submanifold geometry (2/2)

We consider:

- $\mathbb{R}^\times \text{Aut}(\mathfrak{g}) \curvearrowright \tilde{\mathfrak{M}} = \text{GL}_n(\mathbb{R})/\text{O}(n)$.
- This gives “isometry up to scalar”,
and hence preserves all Riemannian geometric properties.

Def:

- $\mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle, \rangle$: the isometry and scaling class of \langle, \rangle .
- The orbit space $\mathfrak{PM} := \mathbb{R}^\times \text{Aut}(\mathfrak{g}) \backslash \tilde{\mathfrak{M}}$: the moduli space.

Remark:

- $\mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle, \rangle$ is an invariant of (G, \langle, \rangle) .
- \mathfrak{PM} (or $\mathbb{R}^\times \text{Aut}(\mathfrak{g}) \curvearrowright \tilde{\mathfrak{M}}$) is an (algebraic) invariant of G .

1.8 Result :

Expectation:

- \langle, \rangle is distinguished (as Riemannian metrics)
- $\Leftrightarrow \mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle, \rangle$ is distinguished (as submanifolds)?

Result:

- In some cases, our expectation is true.

One of Thm. (Hashinaga-T.):

- Let \mathfrak{g} : solvable, $\dim = 3$.
- $(\mathfrak{g}, \langle, \rangle)$: an algebraic Ricci soliton
- $\Leftrightarrow \mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle, \rangle$: a minimal submanifold.

2 Preliminaries on submanifold geometry

2.1 Cohomogeneity one actions (1/3)

Note:

- $\mathbb{R}^\times \text{Aut}(\mathfrak{g}) \simeq \tilde{\mathfrak{M}}$.
- $\tilde{\mathfrak{M}} = \text{GL}_n(\mathbb{R})/\text{O}(n)$: a noncompact symmetric space.
- Well-studied for “cohomogeneity one” case.

Def.:

- $H \curvearrowright (M, g)$: of cohomogeneity one
: \Leftrightarrow maximal dimensional orbits have codimension one
(\Leftrightarrow the orbit space $H \backslash M$ has dimension one).

2.2 Cohomogeneity one actions (2/3) - on $\mathbb{R}H^2$

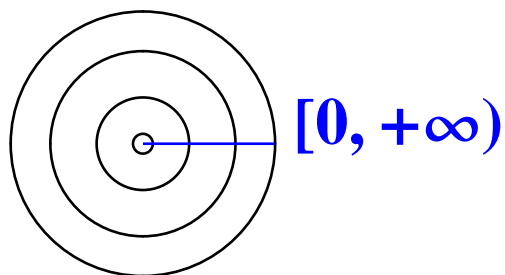
Fact (due to Cartan):

- $H \curvearrowright \mathbb{R}H^2$: cohomogeneity one action (with H connected)

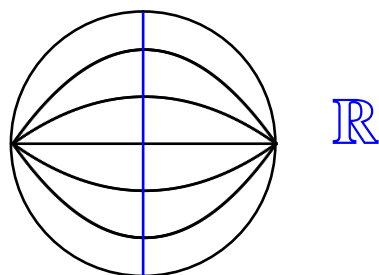
\Rightarrow this is orbit equivalent to the actions of

$$K = \mathbf{SO}(2), \quad A = \left\{ \begin{pmatrix} a & \mathbf{0} \\ \mathbf{0} & a^{-1} \end{pmatrix} \mid a > \mathbf{0} \right\}, \quad N = \left\{ \begin{pmatrix} 1 & b \\ \mathbf{0} & 1 \end{pmatrix} \right\}$$

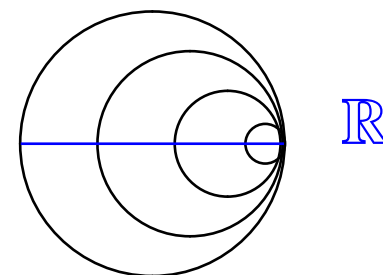
Picture:



type (K)



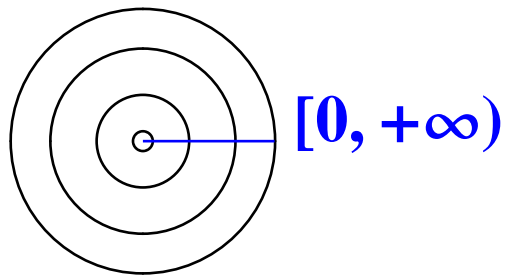
type (A)



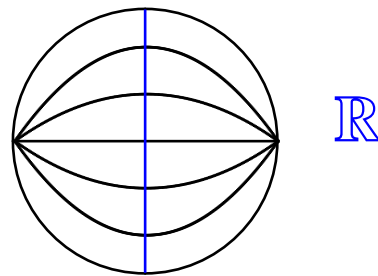
type (N)

2.3 Cohomogeneity one actions (3/3) - geometry of orbits

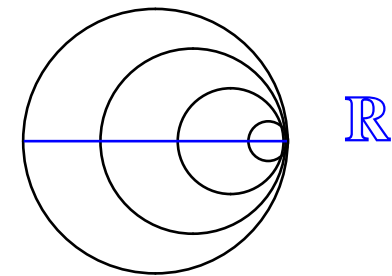
Recall:



type (K)



type (A)



type (N)

Thm. (Berndt-Brück 2002, Berndt-T. 2003):

- M : noncompact symmetric space, irreducible.
- $H \curvearrowright M$: cohomogeneity one
 - \Rightarrow (type (K)) \exists 1 singular orbit,
 - (type (A)) \nexists singular orbit, \exists 1 minimal orbit, or
 - (type (N)) \nexists singular orbit, all orbits are congruent.

3 Three-dimensional case

3.1 Three-dimensional case (1/4) - table

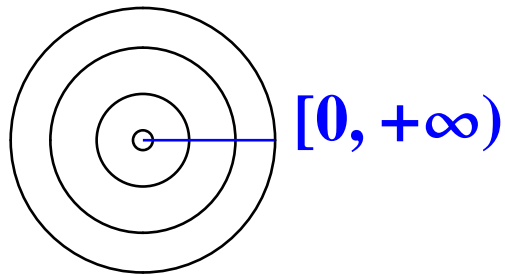
Fact:

- $\mathfrak{g} = \text{span}\{e_1, e_2, e_3\}$: 3-dim. solvable (non abelian)
is isomorphic to one of the following:

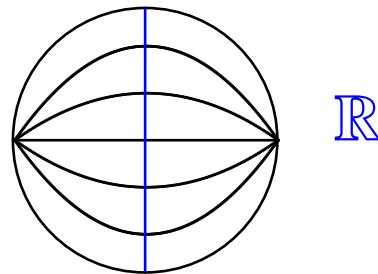
name	comment	brackets
\mathfrak{h}^3	Heisenberg	$[e_1, e_2] = e_3$
$\mathfrak{r}_{3,1}$	$\mathfrak{g}_{\mathbb{R}H^3}$	$[e_1, e_2] = e_2, [e_1, e_3] = e_3$
\mathfrak{r}_3		$[e_1, e_2] = e_2 + e_3, [e_1, e_3] = e_3$
$\mathfrak{r}_{3,a}$	$-1 \leq a < 1$	$[e_1, e_2] = e_2, [e_1, e_3] = ae_3$
$\mathfrak{r}'_{3,a}$	$a \geq 0$	$[e_1, e_2] = ae_2 - e_3, [e_1, e_3] = e_2 + ae_3$

3.2 Three-dimensional case (2/4) - action

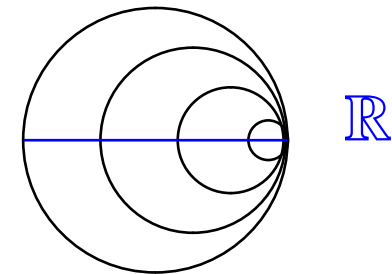
Recall:



type (K)



type (A)



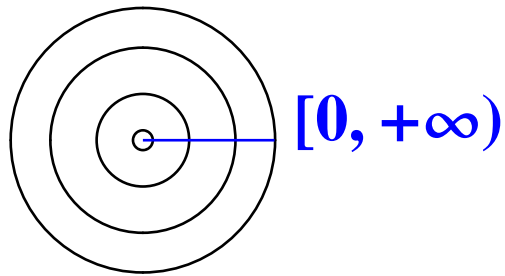
type (N)

Prop. (Hashinaga-T.):

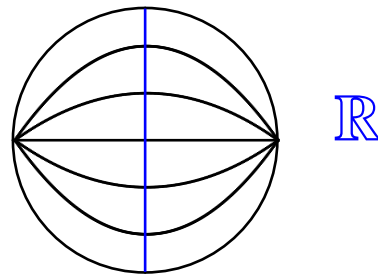
name	$(\mathbb{R}^\times \text{Aut}(\mathfrak{g}))^0 \curvearrowright \tilde{\mathfrak{M}}$	comment
\mathfrak{h}^3	transitive	$\mathbb{R}^\times \text{Aut}(\mathfrak{g})$: parabolic
$\mathfrak{r}_{3,1}$	transitive	$\mathbb{R}^\times \text{Aut}(\mathfrak{g})$: parabolic
\mathfrak{r}_3	type (N)	all orbits are congruent
$\mathfrak{r}_{3,a}$	type (A)	\nexists singular orbit, \exists 1 minimal orbit
$\mathfrak{r}'_{3,a}$	type (K)	\exists 1 singular orbit

3.3 Three-dimensional case (3/4) - metric

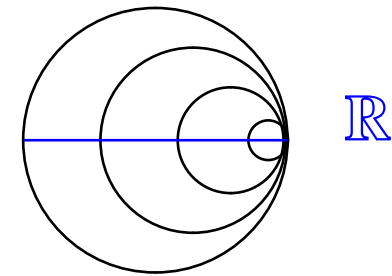
Recall:



type (K)



type (A)



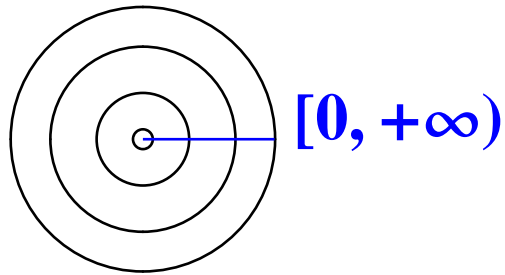
type (N)

Prop.:

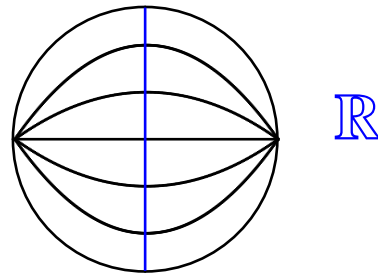
name	action	metric
\mathfrak{h}^3	transitive	algebraic Ricci soliton
$\mathfrak{r}_{3,1}$	transitive	const. negative curvature
\mathfrak{r}_3	type (N)	\nexists Ricci soliton
$\mathfrak{r}_{3,a}$	type (A)	algebraic Ricci soliton \Leftrightarrow minimal
$\mathfrak{r}'_{3,a}$	type (K)	Einstein \Leftrightarrow singular

3.4 Three-dimensional case (4/4) - Theorem

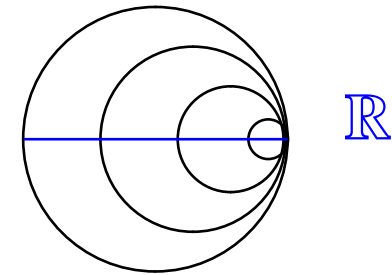
Recall:



type (K)



type (A)



type (N)

Thm. (Hashinaga-T.):

- \mathfrak{g} : solvable, $\dim = 3$.
- $(\mathfrak{g}, \langle, \rangle)$: an algebraic Ricci soliton
 - $\Leftrightarrow \mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle, \rangle$: minimal in $\tilde{\mathfrak{M}} = \text{GL}_n(\mathbb{R})/\text{O}(n)$.

4 Higher dimensional examples

4.1 Higher dimensional examples (1/4)

Note:

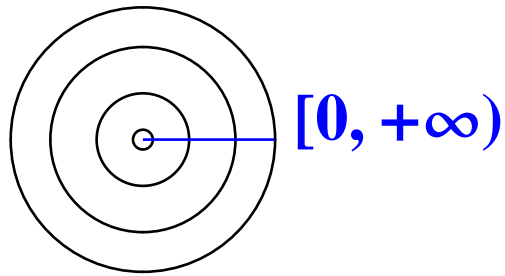
- \mathfrak{g} : three-dimensional solvable
 - \Rightarrow • $\mathbb{R}^\times \text{Aut}(\mathfrak{g}) \curvearrowright \tilde{\mathfrak{M}}$: cohomogeneity at most one,
 - \langle, \rangle : algebraic Ricci soliton $\Leftrightarrow \mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle, \rangle$: minimal.

Natural question:

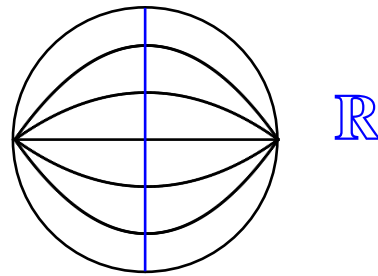
- How are the higher dimensional cases?
- Still $\mathbb{R}^\times \text{Aut}(\mathfrak{g}) \curvearrowright \tilde{\mathfrak{M}}$ can be of cohomogeneity one?

4.2 Higher dimensional examples (2/4)

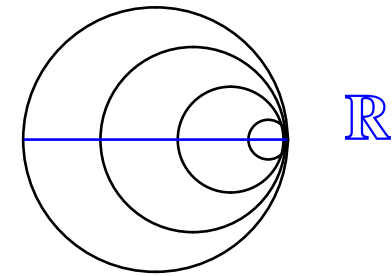
Recall:



type (K)



type (A)



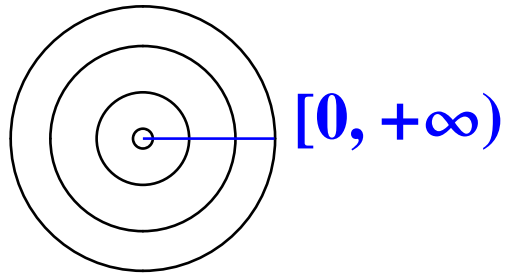
type (N)

Thm. (Hashinaga-T.-Terada):

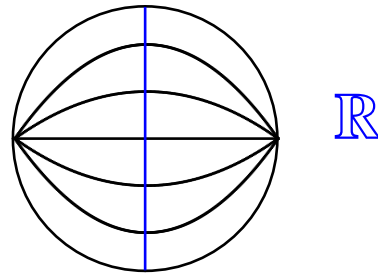
- $\forall n \geq 4, \exists \mathfrak{g} : \text{solvable}, \dim \mathfrak{g} = n$
 - s.t. • $\mathbb{R}^\times \text{Aut}(\mathfrak{g}) \curvearrowright \tilde{\mathfrak{M}} : \text{cohomogeneity one, type (K),}$
 - $\langle, \rangle : \text{Ricci soliton} \Leftrightarrow \mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle, \rangle : \text{singular.}$

4.3 Higher dimensional examples (3/4)

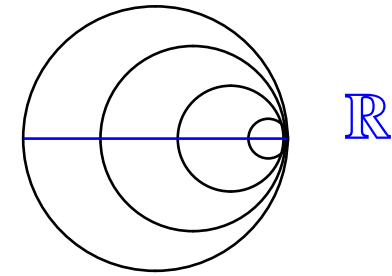
Recall:



type (K)



type (A)



type (N)

Thm. (Taketomi-T.):

- $\forall n \geq 4, \exists \mathfrak{g} : \text{solvable}, \dim \mathfrak{g} = n$
s.t. • $\mathbb{R}^\times \text{Aut}(\mathfrak{g}) \curvearrowright \tilde{\mathfrak{M}} : \text{cohomogeneity one, type (N),}$
• $\nexists \langle, \rangle : \text{Ricci soliton.}$

4.4 Higher dimensional examples (4/4)

Expectation:

- \langle, \rangle is distinguished (as Riemannian metrics)
- $\Leftrightarrow \mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle, \rangle$ is distinguished (as submanifolds)?

Comments:

- This expectation is true for several cases.
- All results are “experimentally”.
- Are there any theoretical reason behind?

5 A pseudo-Riemannian version

5.1 A pseudo version (1/4) - the moduli space

Note:

- Our framework can also be applied to left-invariant pseudo-Riemannian metrics.

Def:

- \mathfrak{g} : $(p + q)$ -dim. Lie algebra
- $\tilde{\mathfrak{M}}_{(p,q)} := \{ \langle \cdot, \cdot \rangle : \text{an inner product on } \mathfrak{g} \text{ with signature } (p, q) \}$
 $\cong \text{GL}_{p+q}(\mathbb{R}) / \text{O}(p, q)$.
- $\mathfrak{M}_{(p,q)} := \mathbb{R}^\times \text{Aut}(\mathfrak{g}) \backslash \tilde{\mathfrak{M}}_{(p,q)}$: the Moduli space.

5.2 A pseudo version (2/4) - preliminary

Def.:

- $\mathfrak{g}_{\mathbb{R}H^n} := \text{span}\{e_1, \dots, e_n\}$ with $[e_1, e_j] = e_j$ ($j \geq 2$)
is called the Lie algebra of $\mathbb{R}H^n$.

Thm.:

- $\forall \langle, \rangle$: Riemannian on $\mathfrak{g}_{\mathbb{R}H^n}$,
 $(\mathfrak{g}_{\mathbb{R}H^n}, \langle, \rangle)$ has constant curvature $c < 0$.

Comments on Proof:

- **Original Proof: Milnor (1976).**
- **Another Proof: Lauret (2003).**
- **Simpler Proof using $\mathfrak{B}\mathfrak{M} = \{\text{pt}\}$: Kodama-Takahara-T. (2011).**

5.3 A pseudo version (3/4) - our result

Thm. (Kubo-Onda-Taketomi-T.):

- $\forall \langle, \rangle$: pseudo-Riemannian on $\mathfrak{g}_{\mathbb{R}H^n}$,
($\mathfrak{g}_{\mathbb{R}H^n}, \langle, \rangle$) has constant curvature c .
(c can take any signature, $c > 0, c = 0, c < 0$)

Comments on Proof:

- Lorentz case has been known.
(Nomizu 1979; his proof is very direct.)
- Our proof: $\#\mathfrak{PM}_{(p,q)} = 3$.
- Recall:
 - $\mathfrak{PM}_{(p,q)} := \mathbb{R}^\times \text{Aut}(\mathfrak{g}) \backslash \tilde{\mathfrak{M}}_{(p,q)}$.
 - $\tilde{\mathfrak{M}}_{(p,q)} = \text{GL}_{p+q}(\mathbb{R}) / \text{O}(p, q)$.

5.4 A pseudo version (4/4) - comments

Our result:

- **It is only on $\mathfrak{g}_{\mathbb{R}H^n}$, and a little bit more...**

Comment:

- **For further studies, we need to know**

the geometry of isometric actions $H \curvearrowright \mathrm{GL}_{p+q}(\mathbb{R})/\mathrm{O}(p, q)$.

(submanifold geometry in pseudo-Riem. symmetric spaces...)

6 Summary and Problems

6.1 Summary

Story:

- We are interested in geometry of left-invariant metrics
(both Riemannian and pseudo-Riemannian)
- An approach from submanifold geometry
 - one can study all metrics efficiently,
 - left-invariant metrics \leftrightarrow properties of submanifolds?

Results:

- In several cases,
 \langle, \rangle : (algebraic) Ricci soliton $\Leftrightarrow \mathbb{R}^\times \text{Aut}(\mathfrak{g})$. \langle, \rangle : distinguished.

6.2 Problems

Problem 1:

- **A continuation of this study, i.e.,**
 - **check the correspondence for other cases,**
((algebraic) Ricci soliton \leftrightarrow distinguished orbit?)
 - **are there any theoretical reason behind?**

Problem 2:

- **Apply our method to other geometric structures, e.g.,**
 - **left-invariant pseudo-Riemannian metrics (have just started),**
 - **left-invariant complex structures,**
 - **left-invariant symplectic structures, ...**