左不変計量の成す空間
— ミルナー枠の一般化について

田丸 博士 (広島大学)

松江セミナー 兼 談話会
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1 Introduction

1.1 Abstract

- **We are studying:**
  - **geometry of left-invariant metrics on Lie groups, from the viewpoint of the space of left-invariant metrics.**
- **Our Results:**
  - **A study of this space provides Milnor-type theorems, a generalization of Milnor frames.**
  - **They provide several existence and nonexistence results of "distinguished" left-invariant metrics.**

1.2 Contents

- **This talk is organized as follows:**

1.3 Theme : Left-invariant metrics (1/2)

- **Our theme:**
  - **Left-invariant Riemannian metrics g on Lie groups G.**

- **They can be studied in terms of:**
  - **$(\mathfrak{g}, \langle , \rangle)$: the corresponding metric Lie algebras.**
  - **The Levi-Civita connection $\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfies**

$$2\langle \nabla_X Y, Z \rangle = \langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle.$$

- **They provide examples of:**
  - **Einstein** $:\Leftrightarrow \ \mathrm{Ric} = c \cdot \mathrm{id} \ \ (\exists c \in \mathbb{R})$,
  - **algebraic Ricci soliton**
    $:\Leftrightarrow \ \mathrm{Ric} = c \cdot \mathrm{id} + D \ \ (\exists c \in \mathbb{R}, \ \exists D \in \mathrm{Der}(\mathfrak{g}))$,
  - **Ricci soliton** $:\Leftrightarrow \ \mathrm{ric} = cg + \mathfrak{L}_X g \ \ (\exists c \in \mathbb{R}, \ \exists X \in \mathfrak{X}(G))$.

1.4 Theme : Left-invariant metrics (2/2)

- **Fact:**
  - Einstein $\Rightarrow$ algebraic Ricci soliton $\Rightarrow$ Ricci soliton.

- **Example (Heisenberg Lie algebra):**
  - $\mathfrak{h}^3 := \mathrm{span}\{e_1, e_2, e_3\}$ with $[e_1, e_2] = e_3$, $[e_1, e_3] = [e_2, e_3] = 0$.
  - $\langle , \rangle$: canonical inner product ($\{e_1, e_2, e_3\}$: o.n.b.)

$\Rightarrow$
  - $$\mathrm{Ric} = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
  - $$\mathrm{Der}(\mathfrak{h}^3) = \left\{ \left(\begin{array}{cc|c} a & * & 0 \\ * & b & 0 \\ \hline * & * & a+b \end{array} \right) \mid a, b \in \mathbb{R} \right\}.$$
  - Hence, $(\mathfrak{h}^3, \langle , \rangle)$ is an algebraic Ricci soliton ($\mathrm{Ric} = c \cdot \mathrm{id} + D$).

1.5 Problem : The existence of distinguished metrics (1/2)

- **Problem:**
  - **For a given Lie group G, examine whether G admits a "distinguished" left-invariant metric.**
    (e.g., Einstein, (algebraic) Ricci soliton)

- **Philosophy:**
  - **This would be related to algebraic structure of the Lie group.**
    ((Alekseevskii conjecture) G : noncompact Einstein $\Rightarrow$ solvable)
  - **This is also related to GIT. (cf. Lauret)**

- **Known results:**
  - **Classification for low dimensional ones.**
    (Einstein when dim ≤ 5, alg. Ricci soliton when dim ≤ 4, ...)
  - **Some higher-dimensional examples.**

1.6 Problem : The existence of distinguished metrics (2/2)

- **Problem:**
  - **Examine whether G admits a distinguished left-invariant metric.**
- **Note:**
  - **In general, this problem is difficult.**
  - **One of the reasons: there are so many left-invariant metrics...**

$$\widetilde{\mathfrak{M}} := \{\text{left-invariant metrics on } G\}$$
$$\cong \{\text{inner products on } \mathfrak{g}\}$$
$$\cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n).$$

(Note that $n := \dim G$, and $g.\langle \cdot, \cdot \rangle := \langle g^{-1} \cdot, g^{-1} \cdot \rangle$)

1.7 Approach : The space of left-invariant metrics (1/1)

- **Recall:**
  - $\widetilde{\mathfrak{M}} := \{\langle , \rangle : \text{an inner product on } \mathfrak{g}\} \cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$.

- **Observation:**
  - $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ preserves all Riemannian geometric properties.
  - Hence, in order to examine the existence of distinguished metrics, we have only to study the orbit space $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}$.

- **Def.:**
  - $\mathfrak{PM} := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}$ (the orbit space) is called the moduli space.

1.8 Result : a generalization of Milnor frames (1/2)

- **Thm.:**
  - **An expression of** $\mathfrak{PM} = \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}}$

    $\Rightarrow$ **A "Milnor-type theorem" for** $\mathfrak{g}$.

- **Thm. (Milnor 1976):**
  - $\mathfrak{g}$ **: 3-dimensional unimodular**
  - $\langle , \rangle$ **: any inner product on** $\mathfrak{g}$

    $\Rightarrow$ $\exists \{x_1, x_2, x_3\}$: **o.n.b. w.r.t.** $\langle , \rangle$, $\exists \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$:

    $$[x_1, x_2] = \lambda_3 x_3, \quad [x_2, x_3] = \lambda_1 x_1, \quad [x_3, x_1] = \lambda_2 x_2.$$

- **Remark:**
  - $\{x_1, x_2, x_3\}$ **is called the Milnor frame.**
  - **For each** $\mathfrak{g}$**, possible values of** $\lambda_1, \lambda_2, \lambda_3$ **are determined.**
  - **All inner products on** $\mathfrak{g}$ **can be studied using up to 3 parameters.**

1.9 Result : a generalization of Milnor frames (2/2)

- **Thm.:**
  - **An expression of $\mathfrak{PM} = \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}}$**

    $\Rightarrow$ A "Milnor-type theorem" for $\mathfrak{g}$.

- **A format of Milnor-type theorems:**
  - **$\langle , \rangle$: any inner product on $\mathfrak{g}$**

    $\Rightarrow$ $\exists k > 0,\ \exists\{x_1, \ldots, x_n\}$: o.n.b. w.r.t. $k\langle , \rangle$:

    the bracket relations contain only l ($:=$ dim $\mathfrak{PM}$) parameters.

- **Comment:**
  - **All inner products on $\mathfrak{g}$ can be studied using l parameters.**

    (Note: l = dim $\mathfrak{PM}$ << dim $\widetilde{\mathfrak{M}}$ in general.)
  - **We can obtain several existence and nonexistence results.**

2 How to get Milnor-type theorems

2.1 How to get Milnor-type theorems (1/3)

- **Recall:**
  - $\mathfrak{g}$: a Lie algebra, $\dim \mathfrak{g} = n$.
  - $\widetilde{\mathfrak{M}} := \{\langle , \rangle : \text{an inner product on } \mathfrak{g}\} \cong GL_n(\mathbb{R})/O(n)$.
  - $\mathfrak{PM} := \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}}$ (the orbit space) : the moduli space.

- **Def.:**
  - $\langle , \rangle_0$: the origin.
  - $GL_n(\mathbb{R}) \supset \mathfrak{U}$: a set of representatives of $\mathfrak{PM}$

    $:\Leftrightarrow \mathfrak{PM} = \{\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).(g.\langle , \rangle_0) \mid g \in \mathfrak{U}\}$.

    $(\Leftrightarrow \{g.\langle , \rangle_0 \mid g \in \mathfrak{U}\}$ intersects all orbits)

2.2 How to get Milnor-type theorems (2/3)

- **Thm. (Hashinaga-T.-Terada, preprint):**
  - $\{e_1, \ldots, e_n\}$: o.n.b. of $\mathfrak{g}$ w.r.t. $\langle , \rangle_0$
  - $\mathrm{GL}_n(\mathbb{R}) \supset \mathfrak{U}$: a set of representatives of $\mathfrak{PM}$
  - $\langle , \rangle$: an arbitrary inner product

    $\Rightarrow \ \exists k > 0, \ \exists \varphi \in \mathrm{Aut}(\mathfrak{g}), \ \exists g \in \mathfrak{U}$:

    $\{\varphi g e_1, \ldots, \varphi g e_n\}$ is orthonormal w.r.t. $k\langle , \rangle$.

- **A sketch of the proof:**
  - By assumption, $\exists g \in \mathfrak{U} : \langle , \rangle \in \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).(g.\langle , \rangle_0)$.
  - By definition, $\exists c\varphi \in \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) : \langle , \rangle = (c\varphi g).\langle , \rangle_0$.

2.3 How to get Milnor-type theorems (3/3)

- **Recall:**
  - $\mathrm{GL}_n(\mathbb{R}) \supset \mathfrak{U}$: a set of representatives of $\mathfrak{PM}$
  - $\langle , \rangle$: an arbitrary inner product

    $\Rightarrow \exists k > 0, \ \exists \varphi \in \mathrm{Aut}(\mathfrak{g}), \ \exists g \in \mathfrak{U}$:

    $\{\varphi g e_1, \ldots, \varphi g e_n\}$ is orthonormal w.r.t. $k\langle , \rangle$.

- **Comment:**
  - $\{\varphi g e_1, \ldots, \varphi g e_n\}$ is a generalized Milnor frame.
  - Note: φ preserves a bracket product.
  - $l := \dim \mathfrak{U}$

    $\Rightarrow$ the bracket relations among them contain only l variables.

3 Trivial case

3.1 Trivial case (1/3)

- **It may happen:**
  - $\mathfrak{PM} := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}} = \{\mathrm{pt}\}$ (i.e., $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$: transitive.)
- **Def.:**
  - $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n} := \mathrm{span}\{e_1, \ldots, e_n\}$ with $[e_1, e_j] = e_j$ $(j \geq 2)$
    is called the **Lie algebra of $\mathbb{R}\mathrm{H}^n$**.
- **Fact:**
  - $\mathfrak{so}(1, n) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$: Iwasawa decomposition
    $\Rightarrow$ $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n} \cong \mathfrak{a} \oplus \mathfrak{n}$.
  - $G_{\mathbb{R}\mathrm{H}^n}$: the simply-connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$
    $\Rightarrow$ $G_{\mathbb{R}\mathrm{H}^n} \curvearrowright \mathbb{R}\mathrm{H}^n$: simply transitive (hence $G_{\mathbb{R}\mathrm{H}^n} \cong \mathbb{R}\mathrm{H}^n$).

3.2 Trivial case (2/3)

- **Thm. (Lauret 2003, Kodama-Takahara-T. 2011):**
  - A Lie algebra $\mathfrak{g}$ satisfies $\mathfrak{PM} = \{pt\}$

    $\Leftrightarrow$ (1) $\mathbb{R}^n$: abelian,

    (2) $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n} := \mathrm{span}\{e_1, \ldots, e_n\}$ with $[e_1, e_j] = e_j$ $(j \geq 2)$,

    (3) $\mathfrak{h}^3 \oplus \mathbb{R}^{n-3} := \mathrm{span}\{e_1, \ldots, e_n\}$ with $[e_1, e_2] = e_3$.

- **Comment:**
  - $\mathfrak{PM} = \{pt\}$

    $\Leftrightarrow$ a left-invariant metric is unique up to isometry and scaling

    $\Leftrightarrow$ $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$: transitive.

3.3 Trivial case (3/3)

- **Thm.:**
  - $\langle , \rangle$: an arbitrary inner product on $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$

    $\Rightarrow \exists k > 0$, $\{x_1, \ldots, x_n\}$: o.n.b. w.r.t. $k\langle , \rangle$:

    $$[x_1, x_j] = x_j \quad (j \geq 2).$$

- **Cor. (cf. Milnor 1976):**
  - $\forall \langle , \rangle$ on $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$ has a constant curvature $c < 0$.

- **Comment:**
  - This was first proved by Milnor (1976), but the original proof is very direct and not short.
  - A Milnor-type theorem provides a very simple proof.

4 Three-dimensional case

4.1 Three-dimensional case (1/3)

- **Comment:**
  - $\mathfrak{g}$: solvable Lie algebra, $\dim \mathfrak{g} = 3$ (not necessary unimodular).
  - We ([Hashinaga-T.]) constructed Milnor-type theorems for all $\mathfrak{g}$.
  - Here we mention some of them.

- **Consider:**
  - $\mathfrak{r}'_{3,a} := \mathrm{span}\{e_1, e_2, e_3\}$ $(a \geq 0)$
    where $[e_1, e_2] = ae_2 - e_3$, $[e_1, e_3] = e_2 + ae_3$.
  - Note: $\mathfrak{r}'_{3,a}$ is solvable.
  - Note: $\mathfrak{r}'_{3,a}$ is unimodular $\Leftrightarrow$ $a = 0$.

4.2 Three-dimensional case (2/3)

- **Thm. (Hashinaga-T.):**
  - $\langle , \rangle$: an (arbitrary) inner product on $\mathfrak{r}'_{3,a}$

    $\Rightarrow \ \exists k > 0, \ \exists \lambda \geq 1, \ \exists \{x_1, x_2, x_3\}$: o.n.b. w.r.t. $k\langle , \rangle$:

    $$[x_1, x_2] = a x_2 - \lambda x_3, \quad [x_1, x_3] = (1/\lambda) x_2 + a x_3.$$

- **Proof:**
  - $\mathfrak{U} := \{\mathrm{diag}(1, 1, 1/\lambda) \mid \lambda \geq 1\}$ is a set of representatives of $\mathfrak{PM}$.

- **Cor.:**
  - $\mathfrak{r}'_{3,a}$ admits a left-invariant Einstein metric.

    (It corresponds to the case $\lambda = 1$.)

4.3 Three-dimensional case (3/3)

- **Recall (when $a = 0$):**
  - $\langle , \rangle$: an (arbitrary) inner product on $\mathfrak{r}'_{3,0}$

    $\Rightarrow\ \exists k > 0,\ \exists \lambda \geq 1,\ \exists \{x_1, x_2, x_3\}$: o.n.b. w.r.t. $k\langle , \rangle$:

    $$[x_1, x_2] = -\lambda x_3, \quad [x_1, x_3] = (1/\lambda) x_2.$$

- **Recall (Milnor's Theorem):**
  - $\langle , \rangle$: an (arbitrary) inner product on $\mathfrak{r}'_{3,0}$

    $\Rightarrow\ \exists \lambda_1, \lambda_2 > 0,\ \exists \{y_1, y_2, y_3\}$: o.n.b. w.r.t. $\langle , \rangle$:

    $$[y_1, y_2] = 0, \quad [y_2, y_3] = \lambda_1 y_1, \quad [y_3, y_1] = \lambda_2 y_2.$$

- **Our theorem recovers Milnor's theorem for $\mathfrak{r}'_{3,0}$:**
  - It is enough to put (and suitable change of indices)

    $$y_i := k^{-1/2} x_i,\ \lambda_1 := 0,\ \lambda_2 := \lambda^{-1} k^{-1/2},\ \lambda_3 := \lambda k^{-1/2}.$$

5 Higher dimensional examples

5.1 Higher dimensional examples (1/3)

- **Comment:**
  - In general, it is not easy to express $\mathfrak{PM}$.
  - We here mention some Milnor-type theorems for $\mathfrak{g}$, where $\mathfrak{g}$ is higher dimensional, but $\dim \mathfrak{PM} = 1$.
- **Fact (Berndt-Brück 2002):**
  - M : a Hadamard manifold (e.g., $M = \widetilde{\mathfrak{M}}$)
  - $H \curvearrowright M$: of cohomogeneity one

    $\Rightarrow \; H \backslash M \cong \mathbb{R}$ or $[0, +\infty)$.
- **Recall:**
  - $\mathfrak{U} := \{\mathrm{diag}(1, 1, 1/\lambda) \mid \lambda \geq 1\} \cong [1, +\infty)$ for $\mathfrak{r}'_{3,a}$.

5.2 Higher dimensional examples (2/3)

- **Consider:**
  - $\mathfrak{g}^n_{1,1} := \mathrm{span}\{e_1, \ldots, e_n\}$,
    where $[e_i, e_n] = e_i$ (for $i = 1, \ldots, n-2$),
    $[e_{n-1}, e_n] = e_1 + e_{n-1}$.

- **Thm. (Taketomi-T.):**
  - $\langle , \rangle$: an arbitrary inner product on $\mathfrak{g}^n_{1,1}$
    $\Rightarrow \exists k > 0, \ \exists \lambda > 0, \ \exists\{x_1, \ldots, x_n\}$: o.n.b. w.r.t. $k\langle , \rangle$:
    $[x_i, x_n] = x_i$ (for $i = 1, \ldots, n-2$),
    $[x_{n-1}, x_n] = \lambda x_1 + x_{n-1}$.

- **Cor.:**
  - The above $\mathfrak{g}^n_{1,1}$ does not admit left-invariant Ricci solitons.

5.3 Higher dimensional examples (3/3)

- **Prop. (Taketomi-T.):**
  - $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}^{n}_{1,1}) \curvearrowright \widetilde{\mathfrak{M}}$ **satisfies:**
    - **The orbit space** $\cong \mathbb{R}_{>0}$.
    - **All orbits** $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}^{n}_{1,1}).\langle , \rangle$ **are hypersurfaces in** $\widetilde{\mathfrak{M}}$.
    - **More strongly, all orbits are congruent to each other.**

      (Looks like a horosphere foliation.)

- **Comment:**
  - **Our expectation:** $\langle , \rangle$ **is special** $\Leftrightarrow$ $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ **is special?**
  - **The above observation certificates this expectation.**

    ($\nexists$ **Ricci soliton,** $\nexists$ **special orbits)**

6 A pseudo-Riemannian version

6.1 A pseudo-Riemannian version (1/4)

- **Def:**
  - $\mathfrak{g}$: $(p+q)$-dim. Lie algebra
  - $\widetilde{\mathfrak{M}}_{(p,q)} := \{\langle , \rangle$: an inner product on $\mathfrak{g}$ with signature $(p,q)\}$

    $\cong \mathrm{GL}_{p+q}(\mathbb{R})/\mathrm{O}(p,q)$.
  - $\mathfrak{PM}_{(p,q)} := \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}}_{(p,q)}$: the Moduli space.
- **Thm. (Kubo-Onda-Taketomi-T.):**
  - A set of representatives of $\mathfrak{PM}_{(p,q)}$

    $\Rightarrow$ a pseudo-Riemannian version Milnor-type theorem

    (by the same procedure)

6.2 A pseudo-Riemannian version (2/4)

- **Consider:**
  - $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$: the Lie algebra of $\mathbb{R}\mathrm{H}^n$ ($n = p + q$).

- **Thm. (Kubo-Onda-Taketomi-T.):**
  - $\langle , \rangle$: an arbitrary inner product on $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$ with signature (p, q)

    $\Rightarrow$ $\exists k > 0$, $\exists \lambda \in \{0, 1, 2\}$,

    $\exists \{x_1, \ldots, x_n\}$: pseudo-o.n.b. w.r.t. $k\langle , \rangle$:

    $[x_1, x_j] = x_j$, $[x_1, x_n] = -\lambda x_1 + x_n$, $[x_j, x_n] = -\lambda x_j$

    (for $j \geq 2$)

- **Idea of Proof:**
  - The orbit space $\mathfrak{PM}_{(p,q)} := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}_{(p,q)}$ consists of 3 points.

6.3 A pseudo-Riemannian version (3/4)

- **Recall:**
  - $\#\mathfrak{PM}_{(p,q)} = 3$ for $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$.
- **Cor.:**
  - $\mathfrak{g} := \mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$

    $\Rightarrow$ $\forall \langle , \rangle$: pseudo-Riemannian, it has a constant curvature c.

    (c can take any signature, $c > 0$, $c = 0$, $c < 0$)
- **Comment:**
  - Lorentz version of this corollary has been known (Nomizu 1979).
  - The (pseudo-Riemannian version of) Milnor-type theorem simplifies the proof, and extends it to an arbitrary signature.

6.4 A pseudo-Riemannian version (4/4)

- **Question:**
  - **Why $\#\mathfrak{PM}_{(p,q)} = 3$ for $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$?**

- **In general:**
  - $\mathfrak{PM}_{(p,q)} := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}_{(p,q)} \cong \mathrm{O}(p,q) \backslash (\mathrm{GL}_{p+q}(\mathbb{R}) / \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))$.

- **For $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$:**
  - **$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is parabolic so that $\mathrm{GL}_{p+q}(\mathbb{R}) / \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \cong \mathbb{R}\mathrm{P}^{p+q-1}$.**
  - **The above 3 orbits correspond to $\{[v] \in \mathbb{R}\mathrm{P}^{p+q-1}\}$ with v : timelike, lightlike, spacelike.**

- **Comment:**
  - **This is a very special case...**
  - **For further studies, we need to know actions on $\mathrm{GL}_{p+q}(\mathbb{R}) / \mathrm{O}(p,q)$.**

7 Summary and Problems

7.1 Summary

- **Story:**
  - **The space of left-invariant metrics**
    (both Riemannian and pseudo-Riemannian settings)
    $\Rightarrow$ **the moduli space (= the orbit space)**
    $\Rightarrow$ **Milnor-type theorems**
    $\Rightarrow$ **one can examine ALL left-invariant metrics.**
  - **This can be applied to the existence and nonexistence problem of distinguished (e.g., Einstein, Ricci soliton) metrics.**

- **Point:**
  - **Actions on symmetric spaces play important roles.**

7.2 Problems

- **Problem 1:**
  - **A continuation of this study, i.e.,**
    - **Get more Milnor-type theorems,**
    - **Study isometric actions on symmetric spaces (both Riemannian and pseudo-Riemannian cases).**
- **Problem 2:**
  - **Apply our method to other geometric structures, e.g.,**
    - **left-invariant complex structures,**
    - **left-invariant symplectic structures, ...**

- **Closing:**
  - ご清聴ありがとうございました.
  - 松江セミナー第 **100** 回おめでとうございます.
  - 今後益々のご発展をお祈りいたします.