Left-invariant metrics on Lie groups and submanifold geometry

Hiroshi Tamaru (Hiroshima University)

International Workshop on
Special Geometry and Minimal Submanifolds,
Tohoku University,
09/August/2013

1 Introduction

1.1 Abstract

- We propose a framework to study
  - geometry of **left-invariant metrics on Lie groups**, from the view point of **submanifold geometry**.
- This talk is organized as follows:

1.2 Theme : Left-invariant metrics (1/2)

- **Our theme:**
  - **Left-invariant Riemannian metrics g on Lie groups G.**
- **They can be studied in terms of:**
  - $(\mathfrak{g}, \langle , \rangle)$ **: the corresponding metric Lie algebras.**
  - **The Levi-Civita connection $\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfies**

$$2\langle \nabla_X Y, Z \rangle = \langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle.$$

- **They provide examples of:**
  - **Einstein** $:\Leftrightarrow$ $\mathrm{Ric} = c \cdot \mathrm{id}$ $(\exists c \in \mathbb{R})$,
  - **algebraic Ricci soliton**
    $:\Leftrightarrow$ $\mathrm{Ric} = c \cdot \mathrm{id} + D$ $(\exists c \in \mathbb{R},\ \exists D \in \mathrm{Der}(\mathfrak{g}))$,
  - **Ricci soliton** $:\Leftrightarrow$ $\mathrm{ric} = cg + \mathfrak{L}_X g$ $(\exists c \in \mathbb{R},\ \exists X \in \mathfrak{X}(G))$.

(Fact: Einstein $\Rightarrow$ algebraic Ricci soliton $\Rightarrow$ Ricci soliton)

1.3 Theme : Left-invariant metrics (2/2)

▼ **Example (Heisenberg Lie algebra):**

- $\mathfrak{h}^3 := \mathrm{span}\{e_1, e_2, e_3\}$ with $[e_1, e_2] = e_3$, $[e_1, e_3] = [e_2, e_3] = 0$.
- $\langle , \rangle$: canonical inner product ($\{e_1, e_2, e_3\}$: o.n.b.)

$\Rightarrow$

- $$\mathrm{Ric} = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
- $$\mathrm{Der}(\mathfrak{h}^3) = \left\{ \left(\begin{array}{cc|c} a & * & 0 \\ * & b & 0 \\ \hline * & * & a+b \end{array} \right) \mid a, b \in \mathbb{R} \right\}.$$
- Hence, $(\mathfrak{h}^3, \langle , \rangle)$ is an algebraic Ricci soliton ($\mathrm{Ric} = c \cdot \mathrm{id} + D$).

$$(\because)\ \mathrm{Ric} = -\frac{3}{2} \cdot \mathrm{id} + \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}.$$

1.4 Problem : The existence of distinguished metrics (1/2)

- **Problem:**
  - For a given Lie group G, examine whether G admits a "distinguished" left-invariant metric. (e.g., Einstein, (algebraic) Ricci soliton)

- **Note:**
  - In general, this problem is difficult.
  - Reason: there are so many left-invariant metrics...

$$\widetilde{\mathfrak{M}} := \{\text{left-invariant metrics on } G\}$$
$$\cong \{\text{inner products on } \mathfrak{g}\}$$
$$\cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n).$$

(Note that $n := \dim G$, and $g.\langle\cdot,\cdot\rangle := \langle g^{-1}\cdot, g^{-1}\cdot\rangle$)

1.5 Problem : The existence of distinguished metrics (2/2)

- **Problem:**
  - Examine whether G admits a distinguished left-invariant metric.

- **Philosophy:**
  - It would be related to the algebraic structure of the Lie group.
  - (Alekseevskii conjecture) G : noncompact Einstein $\Rightarrow$ solvable.

- **Related studies:**
  - (Lauret) Approach from GIT (alg. Ricci soliton for solvable case).
  - Classification for low-dimensional cases.
  - Some higher-dimensional examples have been known.

1.6 Strategy : Approach from submanifold geometry (1/2)

- **Recall:**
  - **Examine whether *G* admits a distinguished left-invariant metric.**
  - **It would be related to the algebraic structure of *G*.**

- **Opportunistic hope:**
  - **Are there any invariants (based on the algebraic structure) which determine the existence of distinguished metrics?**

- **Observation:**
  - **$\mathrm{Aut}(\mathfrak{g})$ is an (algebraic) invariant of a Lie algebra $\mathfrak{g}$.**

    (This is a strong invariant, but not so useful...)
  - **$\widetilde{\mathfrak{M}} :=$ {left-invariant metrics on G} $\cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$.**
  - **$\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ gives an isometry (hence preserves curvatures).**

1.7 Strategy : Approach from submanifold geometry (2/2)

- **We consider:**
  - $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$.
  - This gives "isometry up to scalar",
    and hence preserves all Riemannian geometric properties.

- **Def:**
  - $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$: the isometry and scaling class of $\langle , \rangle$.
  - The orbit space $\mathfrak{PM} := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}$: the moduli space.

- **Remark:**
  - $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ is an invariant of $(G, \langle , \rangle)$.
  - $\mathfrak{PM}$ (or $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$) is an (algebraic) invariant of G.

1.8 Result :

- **Expectation:**
  - $\langle , \rangle$ **is distinguished (as Riemannian metrics)**

    $\Leftrightarrow$ $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ **is distinguished (as submanifolds)?**

- **Our Results:**
  - **In some cases, our expectation is true.**

- **One of Thm. (Hashinaga-T.):**
  - **Let** $\mathfrak{g}$ **: solvable, dim = 3.**
  - $(\mathfrak{g}, \langle , \rangle)$ **: an algebraic Ricci soliton**

    $\Leftrightarrow$ $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ **: a minimal submanifold.**

2 Preliminaries on submanifold geometry

2.1 Cohomogeneity one actions (1/3)

- **Note:**
  - $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$.
  - $\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$: a noncompact symmetric space.
  - Well-studied for "cohomogeneity one" case.

- **Def.:**
  - $H \curvearrowright (M, g)$: of cohomogeneity one

    :⇔ maximal dimensional orbits have codimension one

    (⇔ the orbit space $H \backslash M$ has dimension one).

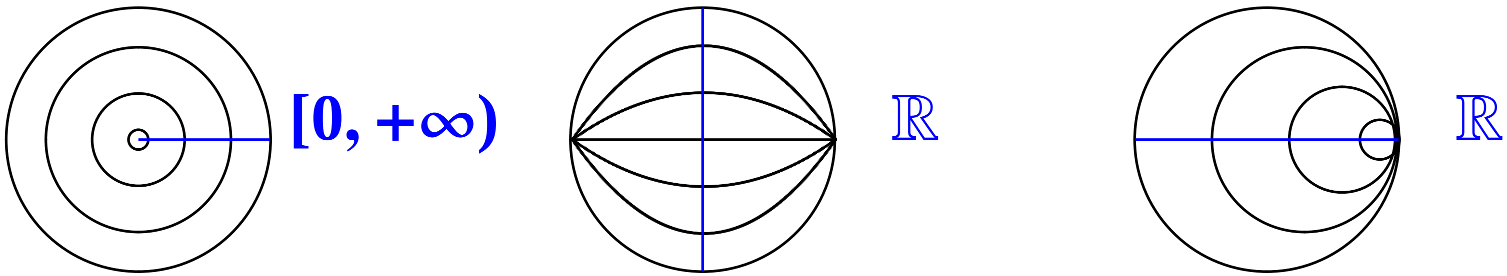
2.2 Cohomogeneity one actions (2/3) - on $\mathbb{R}H^2$

- **Fact (due to Cartan):**
  - $H \curvearrowright \mathbb{R}H^2$: cohomogeneity one action (with H connected)

    $\Rightarrow$ this is orbit equivalent to the actions of

    $$K = SO(2),\ A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a > 0 \right\},\ N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}$$

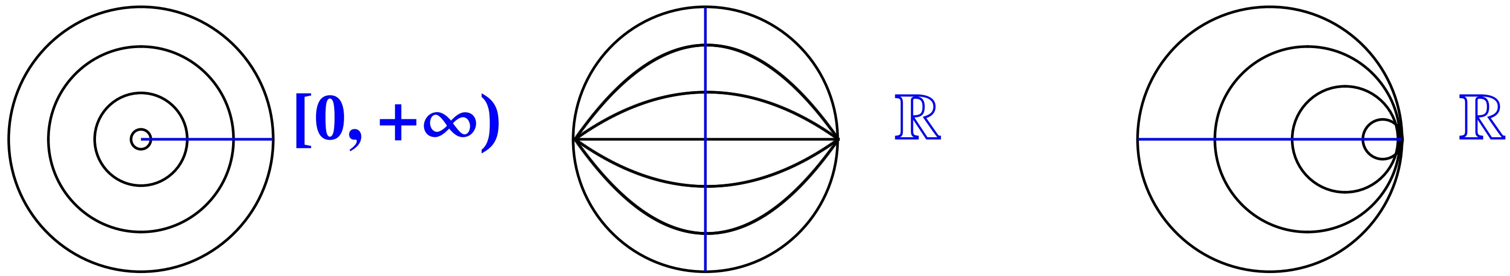
- **Picture:**



type (K) type (A) type (N)

2.3 Cohomogeneity one actions (3/3) - geometry of orbits

- **Recall:**



type (K) type (A) type (N)

- **Thm. (Berndt-Brück 2002, Berndt-T. 2003):**
  - M : noncompact symmetric space, irreducible.
  - $H \curvearrowright M$: cohomogeneity one (H : connected)

  $\Rightarrow$ (type (K)) $\exists 1$ singular orbit,

  (type (A)) $\nexists$ singular orbit, $\exists 1$ minimal orbit, or

  (type (N)) $\nexists$ singular orbit, all orbits are congruent.

3 Three-dimensional case

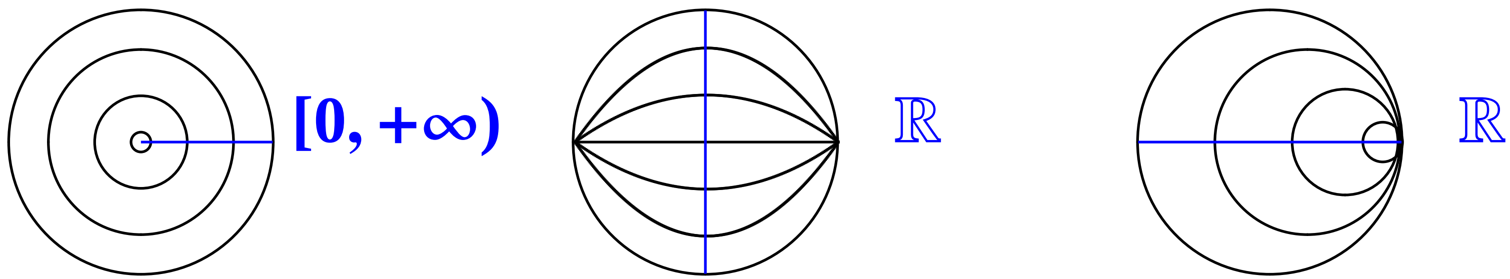
3.1 Three-dimensional case (1/4) - table

- **Fact:**
  - $\mathfrak{g} = \mathrm{span}\{e_1, e_2, e_3\}$: 3-dim. solvable (non abelian) is isomorphic to one of the following:

name	comment	brackets
$\mathfrak{h}^3$	Heisenberg	$[e_1, e_2] = e_3$
$\mathfrak{r}_{3,1}$	$\mathfrak{g}_{\mathbb{R}H^3}$	$[e_1, e_2] = e_2,\ [e_1, e_3] = e_3$
$\mathfrak{r}_3$		$[e_1, e_2] = e_2 + e_3,\ [e_1, e_3] = e_3$
$\mathfrak{r}_{3,a}$	$-1 \leq a < 1$	$[e_1, e_2] = e_2,\ [e_1, e_3] = ae_3$
$\mathfrak{r}'_{3,a}$	$a \geq 0$	$[e_1, e_2] = ae_2 - e_3,\ [e_1, e_3] = e_2 + ae_3$

3.2 Three-dimensional case (2/4) - action

- **Recall:**



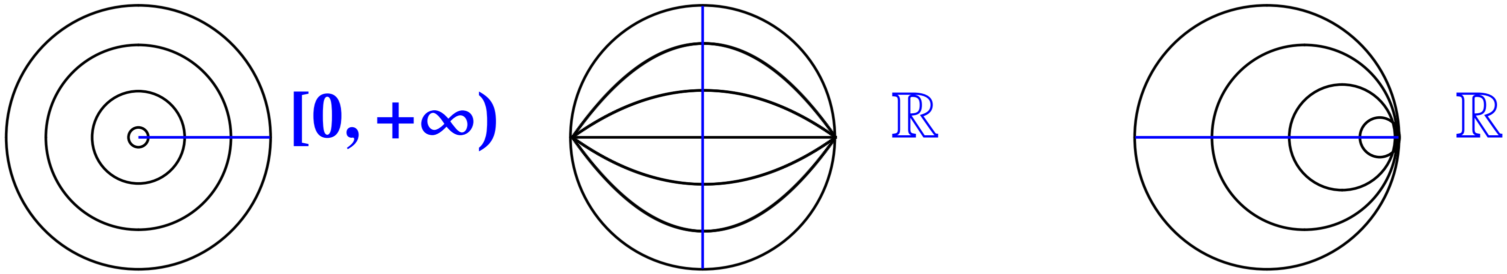
type (K) type (A) type (N)

- **Prop. (Hashinaga-T.):**

name	$(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))^0 \curvearrowright \widetilde{\mathfrak{M}}$	comment
$\mathfrak{h}^3$	transitive	$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$: parabolic
$\mathfrak{r}_{3,1}$	transitive	$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$: parabolic
$\mathfrak{r}_3$	type (N)	all orbits are congruent
$\mathfrak{r}_{3,a}$	type (A)	$\nexists$ singular orbit, $\exists 1$ minimal orbit
$\mathfrak{r}'_{3,a}$	type (K)	$\exists 1$ singular orbit

3.3 Three-dimensional case (3/4) - metric

- **Recall:**



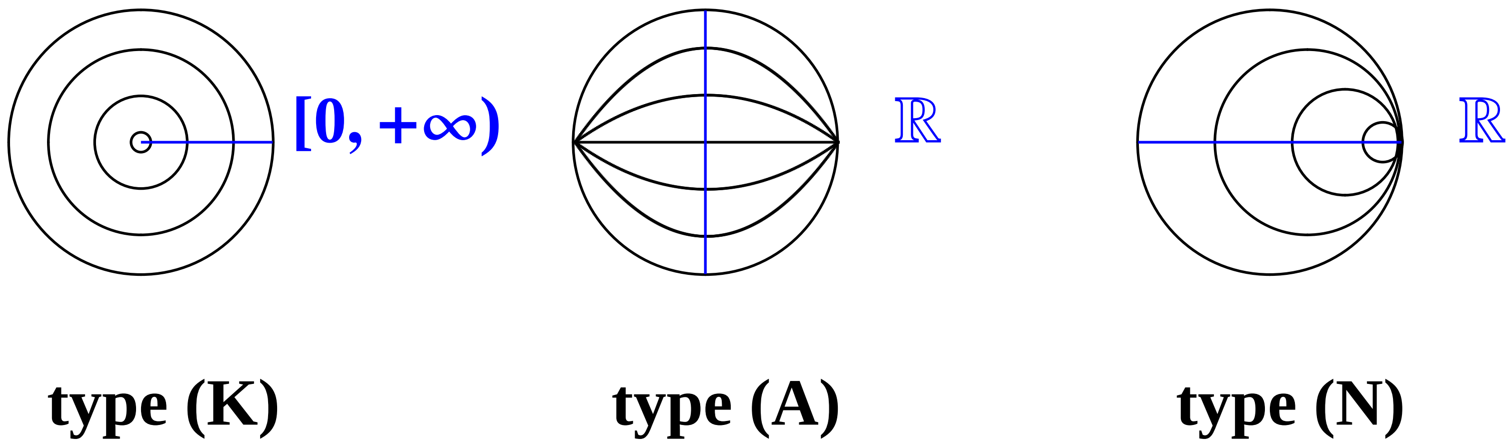
type (K) type (A) type (N)

- **Prop.:**

name	action	metric
$\mathfrak{h}^3$	transitive	algebraic Ricci soliton
$\mathfrak{r}_{3,1}$	transitive	const. negative curvature
$\mathfrak{r}_3$	type (N)	$\nexists$ Ricci soliton
$\mathfrak{r}_{3,a}$	type (A)	algebraic Ricci soliton $\Leftrightarrow$ minimal
$\mathfrak{r}'_{3,a}$	type (K)	Einstein $\Leftrightarrow$ singular

3.4 Three-dimensional case (4/4) - Theorem

- **Recall:**



- **Thm. (Hashinaga-T.):**
  - $\mathfrak{g}$: solvable, dim = 3.
  - $(\mathfrak{g}, \langle , \rangle)$: an algebraic Ricci soliton

    $\Leftrightarrow$ $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).\langle , \rangle$: minimal in $\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$.

4 Higher dimensional examples

4.1 Higher dimensional examples (1/4)

- **Note:**
  - $\mathfrak{g}$: three-dimensional solvable

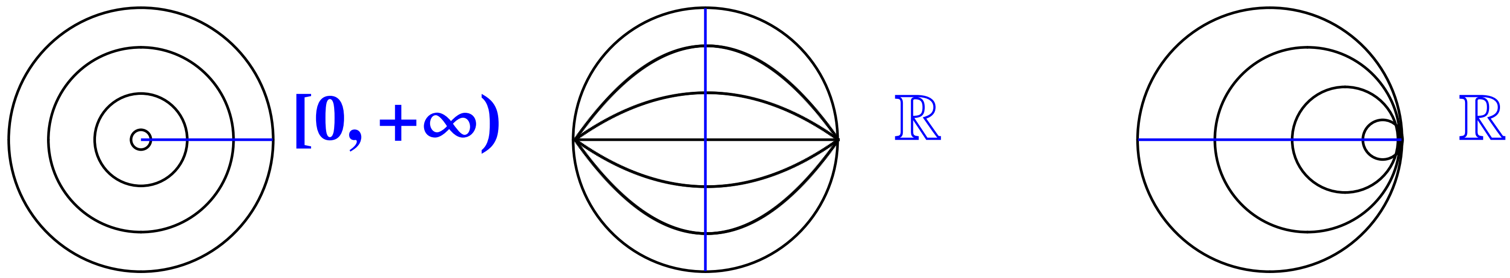
    $\Rightarrow$ • $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$: cohomogeneity at most one,

    • $\langle , \rangle$: algebraic Ricci soliton $\Leftrightarrow$ $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).\langle , \rangle$: minimal.

- **Natural question:**
  - How are the higher dimensional cases?
  - Still $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ can be of cohomogeneity one?

4.2 Higher dimensional examples (2/4)

- **Recall:**

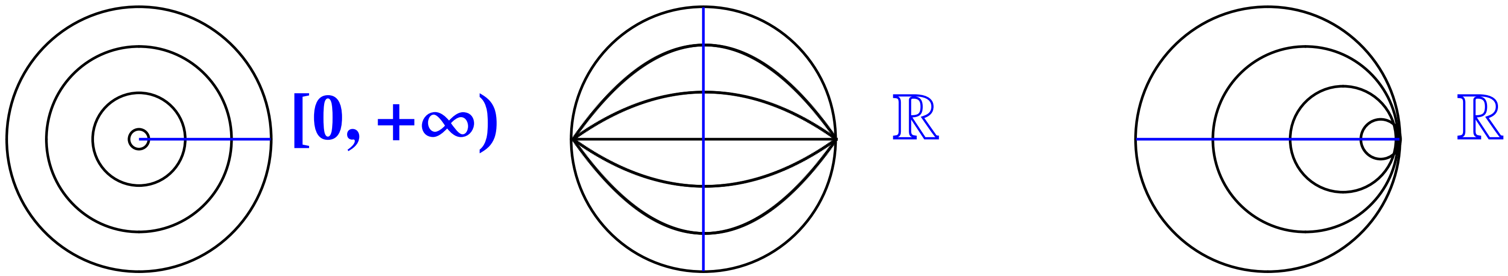


type (K) **type (A)** **type (N)**

- **Thm. (Hashinaga-T.-Terada):**
  - $\forall n \geq 4$, $\exists \mathfrak{g}$: solvable, $\dim \mathfrak{g} = n$
    s.t. • $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$: cohomogeneity one, type (K),
    • $\langle , \rangle$: Ricci soliton $\Leftrightarrow$ $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).\langle , \rangle$: singular.

4.3 Higher dimensional examples (3/4)

▼ **Recall:**



type (K) **type (A)** **type (N)**

▼ **Thm. (Taketomi-T.):**

- $\forall n \geq 4$, $\exists \mathfrak{g}$: solvable, $\dim \mathfrak{g} = n$

  s.t. • $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$: cohomogeneity one, type (N),

  • $\nexists \langle , \rangle$: Ricci soliton.

4.4 Higher dimensional examples (4/4)

- **Expectation:**
  - $\langle , \rangle$ **is distinguished (as Riemannian metrics)**

    $\Leftrightarrow$ $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ **is distinguished (as submanifolds)?**

- **Comments:**
  - **This expectation is true for several cases.**
  - **All results are "experimentally".**
  - **Are there any theoretical reason behind?**

5 A pseudo-Riemannian version

5.1 A pseudo-Riemannian version (1/4) - the moduli space

- **Note:**
  - **Our framework can also be applied to left-invariant pseudo-Riemannian metrics.**
- **Def:**
  - $\mathfrak{g}$: $(p+q)$-dim. Lie algebra
  - $\widetilde{\mathfrak{M}}_{(p,q)} := \{\langle, \rangle$: an inner product on $\mathfrak{g}$ with signature $(p,q)\}$ $\cong \mathrm{GL}_{p+q}(\mathbb{R})/\mathrm{O}(p,q)$.
  - $\mathfrak{PM}_{(p,q)} := \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}}_{(p,q)}$: the Moduli space.

5.2 A pseudo-Riemannian version (2/4) - preliminary

- **Def.:**
  - $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n} := \mathrm{span}\{e_1, \ldots, e_n\}$ with $[e_1, e_j] = e_j$ $(j \geq 2)$ is called the **Lie algebra of $\mathbb{R}\mathrm{H}^n$**.

- **Thm.:**
  - $\forall \langle , \rangle$: Riemannian on $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$, $(\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}, \langle , \rangle)$ has constant curvature $c < 0$.

- **Comments on Proof:**
  - Original Proof: Milnor (1976).
  - Another Proof: Lauret (2003).
  - Simpler Proof using $\mathfrak{PM} = \{\mathrm{pt}\}$: Kodama-Takahara-T. (2011).

5.3 A pseudo-Riemannian version (3/4) - our result

- **Thm. (Kubo-Onda-Taketomi-T.):**
  - $\forall \langle , \rangle$: pseudo-Riemannian on $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$,

    $(\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}, \langle , \rangle)$ has constant curvature c.

    (c can take any signature, $c > 0$, $c = 0$, $c < 0$)

- **Comments on Proof:**
  - Lorentz case has been known.

    (Nomizu 1979; his proof is very direct.)
  - Our proof: $\#\mathfrak{PM}_{(p,q)} = 3$.
  - Recall:
    - $\mathfrak{PM}_{(p,q)} := \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}}_{(p,q)}$.
    - $\widetilde{\mathfrak{M}}_{(p,q)} = \mathrm{GL}_{p+q}(\mathbb{R})/\mathrm{O}(p, q)$.

5.4 A pseudo-Riemannian version (4/4) - comments

- **Our result:**
  - **It is only on $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$, and a little bit more...**
- **Comment:**
  - **For further studies, we need to know the geometry of isometric actions $H \curvearrowright \mathrm{GL}_{p+q}(\mathbb{R})/\mathrm{O}(p, q)$.**

    (submanifold geometry in pseudo-Riem. symmetric spaces...)

6 Summary and Problems

6.1 Summary

- **Story:**
  - **We are interested in geometry of left-invariant metrics (both Riemannian and pseudo-Riemannian)**
  - **An approach from submanifold geometry**
    - **one can study all metrics efficiently,**
    - **left-invariant metrics $\leftrightarrow$ properties of submanifolds?**

- **Results:**
  - **In several cases,**

    $\langle , \rangle$: (algebraic) Ricci soliton $\Leftrightarrow$ $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).\langle , \rangle$: distinguished.

6.2 Problems

- **Problem 1:**
  - **A continuation of this study, i.e.,**
    - **check the correspondence for other cases,**
      ((algebraic) Ricci soliton ↔ distinguished orbit?)
    - **are there any theoretical reason behind?**

- **Problem 2:**
  - **Apply our method to other geometric structures, e.g.,**
    - **left-invariant pseudo-Riemannian metrics (have just started),**
    - **left-invariant complex structures,**
    - **left-invariant symplectic structures, ...**