Group actions on symmetric spaces related to left-invariant geometric structures

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Abstract

In view of left-invariant geometric structures, the following actions are interesting:

(I, II) isometric actions on noncpt Riem. symmetric sp.
(III) nonisometric actions on (symmetric) R-spaces.
(IV) isometric actions on pseudo-Riem. symmetric sp.

Aim of this talk

In this talk, we
- mention our framework and results.
- propose some problems on groups actions.
We are interested in left-invariant geometric structures.

Good Point (1)
Relatively easier to treat.
Many arguments can be reduced to the Lie algebra level.

Good Point (2)
Left-invariant geometric structures provide examples of
- Einstein / Ricci soliton metrics,
- (generalized) complex / symplectic structures, ...
Central Problem
For a given Lie group, determine whether it admits a “good” left-invariant geometric structure or not.

Remark
For a given Lie group and a given geometric structure, one can directly study its property, e.g.,
- calculate the curvatures of a left-invariant metric,
- check the integrability condition of an almost complex structure, ...

But this does not mean that the above problem is easy.
Central Problem (recall)

For a given Lie group, determine whether it admits a “good” left-invariant geometric structure or not.

What is difficult? (1)

There are so many Lie algebras...

\[
\{ G : \text{simply-connected Lie group with } \dim G = n \} \\
\cong \{ g : \text{Lie algebra with } \dim g = n \} \\
\cong \{ [ , ] \in \wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n : \text{satisfying Jacobi identity} \}.
\]
What is difficult? (2)

The space of the left-invariant structures on $G$ is large. For examples, if $\dim G = n$ then

$$\{\text{left-invariant Riem. metrics on } G\}$$

$$\cong \{\text{inner products on } g := \text{Lie}(G)\}$$

$$\cong \text{GL}_n(\mathbb{R})/\text{O}(n).$$

$$\{\text{left-invariant metrics on } G \text{ with signature } (p, q)\}$$

$$\cong \text{GL}_n(\mathbb{R})/\text{O}(p, q).$$

These spaces are symmetric spaces. We study our problem in terms of groups actions!
Slogan

Left-invariant metrics vs isom. actions on $\text{GL}_n(\mathbb{R})/\text{O}(n)$.

Problem (I)

$M = G/K$ : Riem. symmetric space of noncompact type.
Consider $H \curvearrowright G/K$ : isometric action.
Then, what the orbit space $H \backslash M$ can be?

Note

In this section, we mention:

- What are known?
- How related to the study of left-invariant metrics?
Consider $K, A, N \sim \mathbb{R}H^2 = \text{SL}_2(\mathbb{R})/\text{SO}(2)$, where

$$K = \text{SO}(2),$$

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mid a > 0 \right\},$$

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}.$$

Then the orbits look like

- type (K)
- type (A)
- type (N)
Riemannian metrics (I) - (3/9)

Picture for $\mathbb{R}H^2$

- type (K)
- type (A)
- type (N)

Theorem (Berndt-Brück (2001))

$M = G/K$: Riem. symmetric space of noncompact type.
$H \lhd G/K$: cohomogeneity one (with $H$ connected).
$\Rightarrow H \backslash M \cong \mathbb{R}$ or $[0, +\infty)$. 
Fix $G$: Lie group with $\dim G = n$.
We now consider left-invariant Riemannian metrics on $G$.

**Def.**
The space of left-invariant Riemannian metrics:
\[
\widehat{M} := \{\text{left-invariant Riem. metrics on } G\}
\cong \{\text{inner products on } g := \text{Lie}(G)\}
\cong \text{GL}_n(\mathbb{R})/\text{O}(n).
\]

**Note**
\[
\text{GL}_n(\mathbb{R}) = \text{GL}(g) \curvearrowright \widehat{M} \text{ by } g.\langle \cdot, \cdot \rangle := \langle g^{-1}(\cdot), g^{-1}(\cdot) \rangle.
\]
\[ \mathbb{R}^\times := \{ c \cdot \text{id} \in \text{GL}_n(\mathbb{R}) \mid c \in \mathbb{R} \neq 0 \}, \]
\[ \text{Aut}(g) := \{ \varphi \in \text{GL}_n(\mathbb{R}) \mid \varphi[\cdot, \cdot] = [\varphi(\cdot), \varphi(\cdot)] \}. \]

**Def.**

The **moduli space of left-invariant Riem. metrics**:

\[ \mathcal{PM} := \mathbb{R}^\times \text{Aut}(g) \backslash \widetilde{M} \quad \text{(orbit space)}. \]

**Remark**

\[ \mathbb{R}^\times \text{Aut}(g) \curvearrowright \widetilde{M} \text{ gives rise to an isometry up to scaling.} \]
Prop. (Hashinaga-T. (preprint))

\( \mathfrak{g} := (\mathbb{R}^3, [, ,]) \) with \([e_1, e_2] = e_2\) (others = 0).

\( \langle , \rangle_0 : \) the canonical inner product.

\[ \Rightarrow \]

\( \mathbb{R} \times \text{Aut}(\mathfrak{g}) \curvearrowright \tilde{M} \) is of cohomogeneity one.

\[ \mathcal{PM} = \{ \mathbb{R} \times \text{Aut}(\mathfrak{g}).( \begin{pmatrix} 1 & 1 \\ \lambda & 1 \end{pmatrix} . \langle , \rangle_0 | \lambda \in \mathbb{R} \} \].

Proof

Very direct matrix calculations.
A general theory gives a certification.
Cor. (Hashinaga-T. (preprint))

\[ g := (\mathbb{R}^3, [, ]) \] with \([e_1, e_2] = e_2 \) (others = 0).
\[
\langle , \rangle : \text{any inner product.}
\Rightarrow
\exists \lambda \in \mathbb{R}, \exists k > 0, \exists \{x_1, x_2, x_3\} \text{ o.n.b. w.r.t. } k\langle , \rangle:
\]

\[ [x_1, x_2] = x_2 - \lambda x_3, \quad \text{others} = 0. \]

Note

\{x_1, x_2, x_3\} is a generalization of “Milnor frames”.
Hence, this is called a Milnor-type theorem.
Result (Hashinaga-T.-Terada)

∀n ≥ 3,

- we construct $g$ of dimension $n$ such that $\mathbb{R} \times \text{Aut}(g) \curvearrowright \mathcal{M}$: cohomogeneity one.
- we determine all possible Ricci signatures on them.

Result (Hashinaga-T.)

∀g : 3-dim., solvable,

- we construct Milnor-type theorems for $g$.
- we (re)classify left-invariant Ricci soliton metrics.

Key Point: for cohomogeneity one actions, $\mathcal{PM}$ is easy!
Problem (I)-1
For Riemannian symmetric spaces of noncompact type, classify (possible topological type of) orbit spaces of
- cohomogeneity two actions,
- (hyper)polar actions, ...

Problem (I)-2
Classify $g$ such that $\mathbb{R}^\times \text{Aut}(g) \curvearrowright \tilde{M}$ are
- cohomogeneity one or two actions,
- (hyper)polar actions, ...
(∃ examples by Taketomi (2014))
Riemannian metrics (II) - (1/8)

Slogan
Left-invariant metrics vs isom. actions on $\mathbb{GL}_n(\mathbb{R})/O(n)$.

Problem (II)
$M = G/K$ : Riem. symmetric space of noncompact type.
Consider $H \ltimes G/K$ : isometric action.
Then, are there “distinguished” orbits?

Note
In this section, we mention:
- What are known?
- How related to the study of left-invariant metrics?
Thm. (Berndt-T. (2003))

\[ M = G/K : \text{irr. Riem. symmetric space of noncpt type.} \]

\[ H \curvearrowright G/K : \text{cohomogeneity one (with } H \text{ connected).} \]

\[ \Rightarrow \text{it satisfies one of the following:} \]

\( (K) \) \( \exists! \) singular orbit.

\( (A) \) \( \nexists \) singular orbit, \( \exists! \) minimal orbit.

\( (N) \) \( \nexists \) singular orbit, all orbits are congruent.

\[ \begin{align*}
\text{type (K)} & \quad [0, +\infty) \\
\text{type (A)} & \quad \mathbb{R} \quad \mathbb{R} \\
\text{type (N)} & \quad \mathbb{R}
\end{align*} \]
This picture fits very nicely to “algebraic Ricci solitons”.

Def.

\[(\mathfrak{g}, \langle \cdot, \cdot \rangle) : \text{algebraic Ricci soliton (ARS)} \]
\[\Leftrightarrow \exists c \in \mathbb{R}, \exists D \in \text{Der}(\mathfrak{g}) : \text{Ric} = c \cdot \text{id} + D.\]

\[\text{Der}(\mathfrak{g}) := \{ D \in \mathfrak{gl}(\mathfrak{g}) \mid D[\cdot, \cdot] = [D(\cdot), \cdot] + [\cdot, D(\cdot)]\}.\]

Note

- left-invariant Einstein \(\Rightarrow\) algebraic Ricci soliton.
- algebraic Ricci soliton \(\Rightarrow\) Ricci soliton (next page).

(in many cases, the converse also holds)
Prop. (Lauret (2011))

Let us consider

- \((g, \langle \cdot, \cdot \rangle)\) : algebraic Ricci soliton \((\text{Ric} = c \cdot \text{id} + D)\),
- \((G, g)\) : corresponding simply-connected one.

Then one has

- \(\exp(tD) \in \text{Aut}(g)\) for \(\forall t \in \mathbb{R}\),
- \(\exists \varphi_t \in \text{Aut}(G) : (d \varphi_t)_e = \exp(tD)\),
- \(X \in \mathfrak{X}(G)\) by \(X_p := \frac{d}{dt} \varphi_t(p)|_{t=0}\),
- \(\text{ric}_g = cg - (1/2)\mathcal{L}_X g\) (i.e., Ricci soliton).
Thm. (Hashinaga-T.)

\( \mathfrak{g} \): 3-dimensional, solvable. Then, 
\[ \langle \cdot, \cdot \rangle \text{ on } \mathfrak{g} \text{ is an algebraic Ricci soliton (ARS)} \]
\[ \Leftrightarrow \tilde{\mathcal{M}} \supset \mathbb{R}^\times \text{Aut}(\mathfrak{g}).\langle \cdot, \cdot \rangle: \text{ minimal}. \]

Note

- \( \tilde{\mathcal{M}} = \text{GL}_n(\mathbb{R})/\text{O}(n) \) is a Riem. symmetric space (w.r.t. a natural \( \text{GL}_n(\mathbb{R}) \)-invariant metric).
- In these cases, \( \mathbb{R}^\times \text{Aut}(\mathfrak{g}) \bowtie \tilde{\mathcal{M}} \) is of cohom. \( \leq 1 \).
Thm. (Hashinaga-T.): more precise

Let $g$ : 3-dimensional, solvable. Then,

- (K)-type: $\text{ARS} \iff \mathbb{R}^\times \text{Aut}(g).\langle,\rangle : \text{singular}$.
- (A)-type: $\text{ARS} \iff \mathbb{R}^\times \text{Aut}(g).\langle,\rangle : \text{minimal}$.
- (N)-type: $\not\exists \text{ARS}$.
Fact

For all known $g$ with $\mathbb{R}^\times \text{Aut}(g) \curvearrowright \hat{\mathcal{M}}$ cohom. one, we have checked that “ARS $\Leftrightarrow$ minimal”.

Thm. (Hashinaga (to appear))

$\exists g : 4\text{-dim.}, \text{solvable} :$
both implications of “ARS $\Leftrightarrow$ minimal” do not hold.

Expectation

ARS $\Leftrightarrow \mathbb{R}^\times \text{Aut}(g).\langle,\rangle$ is “distinguished” in some sense? (cf. $\langle,\rangle$ : biinvariant $\Rightarrow \mathbb{R}^\times \text{Aut}(g).\langle,\rangle$ is totally geod.)
Problem (II)-1

For Riemannian symmetric spaces of noncompact type, study the geometry of orbits of
- cohomogeneity one actions (with $H$ not connected),
- cohomogeneity two actions, (hyper)polar actions, ...

Problem (II)-2

Property of $\mathfrak{g}$ can be understood by group actions?
- ARS can be characterized by submanifolds?
- Tasaki-Umehara invariant for 3-dim. Lie algebras?
**Def.**

$L$ : semisimple Lie group, with trivial center,  
$L \supset Q$ : parabolic subgroup.  
Then $M := L/Q$ is called an **R-space**.

**Problem (III)**

Let $M = L/Q : R$-space, $H \subset L$.  
Study the action $H \curvearrowright M = L/Q$.

**Note**

In this section, we mention: An easy example.  
(Motivation will be mentioned in the next section.)
Example

$\mathbb{R}P^n$ is an R-space:

$$\mathbb{R}P^{n-1} = \text{SL}_n(\mathbb{R})/\left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix} \right\}.$$

Side Remark

- R-spaces can be realized as orbits of s-reps.
- $\mathbb{R}P^{n-1}$ is an orbit of the s-rep. of $\text{SL}_n(\mathbb{R})/\text{SO}(n)$. 
Set Up

We consider $SO(p, q) \curvearrowright \mathbb{R}P^{n-1}$ (with $n = p + q$).

- $\mathbb{R}P^{n-1} = SL_n(\mathbb{R})/Q$, and
- $SL_n(\mathbb{R}) \supset SO(p, q)$: symmetric.

(an analogy of “Hermann actions”?)

Remark

$\langle , \rangle_0$: canonical inner product on $\mathbb{R}^n$, signature $(p, q)$. Then, $SO(p, q) \curvearrowright \mathbb{R}^n$ preserves $\langle , \rangle_0$. 
Prop.

$\text{SO}(p, q) \ltimes \mathbb{RP}^{n-1}$ has three orbits:

$O^+ := \{ [v] \in \mathbb{RP}^{n-1} \mid \langle v, v \rangle_0 > 0 \},$

$O^0 := \{ [v] \in \mathbb{RP}^{n-1} \mid \langle v, v \rangle_0 = 0 \},$

$O^- := \{ [v] \in \mathbb{RP}^{n-1} \mid \langle v, v \rangle_0 < 0 \}.$

Note

$O^+, O^-$ are open.
Problem (III)-1

Let $M = L/Q : \text{R-space}$, $H \subset L$. Study the action $H \ltimes M = L/Q$.

- Construct interesting examples.
- What happens if $L \supset H$ is symmetric?
- When it has an open orbit?

Problem (III)-2

Let $M = L/Q : \text{R-space}$, $H \subset L$.

- Study the geometry of orbits.
- $H.p$ can be inhomogeneous w.r.t. $\text{Isom}(M)$, but have some “nice” properties?
Slogan
Left-invariant pseudo-Riemannian metrics vs isometric actions on $\text{GL}_n(\mathbb{R})/\text{O}(p, q)$.

Problem (IV)

$M = G/K$: pseudo-Riemannian symmetric space.
Study isometric actions $H \curvearrowright M$.

In this section, we mention:

- An easy example.
- How related to left-invariant pseudo-Riem. metrics.
- How related to R-spaces.
The following action has exactly three orbits:

\[
Q := \left\{ \begin{pmatrix}
* & * & \cdots & * \\
0 & \ddots & & \\
\vdots & & * & \\
0 & & & \\
\end{pmatrix} \right\} \curvearrowright \frac{\text{GL}_{p+q}(\mathbb{R})}{O(p, q)}.
\]

Proof

The orbit space coincides with the orbit space of

\[
O(p, q) \curvearrowright \frac{\text{GL}_{p+q}(\mathbb{R})}{Q} = \mathbb{RP}^{p+q-1}.
\]
Fix $G$ : Lie group with $\dim G = n = p + q$.

**Def.**

The **space of left-inv. metrics with signature** $(p, q)$:

$$\widetilde{M}_{p,q} := \{\text{left-invariant metrics on } G \text{ with } (p, q)\}$$

$$\cong \{\text{inner products on } g := \text{Lie}(G) \text{ with } (p, q)\}$$

$$\cong \text{GL}_n(\mathbb{R})/\text{O}(p, q).$$

The **moduli space of left-inv. metrics with** $(p, q)$:

$$\mathcal{PM}_{p,q} := \mathbb{R}^\times \text{Aut}(g) \backslash \widetilde{M}_{p,q} \quad \text{(orbit space)}.$$
Lem.

Let \( g \) be one of the following:

- \( H^3 \): Heisenberg group.
- \( G_{RH^n} \): the group acting simply-transitively on \( RH^n \).

Then \( \mathbb{R}^\times \text{Aut}(g) \) is parabolic (\( = \) the previous \( Q \)).

Thm. (Kubo-Onda-Taketomi-T.)

On the above Lie groups,

\[
\forall (p, q) \in \mathbb{N}^2 \text{ with } n = p + q,
\exists \text{ exactly three left-invariant metrics with } (p, q).
\]
Thm. (Kubo-Onda-Taketomi-T.): recall

On the above Lie groups,
\[ \forall (p, q) \in \mathbb{N}^2 \text{ with } n = p + q, \]
\[ \exists \text{ exactly three left-invariant metrics with } (p, q). \]

Comments

- \( H^3 \): known by Rahmani (1992), method is different.
- \( G_{RH^n} \): known by Nomizu (1979) for Lorentzian case. The case of generic signature is new.

Side Remark

For \( G_{RH^n} \), any left-invariant metric has const. curvature.
### Problem (IV)-1

\[ M = G/K : \text{pseudo-Riemannian symmetric space.} \]

Study isometric actions \( H \curvearrowright M \).

- First of all, study the case when \( H \) is parabolic. 

(\( \Leftrightarrow \) symmetric actions on R-spaces.)

### Problem (IV)-2

\[ M = G/K : \text{pseudo-Riemannian symmetric space.} \]

- Construct “nice” actions on \( M = G/K \).
- Let \( M' = G/K' : \text{Riemannian symmetric.} \)
  
  If \( H \curvearrowright M' \) is nice, then so is \( H \curvearrowright M \)?
Comment

Our framework is

- left-inv. metrics vs actions on symmetric spaces.
- theory of symmetric spaces is quite useful.

This would also be useful to study

- left-invariant complex structures:
  \[ \{ \text{almost complex strs} \} \cong \text{GL}_{2n}(\mathbb{R})/\text{GL}_n(\mathbb{C}). \]
- left-invariant symplectic structures:
  \[ \{ \text{nondegenerate 2-forms} \} \cong \text{GL}_{2n}(\mathbb{R})/\text{Sp}_{2n}(\mathbb{R}). \]
- and so on...
Our framework motivates to study:
- isometric actions on noncompact Riem. symmetric sp.
- nonisometric actions on (symmetric) R-spaces.
- isometric actions on pseudo-Riem. symmetric sp.

This would provide:
- new examples of actions (from \text{Aut}(g)).
- particular examples of actions with applications.
- new ideas or new notions?
References


Thank you!