Group actions on symmetric spaces related to left-invariant geometric structures

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Abstract

In view of left-invariant geometric structures,
the following actions are interesting:

(I, II) isometric actions on noncpt Riem. symmetric sp.

(III) nonisometric actions on (symmetric) R-spaces.

(IV) isometric actions on pseudo-Riem. symmetric sp.

Aim of this talk

In this talk, we

- mention our framework and results.
- propose some problems on groups actions.

Introduction - (1/4)

We are interested in left-invariant geometric structures.

Good Point (1)

Relatively easier to treat.
Many arguments can be reduced to the Lie algebra level.

Good Point (2)

Left-invariant geometric structures provide examples of

- Einstein / Ricci soliton metrics,
- (generalized) complex / symplectic structures, ...

Introduction - (2/4)

Central Problem

For a given Lie group, determine whether it admits a "good" left-invariant geometric structure or not.

Remark

For a given Lie group and a given geometric structure, one can directly study its property, e.g.,

- calculate the curvatures of a left-invariant metric,
- check the integrability condition of an almost complex structure, ...

But this does not mean that the above problem is easy.

Introduction - (3/4)

Central Problem (recall)

For a given Lie group, determine whether it admits a "good" left-invariant geometric structure or not.

What is difficult? (1)

There are so many Lie algebras...

$$\begin{aligned}&\{G : \text{simply-connected Lie group with } \dim G = n\}\\ &\quad\cong \{\mathfrak{g} : \text{Lie algebra with } \dim \mathfrak{g} = n\}\\ &\quad\cong \{[,] \in \wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n : \text{satisfying Jacobi identity}\}.\end{aligned}$$

Introduction - (4/4)

What is difficult? (2)

The space of the left-invariant structures on G is large. For examples, if $\dim G = n$ then

$$\begin{aligned}&\{\text{left-invariant Riem. metrics on } G\}\\ &\quad \cong \{\text{inner products on } \mathfrak{g} := \mathrm{Lie}(G)\}\\ &\quad \cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n).\end{aligned}$$

$$\begin{aligned}&\{\text{left-invariant metrics on } G \text{ with signature } (p,q)\}\\ &\quad \cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(p,q).\end{aligned}$$

These spaces are symmetric spaces.
We study our problem in terms of groups actions!

Riemannian metrics (I) - (1/9)

Slogan

Left-invariant metrics vs isom. actions on $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$.

Problem (I)

$M = G/K$: Riem. symmetric space of noncompact type.
Consider $H \curvearrowright G/K$: isometric action.
Then, what the orbit space $H \backslash M$ can be?

Note

In this section, we mention:

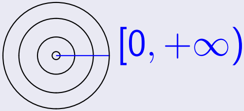
- What are known?
- How related to the study of left-invariant metrics?

Riemannian metrics (I) - (2/9)

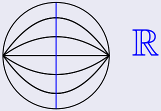
Consider $K, A, N \curvearrowright \mathbb{R}\mathrm{H}^2 = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$, where

$$K = \mathrm{SO}(2),$$
$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mid a > 0 \right\},$$
$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}.$$

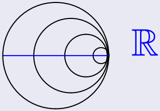
Then the orbits look like



type (K)



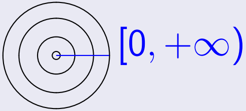
type (A)



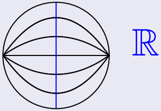
type (N)

Riemannian metrics (I) - (3/9)

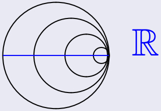
Picture for $\mathbb{R}\mathrm{H}^2$



type (K)



type (A)



type (N)

Theorem (Berndt-Brück (2001))

$M = G/K$: Riem. symmetric space of noncompact type.
$H \curvearrowright G/K$: cohomogeneity one (with H connected).
$\Rightarrow \; H\backslash M \cong \mathbb{R}$ or $[0, +\infty)$.

Fix G : Lie group with dim $G = n$.
We now consider left-invariant Riemannian metrics on G.

Def.

The **space of left-invariant Riemannian metrics**:

$$\begin{aligned}\widetilde{\mathfrak{M}} &:= \{\text{left-invariant Riem. metrics on } G\} \\ &\cong \{\text{inner products on } \mathfrak{g} := \mathrm{Lie}(G)\} \\ &\cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n).\end{aligned}$$

Note

$\mathrm{GL}_n(\mathbb{R}) = \mathrm{GL}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ by $g.\langle\cdot,\cdot\rangle := \langle g^{-1}(\cdot), g^{-1}(\cdot)\rangle$.

Riemannian metrics (I) - (5/9)

$$\mathbb{R}^\times := \{c \cdot \mathrm{id} \in \mathrm{GL}_n(\mathbb{R}) \mid c \in \mathbb{R}_{\neq 0}\},$$
$$\mathrm{Aut}(\mathfrak{g}) := \{\varphi \in \mathrm{GL}_n(\mathbb{R}) \mid \varphi[\cdot,\cdot] = [\varphi(\cdot),\varphi(\cdot)]\}.$$

Def.

The **moduli space of left-invariant Riem. metrics**:

$$\mathfrak{P}\mathfrak{M} := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}} \quad \text{(orbit space)}.$$

Remark

$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ gives rise to an isometry up to scaling.

Riemannian metrics (I) - (6/9)

Prop. (Hashinaga-T. (preprint))

$\mathfrak{g} := (\mathbb{R}^3, [,])$ with $[e_1, e_2] = e_2$ (others $= 0$).
$\langle , \rangle_0$: the canonical inner product.
$\Rightarrow$

- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is of cohomogeneity one.
- $\mathfrak{PM} = \left\{ \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\left(\begin{pmatrix} 1 & & \\ & 1 & \\ & \lambda & 1 \end{pmatrix} .\langle , \rangle_0 \right) \mid \lambda \in \mathbb{R} \right\}$.

Proof

Very direct matrix calculations.
A general theory gives a certification.

Riemannian metrics (I) - (7/9)

Cor. (Hashinaga-T. (preprint))

$\mathfrak{g} := (\mathbb{R}^3, [,])$ with $[e_1, e_2] = e_2$ (others $= 0$).
$\langle , \rangle$: any inner product.
$\Rightarrow$
$\exists \lambda \in \mathbb{R}$, $\exists k > 0$, $\exists \{x_1, x_2, x_3\}$ o.n.b. w.r.t. $k\langle , \rangle$:

$$[x_1, x_2] = x_2 - \lambda x_3, \quad \text{others} = 0.$$

Note

$\{x_1, x_2, x_3\}$ is a generalization of "Milnor frames".
Hence, this is called a **Milnor-type theorem**.

Riemannian metrics (I) - (8/9)

Result (Hashinaga-T.-Terada)

$\forall n \geq 3$,

- we construct $\mathfrak{g}$ of dimension n such that $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$: cohomogeneity one.
- we determine all possible Ricci signatures on them.

Result (Hashinaga-T.)

$\forall \mathfrak{g}$: 3-dim., solvable,

- we construct Milnor-type theorems for $\mathfrak{g}$.
- we (re)classify left-invariant Ricci soliton metrics.

Key Point: for cohomogeneity one actions, $\mathfrak{PM}$ is easy!

Riemannian metrics (I) - (9/9)

Problem (I)-1

For Riemannian symmetric spaces of noncompact type, classify (possible topological type of) orbit spaces of

- cohomogeneity two actions,
- (hyper)polar actions, ...

Problem (I)-2

Classify $\mathfrak{g}$ such that $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ are

- cohomogeneity one or two actions,
- (hyper)polar actions, ...
  ($\exists$ examples by Taketomi (2014))

Riemannian metrics (II) - (1/8)

Slogan

Left-invariant metrics vs isom. actions on $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$.

Problem (II)

$M = G/K$: Riem. symmetric space of noncompact type.
Consider $H \curvearrowright G/K$: isometric action.
Then, are there "distinguished" orbits?

Note

In this section, we mention:

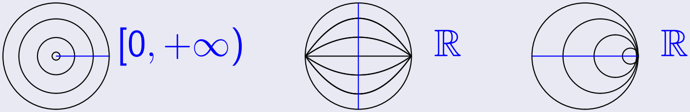
- What are known?
- How related to the study of left-invariant metrics?

Riemannian metrics (II) - (2/8)

Thm. (Berndt-T. (2003))

$M = G/K$: irr. Riem. symmetric space of noncpt type.
$H \curvearrowright G/K$: cohomogeneity one (with H connected).
$\Rightarrow$ it satisfies one of the following:

(K) $\exists!$ singular orbit.
(A) $\nexists$ singular orbit, $\exists!$ minimal orbit.
(N) $\nexists$ singular orbit, all orbits are congruent.



type (K) type (A) type (N)

This picture fits very nicely to “algebraic Ricci solitons”.

Def.

$(\mathfrak{g}, \langle , \rangle)$: **algebraic Ricci soliton (ARS)**
$:\Leftrightarrow \exists c \in \mathbb{R}, \exists D \in \mathrm{Der}(\mathfrak{g}) : \mathrm{Ric} = c \cdot \mathrm{id} + D.$

$$\mathrm{Der}(\mathfrak{g}) := \{ D \in \mathfrak{gl}(\mathfrak{g}) \mid D[\cdot, \cdot] = [D(\cdot), \cdot] + [\cdot, D(\cdot)] \}.$$

Note

- left-invariant Einstein $\Rightarrow$ algebraic Ricci soliton.
- algebraic Ricci soliton $\Rightarrow$ Ricci soliton (next page).
  (in many cases, the converse also holds)

Riemannian metrics (II) - (4/8)

Prop. (Lauret (2011))

Let us consider

- $(\mathfrak{g}, \langle , \rangle)$: algebraic Ricci soliton $(\mathrm{Ric} = c \cdot \mathrm{id} + D)$,
- (G, g) : corresponding simply-connected one.

Then one has

- $\exp(tD) \in \mathrm{Aut}(\mathfrak{g})$ for $\forall t \in \mathbb{R}$,
- $\exists \varphi_t \in \mathrm{Aut}(G)$: $(d\varphi_t)_e = \exp(tD)$,
- $X \in \mathfrak{X}(G)$ by $X_p := \frac{d}{dt}\varphi_t(p)|_{t=0}$,
- $\mathrm{ric}_g = cg - (1/2)\mathfrak{L}_X g$ (i.e., **Ricci soliton**).

Riemannian metrics (II) - (5/8)

Thm. (Hashinaga-T.)

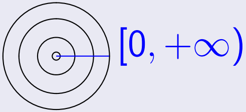
$\mathfrak{g}$: 3-dimensional, solvable. Then,

$\langle , \rangle$ on $\mathfrak{g}$ is an algebraic Ricci soliton (ARS)

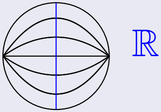
$\Leftrightarrow \widetilde{\mathfrak{M}} \supset \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$: minimal.

Note

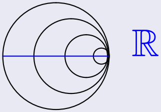
- $\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$ is a Riem. symmetric space (w.r.t. a natural $\mathrm{GL}_n(\mathbb{R})$-invariant metric).
- In these cases, $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is of cohom. ≤ 1.



type (K)



type (A)



type (N)

Thm. (Hashinaga-T.): more precise

Let $\mathfrak{g}$: 3-dimensional, solvable. Then,

- (K)-type: ARS $\Leftrightarrow$ $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).\langle,\rangle$: singular.
- (A)-type: ARS $\Leftrightarrow$ $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).\langle,\rangle$: minimal.
- (N)-type: $\nexists$ ARS.

Riemannian metrics (II) - (7/8)

Fact

For all known $\mathfrak{g}$ with $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ cohom. one, we have checked that "ARS $\Leftrightarrow$ minimal".

Thm. (Hashinaga (to appear))

$\exists \mathfrak{g}$: 4-dim., solvable :
both implications of "ARS $\Leftrightarrow$ minimal" do not hold.

Expectation

ARS $\Leftrightarrow \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ is "distinguished" in some sense?
(cf. $\langle , \rangle$: biinvariant $\Rightarrow \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ is totally geod.)

Riemannian metrics (II) - (8/8)

Problem (II)-1

For Riemannian symmetric spaces of noncompact type, study the geometry of orbits of

- cohomogeneity one actions (with H not connected),
- cohomogeneity two actions, (hyper)polar actions, ...

Problem (II)-2

Property of $\mathfrak{g}$ can be understood by group actions?

- ARS can be characterized by submanifolds?
- Tasaki-Umehara invariant for 3-dim. Lie algebras?

R-spaces - (1/5)

Def.

L : semisimple Lie group, with trivial center,
$L \supset Q$: parabolic subgroup.
Then $M := L/Q$ is called an **R-space**.

Problem (III)

Let $M = L/Q$: R-space, $H \subset L$.
Study the action $H \curvearrowright M = L/Q$.

Note

In this section, we mention: An easy example.
(Motivation will be mentioned in the next section.)

R-spaces - (2/5)

Example

$\mathbb{R}\mathrm{P}^n$ is an R-space:

$$\mathbb{R}\mathrm{P}^{n-1} = \mathrm{SL}_n(\mathbb{R}) / \left\{ \left(\begin{array}{c|ccc} * & * & \cdots & * \\ \hline 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{array} \right) \right\} .$$

Side Remark

- R-spaces can be realized as orbits of *s*-reps.
- $\mathbb{R}\mathrm{P}^{n-1}$ is an orbit of the *s*-rep. of $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$.

R-spaces - (3/5)

Set Up

We consider $\mathrm{SO}(p, q) \curvearrowright \mathbb{RP}^{n-1}$ (with $n = p + q$).

- $\mathbb{RP}^{n-1} = \mathrm{SL}_n(\mathbb{R})/Q$, and
- $\mathrm{SL}_n(\mathbb{R}) \supset \mathrm{SO}(p, q)$: symmetric.

  (an analogy of "Hermann actions"?)

Remark

$\langle , \rangle_0$: canonical inner product on $\mathbb{R}^n$, signature (p, q).
Then, $\mathrm{SO}(p, q) \curvearrowright \mathbb{R}^n$ preserves $\langle , \rangle_0$.

R-spaces - (4/5)

Prop.

$\mathrm{SO}(p, q) \curvearrowright \mathbb{R}\mathrm{P}^{n-1}$ has three orbits:

$$\begin{aligned} \mathcal{O}^+ &:= \{[v] \in \mathbb{R}\mathrm{P}^{n-1} \mid \langle v, v\rangle_0 > 0\}, \\ \mathcal{O}^0 &:= \{[v] \in \mathbb{R}\mathrm{P}^{n-1} \mid \langle v, v\rangle_0 = 0\}, \\ \mathcal{O}^- &:= \{[v] \in \mathbb{R}\mathrm{P}^{n-1} \mid \langle v, v\rangle_0 < 0\}. \end{aligned}$$

Note

$\mathcal{O}^+$, $\mathcal{O}^-$ are open.

R-spaces - (5/5)

Problem (III)-1

Let $M = L/Q$: R-space, $H \subset L$.
Study the action $H \curvearrowright M = L/Q$.

- Construct interesting examples.
- What happens if $L \supset H$ is symmetric?
- When it has an open orbit?

Problem (III)-2

Let $M = L/Q$: R-space, $H \subset L$.

- Study the geometry of orbits.
- $H.p$ can be inhomogeneous w.r.t. $\mathrm{Isom}(M)$, but have some "nice" properties?

Pseudo-Riemannian metrics - (1/6)

Slogan

Left-invariant pseudo-Riemannian metrics
vs isometric actions on $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}(p,q)$.

Problem (IV)

$M = G/K$: pseudo-Riemannian symmetric space.
Study isometric actions $H \curvearrowright M$.

In this section, we mention:

- An easy example.
- How related to left-invariant pseudo-Riem. metrics.
- How related to R-spaces.

Pseudo-Riemannian metrics - (2/6)

Prop.

The following action has exactly three orbits:

$$Q := \left\{ \left(\begin{array}{c|ccc} * & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array} \right) \right\} \curvearrowright \mathrm{GL}_{p+q}(\mathbb{R})/\mathrm{O}(p,q).$$

Proof

The orbit space coincides with the orbit space of

$$\mathrm{O}(p,q) \curvearrowright \mathrm{GL}_{p+q}(\mathbb{R})/Q = \mathbb{RP}^{p+q-1}.$$

Pseudo-Riemannian metrics - (3/6)

Fix G : Lie group with $\dim G = n = p + q$.

Def.

The **space of left-inv. metrics with signature (p, q)**:

$$\widetilde{\mathfrak{M}}_{p,q} := \{\text{left-invariant metrics on } G \text{ with } (p,q)\}$$
$$\cong \{\text{inner products on } \mathfrak{g} := \mathrm{Lie}(G) \text{ with } (p,q)\}$$
$$\cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(p,q).$$

The **moduli space of left-inv. metrics with (p, q)**:

$$\mathfrak{PM}_{p,q} := \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})\backslash\widetilde{\mathfrak{M}}_{p,q} \quad \text{(orbit space)}.$$

Pseudo-Riemannian metrics - (4/6)

Lem.

Let $\mathfrak{g}$ be one of the following:

- H^3 : Heisenberg group.
- $G_{\mathbb{R}\mathrm{H}^n}$: the group acting simply-transitively on $\mathbb{R}\mathrm{H}^n$.

Then $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is parabolic ($=$ the previous Q).

Thm. (Kubo-Onda-Taketomi-T.)

On the above Lie groups,
$\forall (p, q) \in \mathbb{N}^2$ with $n = p + q$,
$\exists$ exactly three left-invariant metrics with (p, q).

Pseudo-Riemannian metrics - (5/6)

Thm. (Kubo-Onda-Taketomi-T.): recall

On the above Lie groups,
$\forall (p, q) \in \mathbb{N}^2$ with $n = p + q$,
$\exists$ exactly three left-invariant metrics with (p, q).

Comments

- H^3 : known by Rahmani (1992), method is different.
- $G_{\mathbb{R}\mathrm{H}^n}$: known by Nomizu (1979) for Lorentzian case. The case of generic signature is new.

Side Remark

For $G_{\mathbb{R}\mathrm{H}^n}$, any left-invariant metric has const. curvature.

Pseudo-Riemannian metrics - (6/6)

Problem (IV)-1

$M = G/K$: pseudo-Riemannian symmetric space.
Study isometric actions $H \curvearrowright M$.

- First of all, study the case when H is parabolic.
  ($\leftrightarrow$ symmetric actions on R-spaces.)

Problem (IV)-2

$M = G/K$: pseudo-Riemannian symmetric space.

- Construct "nice" actions on $M = G/K$.
- Let $M' = G/K'$: Riemannian symmetric.
  If $H \curvearrowright M'$ is nice, then so is $H \curvearrowright M$?

Summary - (1/2)

Comment

Our framework is

- left-inv. metrics vs actions on symmetric spaces.
- theory of symmetric spaces is quite useful.

This would also be useful to study

- left-invariant complex structures:
  $$\{\text{almost complex strs}\} \cong \mathrm{GL}_{2n}(\mathbb{R})/\mathrm{GL}_n(\mathbb{C}).$$
- left-invariant symplectic structures:
  $$\{\text{nondegenerate 2-forms}\} \cong \mathrm{GL}_{2n}(\mathbb{R})/\mathrm{Sp}_{2n}(\mathbb{R}).$$
- and so on...

Summary - (2/2)

Comment

Our framework motivates to study

- isometric actions on noncpt Riem. symmetric sp.
- nonisometric actions on (symmetric) R-spaces.
- isometric actions on pseudo-Riem. symmetric sp.

This would provide

- new examples of actions (from $\mathrm{Aut}(\mathfrak{g})$).
- particular examples of actions with applications.
- new ideas or new notions?

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Thank you!