# Quandles and a discretization of the theory of symmetric spaces

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# Abstract

Quandle:

- An algebraic system, used in knot theory.
- Every symmetric space is a quandle.

## Our Theme

Construct structure theory of quandles,
i.e., structure theory of "discrete" symmetric spaces.

§1: Introduction
§2: Preliminary
§3: Result 1: two-point homogeneity
§4: Result 2: flatness

# Introduction - (1/5)

**Def. (Joyce (1982))**

Let $X$ be a set, and $* : X \times X \to X$ a binary operation.
Then, $(X, *)$ is called a **quandle**

$:\Leftrightarrow$ (Q1) $\forall x \in X$, $x * x = x$.

(Q2) $\forall x, y \in X$, $\exists! z \in X$ : $z * y = x$.

(Q3) $\forall x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

# Introduction - (2/5)

(Q1), (Q2), (Q3) $\leftrightarrow$ Reidemeister moves (I), (II), (III).

## Def.

Let $K$ be an oriented knot in $\mathbb{R}^3$, $[K]$ its diagram,
and $(X, *)$ be a quandle.
Then, a map $[K] \to X$ is **quandle coloring**
$:\Leftrightarrow$ crossing and quandle operation are "compatible".

## Fact

Quandle colorings are invariant by Reidemeister moves.
In particular, the number of colorings are knot invariant.

# Introduction - (3/5)

## Recall

(Q1) $\forall x \in X,\ x * x = x.$

(Q2) $\forall x, y \in X,\ \exists! z \in X : z * y = x.$

(Q3) $\forall x, y, z \in X,\ (x * y) * z = (x * z) * (y * z).$

## Prop.

Let $X$ be a set, and $s : X \to \mathrm{Map}(X, X) : x \mapsto s_x$.
Then, $* : X \times X \mapsto X : (y, x) \mapsto s_x(y)$ gives a quandle

$\Leftrightarrow$ (S1) $\forall x \in X,\ s_x(x) = x.$

(S2) $\forall x \in X,\ s_x$ is bijective.

(S3) $\forall x, y \in X,\ s_x \circ s_y = s_{s_x(y)} \circ s_x.$

# Introduction - (4/5)

## Notation

$(X, s)$ denotes a quandle, where $s : X \to \mathrm{Map}(X, X)$.

## Prop. (Joyce (1982))

A connected Riemannian symmetric space is a quandle.

## Fact

Affine symmetric spaces and $k$-symmetric spaces are also quandles.

# Introduction - (5/5)

## Example

$(X, s)$ is a quandle if $s_x := \mathrm{id}_X$ (**trivial quandle**).

## Example

The following $(X, s)$ is a quandle (**dihedral quandle**):

- $X := \{p_1, \ldots, p_n : n$ equally divided points on $S^1\}$,
- $s_x :=$ [a reflection w.r.t. the central axis $ox$].

## Example

The following $(X, s)$ is the **tetrahedron quandle**:

- $X := \{$vertices of the regular tetrahedron$\}$,
- $s_x :=$ [a rotation by $120^\circ$ in a suitable way].

# Preliminary - (1/6)

**Aim of this section**

$(X, s)$ : a "homogeneous" quandle

$$\updownarrow$$

$(G, K, \sigma)$ : an "analogue" of a symmetric pair.

# Preliminary - (2/6)

What is "homogeneous"?

**Def.**

Let $f : (X, s^X) \to (Y, s^Y)$ be a map.

- $f$ a **homomorphism** $:\Leftrightarrow \forall x \in X,\ f \circ s_x^X = s_{f(x)}^Y \circ f$.
- $f$ an **isomorphism** $:\Leftrightarrow$ a bijective homomorphism.

**Prop.**

$\forall x \in X,\ s_x : X \to X$ is an isomorphism, because of

(S2) $\forall x \in X,\ s_x$ is bijective.

(S3) $\forall x, y \in X,\ s_x \circ s_y = s_{s_x(y)} \circ s_x$.

# Preliminary - (3/6)

## Def.

Define the **(inner) automorphism group** by

- $\mathrm{Aut}(X, s) := \{f : X \to X : \text{isomorphism}\}$.
- $\mathrm{Inn}(X, s) := \langle \{s_x \mid x \in X\} \rangle$.

$(X, s)$ is said to be

- **homogeneous** $:\Leftrightarrow \mathrm{Aut}(X, s) \curvearrowright X$ is transitive.
- **connected** $:\Leftrightarrow \mathrm{Inn}(X, s) \curvearrowright X$ is transitive.

## Example

- $R_n$ (dihedral quandle) is homogeneous.
- $R_n$ (dihedral quandle) is connected $\Leftrightarrow$ $n$ is odd.

# Preliminary - (4/6)

What is an "analogue" of a symmetric pair?

**Def.**

$(G, K, \sigma)$ is a **quandle triplet**

$:\Leftrightarrow$ $G$ is a group,

$K$ is a subgroup of $G$,

$\sigma \in \mathrm{Aut}(G)$,

$K \subset \mathrm{Fix}(\sigma, G)$.

# Preliminary - (5/6)

**Prop.**

$(X, s)$ is a homogeneous quandle

$\Rightarrow$ $(G, K, \sigma)$ is a quandle triplet, where

$$G := \mathrm{Aut}(X, s),\ K := G_x,\ \sigma(g) := s_x \circ g \circ s_x^{-1}.$$

**Prop.**

$(G, K, \sigma)$ is a quandle triplet

$\Rightarrow$ $(G/K, s)$ is a homogeneous quandle, where

$$s_{[e]}([h]) := [\sigma(h)].$$

$$s_{[g]}([h]) := [g\sigma(g^{-1}h)].$$

# Preliminary - (6/6)

**Notation**

$Q(G, K, \sigma)$ : the quandle obtained by $(G, K, \sigma)$.

**Example**

$Q(G, K, \mathrm{id})$ is a trivial quandle.

**Example**

$Q(\mathbb{Z}_n, \{0\}, -\mathrm{id})$ is a dihedral quandle.

# Result 1 - (1/7)

**Result 1 (T., Iwanaga, Vendramin, Wada)**

$(X, s)$ is a "two-point homogeneous" finite quandle
$\Leftrightarrow$ $(X, s) \cong$ "some Alexander quandle".

# Result 1 - (2/7)

What is a "two-point homogeneous" quandle?

## Def. (T. (2013))

$(X, s)$ is **two-point homogeneous**

$:\Leftrightarrow \ \forall (x_1, x_2), (y_1, y_2)$ : distinct,
$\exists f \in \mathrm{Inn}(X, s) : f(x_1, x_2) = (y_1, y_2)$.

Recall that $\mathrm{Inn}(X, s) := \langle \{ s_x \mid x \in X \} \rangle$.

# Result 1 - (3/7)

Two-point homogeneous quandles are analogues of "two-point homogeneous Riemannian manifolds".

## 補足

A Riem. manifold $(M, g)$ is **two-point homogeneous**

$$:\Leftrightarrow \forall (x_1, x_2), (y_1, y_2) : \text{equidistant},$$
$$\exists f \in \mathrm{Isom}(M, g) : f(x_1, x_2) = (y_1, y_2)$$

$\Leftrightarrow$ $(M, g)$ : isotropic
(i.e., $\mathrm{Isom}(M, g)_x \curvearrowright T_x M$ (the isotropy representation) is transitive on the unit sphere)

$\Leftrightarrow$ $(M, g) \cong \mathbb{R}^n$ or rank one symmetric spaces.

# Result 1 - (4/7)

## Recall

Two-point Riem. manifolds can be characterized in terms of the isotropy actions.

## Prop. (T. (2013))

$(X, s)$ is two-point homogeneous

$\Leftrightarrow \ \forall x \in X, \ \mathrm{Inn}(X, s)_x \curvearrowright X \setminus \{x\}$ is transitive.

## Example

- $R_3$ (dihedral quandle) is two-point homogeneous.
- $R_n$ $(n \geq 4)$ is not two-point homogeneous.
- The tetrahedron quandle is two-point homogeneous.

# Result 1 - (5/7)

What are "Alexander quandles"?

## Recall

$G$ is a group, and $\varphi \in \mathrm{Aut}(G)$

$\Rightarrow$ $(G, \{e\}, \varphi)$ is always a quandle triplet

$\Rightarrow$ $Q(G, \varphi) := Q(G, \{e\}, \varphi)$ the associated quandle.

## Def.

The above $Q(G, \varphi)$ is an **Alexander quandle**

$:\Leftrightarrow$ $G$ is abelian.

# Result 1 - (6/7)

## Result 1 (T., I., V., W.) in detail

$(X, s)$ is a two-point homogeneous quandle

$\Leftrightarrow$ $(X, s) \cong Q(\mathbb{F}_q, L_a)$,

where $\mathbb{F}_q$ is the finite field with cardinality $q$,

and $a$ is a primitive root of $\mathbb{F}_q$ (with $L_a(x) := ax$).

## Division of roles

- T. (2013) classified ones with $\# X = p$ (prime).
- Iwanaga (2013) classified ones with $\# X = p^2$.
- Vendramin (in press) showed that $\# X = p^m$.
- Wada (preprint) classified ones with $\# X = p^m$.

# Result 1 - (7/7)

Why it is related to "primitive roots"?

## Def.

$a \in \mathbb{F}_q$ is a **primitive root**

$:\Leftrightarrow$ $a \in \mathbb{F}_q$ generates $\mathbb{F}_q^\times$ $(= \mathbb{F}_q \setminus \{0\} \cong \mathbb{Z}_{q-1})$.

## Prop.

$Q(\mathbb{F}_q, L_a)$ with $a \in \mathbb{F}_q$ satisfies

- the isotropy subgroup: $G_0 = \langle s_0 \rangle = \langle L_a \rangle$.
- the orbit: $(G_0).1 = \{(L_a)^k(1)\} = \{1, a, a^2, a^3, \ldots\}$.
- Therefore, $G_0 \curvearrowright \mathbb{F}_q \setminus \{0\}$ is transitive

  $\Leftrightarrow$ $a$ is a primitive root.

# Result 2 - (1/4)

## Result 2 (Ishihara-T.)

$(X, s)$ is a "flat" finite quandle

$\Leftrightarrow$ $\exists q_1, \ldots, q_n$ (odd prime powers) s.t.
$(X, s) \cong R_{q_1} \times \cdots \times R_{q_n}$.

# Result 2 - (2/4)

What is a "flat" quandle?

**Def.**

A connected quandle $(X, s)$ is **flat**

$:\Leftrightarrow \ G^0(X, s) := \langle \{ s_p \circ s_q \mid p, q \in X \} \rangle$ is abelian.

**Fact (cf. Loos)**

A connected Riem. symmetric space $(M, g)$ is flat

$\Leftrightarrow \ G^0(M, g) := \langle \{ s_p \circ s_q \mid p, q \in M \} \rangle$ is abelian.

# Result 2 - (3/4)

## Example

$S^1$ is a flat symmetric space, and satisfies

- $\mathrm{Inn}(S^1) = \mathrm{O}(2)$ $(= \mathrm{Isom}(S^1))$.
- $G^0(S^1) = \mathrm{SO}(2)$ $(= \mathrm{Isom}^0(S^1))$, abelian.

## Example

The dihedral quandle $R_n$ (with $n$ odd) is flat.

# Result 2 - (4/4)

## Result 2 (Ishihara-T.): recall

$(X, s)$ is a flat finite quandle

$\Leftrightarrow$ $\exists q_1, \ldots, q_n$ (odd prime powers) :
$(X, s) \cong R_{q_1} \times \cdots \times R_{q_n}$.

## Idea of Proof for ($\Rightarrow$)

Let $(X, s)$ be a flat finite quandle.

(Step 1) $G^0(X, s)$ is a finite abelian group.

(Step 2) $(X, s) \cong Q(\mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_n}, \sigma)$.

(Step 3) Show that $\sigma = -\mathrm{id}$.

# Further Problems

## Problem

Study homogeneous but disconnected quandles.

- We only considered connected (TPH, flat) quandles.
- What happens if we just assume "homogeneous"?

## Example

$\{\pm e_1, \pm e_2, \pm e_3\} \subset S^2$ is a "subquandle".
This is flat, homogeneous, but disconnected.

## Comment

$\{e_i \wedge e_j \mid i \neq j\} \subset G_2(\mathbb{R}^n)^{\sim}$ is a "subquandle",
a nice example of homogeneous disconnected quandles.

# Further Problems

## Theme

Study structures of "discrete symmetric spaces".

## Further Plans - on quandles

- Are there congruency of maximal flats?
- Can we define the "rank"?
- Is it related to two-point homogeneity?
- What happens for infinite discrete quandles?

## Further Plans - on symmetric spaces

- Characterize properties in terms of the symmetries.
  (polars, meridians, ... are very very interesting.)

Thank you for your kind attention!

and

Professors Katsuya Mashimo and Yoshihisa Kitagawa,
Congratulations on your 60th birthday!!

and

We would like to thank the organizers!