The space of left-invariant metrics and submanifold geometry

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Abstract

Our theme is

- geometry of left-invariant metrics on Lie groups. (Einstein/(algebraic) Ricci soliton metrics)

Our study is from the viewpoint of

- submanifolds in noncompact symmetric spaces.

In this talk we mention that

- for 3-dim. solvable case, $\exists$ a nice correspondence: algebraic Ricci solitons $\longleftrightarrow$ submanifold geometry.

Introduction - (1/3)

Contents

Notation

Throughout this talk, we denote by

- $(G, \langle , \rangle)$: a simply-connected Lie group with a left-invariant Riemannian metric.
- $(\mathfrak{g}, \langle , \rangle)$: the corresponding metric Lie algebra.

Introduction - (2/3)

Def.

$(G, \langle , \rangle)$ or $(\mathfrak{g}, \langle , \rangle)$ is

- **Einstein** $:\Leftrightarrow \mathrm{Ric} = c \cdot \mathrm{id}$ $(\exists c \in \mathbb{R})$.
- **algebraic Ricci soliton**
  $:\Leftrightarrow \mathrm{Ric} = c \cdot \mathrm{id} + D$ $(\exists c \in \mathbb{R}, \exists D \in \mathrm{Der}(\mathfrak{g}))$
- **Ricci soliton**
  $:\Leftrightarrow \mathrm{ric} = c\langle , \rangle + \mathfrak{L}_X \langle , \rangle$ $(\exists c \in \mathbb{R}, \exists X \in \mathfrak{X}(G))$.

Fact. (Lauret (2001, 2011))

Einstein $\Rightarrow$ algebraic Ricci soliton $\Rightarrow$ Ricci soliton.

Introduction - (3/3)

General Problem

Examine whether G admits a "distinguished" left-invariant metric or not.

Our Approach

We study left-invariant metrics in terms of submanifold geometry in noncompact symmetric spaces.

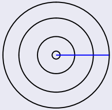
Cohomogeneity One Actions - (1/2)

Def.

An isom. action $H \curvearrowright (M, g)$ is of **cohomogeneity one** $:\Leftrightarrow$ a regular orbit has codimension one.

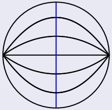
Ex.

Consider $\mathrm{SL}(2, \mathbb{R}) = KAN$: the Iwasawa decomposition. Then, $K, A, N \curvearrowright \mathbb{R}\mathrm{H}^2$ is of cohomogeneity one, and their orbits are as follows:



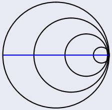
$[0, +\infty)$

type (K)



$\mathbb{R}$

type (A)



$\mathbb{R}$

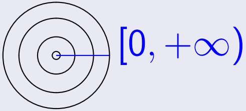
type (N)

Cohomogeneity One Actions - (2/2)

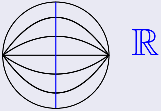
Thm. (Berndt-T., 2003, 2013)

M : irr. Riem. symmetric space of noncompact type.
$H \curvearrowright M$: cohomogeneity one (with H connected).
Then, it is of type (K), (A), or (N):

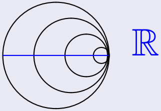
- (K): $\exists 1$ singular orbit.
- (A): $\nexists$ singular orbit, $\exists 1$ minimal orbit.
- (N): $\nexists$ singular orbit, all orbits are congruent.



type (K)



type (A)



type (N)

Main Result - (1/7)

Framework

For each $(\mathfrak{g}, \langle , \rangle)$, we have a submanifold in a noncompact Riemannian symmetric space.

Symmetric spaces:

Let G be given, and put $n := \dim G = \dim \mathfrak{g}$. Then we have a noncompact Riemannian symmetric space:

$$\begin{aligned} \widetilde{\mathfrak{M}} &:= \{\langle , \rangle : \text{an inner product on } \mathfrak{g}\} \\ &\cong \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n). \end{aligned}$$

$\widetilde{\mathfrak{M}} \cong \{\langle , \rangle$: a left-invariant Riemannian metric on $G\}$.

Main Result - (2/7)

Submanifolds:

For each $\langle , \rangle$, we have a (homogeneous) submanifold:

$$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle \subset \widetilde{\mathfrak{M}}$$

Note

$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ is the isometry and scaling class of $\langle , \rangle$. This action preserves Einstein/(algebraic) Ricci soliton.

Note

In some cases, the uniqueness (up to $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$) holds. (e.g., Einstein/algebraic Ricci solitons on solvable $\mathfrak{g}$.)

Main Result - (3/7)

Problem

$\langle , \rangle$ is a "distinguished" left-invariant metric on G
$\Leftrightarrow \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ is a "distinguished" submanifold in $\widetilde{\mathfrak{M}}$?

Main Thm. (Hashinaga-T.)

Let $\mathfrak{g}$ be a solvable Lie algebra of dimension three.
Then, $\langle , \rangle$ is an **algebraic Ricci soliton** on $\mathfrak{g}$
$\Leftrightarrow \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ is a **minimal** submanifold in $\widetilde{\mathfrak{M}}$.

Key

$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is of cohomogeneity at most one.

Main Result - (4/7)

Prop.

If $\mathfrak{g} = \mathbb{R}^3, \mathfrak{h}^3, \mathfrak{g}_{\mathbb{R}\mathrm{H}^3}$, then $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is transitive.

- $\mathbb{R}^3$: abelian,
- $\mathfrak{h}^3$: Heisenberg,
- $\mathfrak{g}_{\mathbb{R}\mathrm{H}^3}$: the Lie algebra of $\mathbb{R}\mathrm{H}^3$.

Otherwise, $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is of cohomogeneity one.

Rem.

Main Theorem is true for $\mathfrak{g} = \mathbb{R}^3, \mathfrak{h}^3, \mathfrak{g}_{\mathbb{R}\mathrm{H}^3}$, since

- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle = \widetilde{\mathfrak{M}}$, which is minimal in $\widetilde{\mathfrak{M}}$.
- $\forall \langle , \rangle$ is known to be an algebraic Ricci soliton.

It remains to study 3 families of Lie algebras, $\mathfrak{r}_3$, $\mathfrak{r}_{3,a}$, $\mathfrak{r}'_{3,a}$.

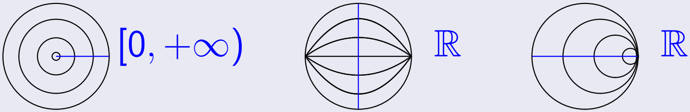
Prop.

$\mathfrak{g} := \mathfrak{r}_3 = \mathrm{span}\{e_1, e_2, e_3\}$ with

$$[e_1, e_2] = e_2 + e_3, \ [e_1, e_3] = e_3, \ [e_2, e_3] = 0.$$

Then we have

- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is of type (N).
- $\nexists \langle , \rangle$: ARS on $\mathfrak{g}$.



Main Result - (6/7)

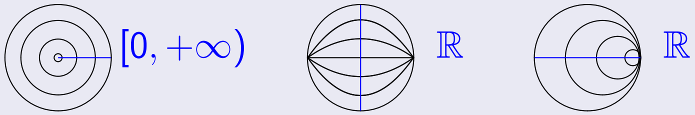
Prop.

$\mathfrak{g} := \mathfrak{r}_{3,a} = \mathrm{span}\{e_1, e_2, e_3\}$ with $-1 \leq a < 1$,

$$[e_1, e_2] = e_2,\ [e_1, e_3] = ae_3,\ [e_2, e_3] = 0.$$

Then we have

- $(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))^0 \curvearrowright \widetilde{\mathfrak{M}}$ is of type (A).
- $\langle , \rangle$ is ARS $\Leftrightarrow$ $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ is a minimal orbit.



type (K) type (A) type (N)

Main Result - (7/7)

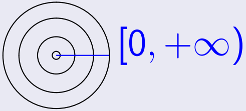
Prop.

$\mathfrak{g} := \mathfrak{r}'_{3,a} = \mathrm{span}\{e_1, e_2, e_3\}$ with $a \geq 0$,

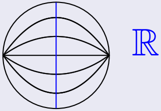
$$[e_1, e_2] = ae_2 - e_3,\ [e_1, e_3] = e_2 + ae_3,\ [e_2, e_3] = 0.$$

Then we have

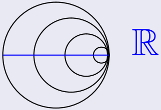
- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is of type (K).
- $\langle , \rangle$ is ARS (in fact, Einstein)
  $\Leftrightarrow \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ is a singular orbit.



type (K)



type (A)



type (N)

Some Remarks - (1/2)

Rem. 1

Main Theorem gives a nice correspondence between geometric structures (ARS) $\longleftrightarrow$ submanifold geometry.

Rem. 2

Main Theorem would lead to study invariants of $\mathfrak{g}$:

- Properties of $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ are invariants of $\mathfrak{g}$.
- Properties of $\mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ are invariants of $\langle , \rangle$.

They are interesting, because it is believed that

$\exists\langle , \rangle$: distinguished $\Rightarrow$ restrictions on algebraic structures on $\mathfrak{g}$.

Some Remarks - (2/2) : recent works

Thm. (Hashinaga-T.-Terada (to appear))

An expression of the orbit space $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}$ derives a generalization of "Milnor frames".

Thm. (Taketomi-T.)

$\forall n \geq 3$, $\exists G$ of dimension n :

- $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \curvearrowright \widetilde{\mathfrak{M}}$ is of type (N),
- $\nexists$ left-invariant Ricci solitons on G.

Thm. (Hashinaga 2014)

"ARS $\leftrightarrow$ minimal" is true for 4-dim. nilpotent case, but not true in general for 4-dim. solvable ones.

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Thank you!